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Contest success functions*

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Summary. Tournaments, conflict, and rent-seeking have been modelled as contests in which participants exert effort to increase their probability of winning a prize. A Contest Success Function (CSF) provides each player's probability of winning as a function of all players' efforts. In this paper the additive CSF employed in most contests is axiomatized, with an independence from irrelevant alternatives property as the key axiom. Two frequently used functional forms are also axiomatized: one in which winning probabilities depend on the ratio of players' efforts and the other in which winning probabilities depend on the difference in efforts.

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1. Introduction

A contest is a game in which the players compete for a prize by exerting effort so as to increase their probability of winning. Contests have been used to study a variety of topics, including rent-seeking (e.g., Tullock (1980), Nitzan (1991), Baye et al. (1993)), tournaments (e.g. Rosen (1986)), conflict (e.g., Hirshleifer (1991), Skaperdas (1992)), and political campaigns (Skaperdas and Grofman, 1995); Dixit (1987) has examined contests in general. A critical component of a contest is the *Contest Success Function* ("CSF") which provides each player's probability of winning for any given level of efforts. CSFs are formally similar to probabilistic choice functions used in other disciplines.[1] A considerable majority of the papers in the contest literature has been employing specific functional forms or classes of functional forms of CSFs without any particular reason other than analytical convenience. One way of putting the use

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[1] See Luce and Suppes (1965) and Suppes et al. (1989, Ch. 17) for general axiomatic developments, Coughlin (1986) in the context of probabilistic voting models, and McFadden (1984) for discrete choice econometrics.

of a CSF on a surer footing and developing a better understanding of any advantages or limitations it might have is to derive the CSF from easily interpretable axioms.[2]

Such an axiomatic approach is followed in this paper. After introducing basic properties that ensure a symmetric CSF across contestants, I axiomatize a convenenient additive representation used in all the papers mentioned above. The representation is partly motivated by the study of coalition formation in contests and an "independence from irrelevant alternatives" axiom is a key axiom. It says that if, for example, two players engage in a contest just by themselves, the outcome of their contest should not depend on the efforts of other players not participating in the contest.

In addition, I axiomatize two functional forms within the class of additive CSFs: one in which winning probabilities in contests depend on the ratio of the efforts of the players and another one in which the winning probabilities depend on the difference in efforts. Hirshleifer (1989) has compared the properties of these two functional forms. The main contribution of the axiomatizations of these functional forms in this paper is that they are also found to be the only ones satisfying the just-mentioned properties. The functional form in which the outcome of contests depends on the ratio of efforts is the one usually employed in the rent-seeking literature.

2. Axiomatizations

Let $N=\{1,2,\ldots,n\}$ be the set of players who may participate in a contest and let y_i denote the "effort," or "strategic endowment" of player $i\in N$. In military contexts this effort could be arms, in rent-seeking environments it could represent the amount of money spent by each player, and in tournaments it could be the amount of effort expended by each participant. In any particular contest with a given prize size, costs of effort, and other characteristics the players would choose a specific vector of efforts. The CSF gives the probability of each player winning the contest for any given vector of efforts. The axiomatizations that follow do not depend on the specification of the contest and, therefore, on the particular realization of efforts; the focus is on deriving CSFs based on axioms that are independent of a particular vector of efforts.[3] I first examine the n-player case, then axiomatize the main functional form by considering contests among a smaller number of players, and finally axiomatize the two functional forms.

2.1. *n-player contests*

Let $y=(y_1,y_2,\ldots,y_n)$ denote a vector of efforts for the n players. Each player i's winning probability is denoted by $p^i(y)$($p^i\colon[0,Y]^n\to\mathbb{R}$ where $Y>y_i$ for all $i\in N$). The following properties are maintained.

[2] Another way to ground a CSF is its derivation from primitive distributional assumptions on the underlying contest. Whereas the logit CSF can be derived in such a way (see McFadden, 1984), the general additive form and the power functional form axiomatized in this paper do not appear to have been derived that way. The two approaches – the axiomatic and the one based on distributions – are clearly complementary in understanding the limitations and advantages of a CSF.

[3] As, for example, axiomatizations of utility functions do not depend on an agent's particular choice or on the model the utility function is embedded.

(A1) $\sum_{i\in N} p^i(y) = 1$ and $p^i(y) \geq 0$ for all $i \in N$ and all y; if $y_i > 0$ then $p^i(y) > 0$.
(A2) For all $i \in N$ $p^i(y)$ is increasing in y_i and decreasing in y_j for all $j \neq i$.
(A3) For any permutation π of N (i.e., a bijection $\pi: N \to N$) we have

$$p^{\pi(i)}(y) = p(y_{\pi_1}, y_{\pi_2}, \ldots, y_{\pi_n}) \, \forall i \in N$$

(A1) says that the contest success function satisfies the conditions of a probability distribution function and, in addition, when the effort of a player is positive that player's probability of success is also positive. The positivity of each player's probability of success, although not needed for the respresentation in (1) below, becomes necessary in the proof of Theorem 1.[4] (A2) states that a player's probability of success is increasing in the player's own effort but decreasing in every other player's effort. (A3) is an anonymity property stating that each player's probability of success should not depend on his or her identity or on the identities of their opponents, but just on the efforts of the players. This property also implies that if in some vector of efforts two players have identical efforts then their probabilities of success must be equal and if all players were to exert identical efforts, then each one of them would have a probability of success equal to $1/n$.

For any $k \in N$, let $y_{-k} = (y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$. Let $p^1, p^2, \ldots, p^n$ be functions satisfying (A1)–(A3). Then it can be shown that there exists a function $p: [0, Y]^n \to \mathbb{R}$ which is increasing in its first argument, decreasing in the remainder $n-1$ arguments, and such that for any given vector of efforts $y^0 = (y_1^0, y_2^0, \ldots, y_n^0)$ we have

$$p^k(y^0) = p(y_k^0, y_{-k}^0) \text{ for all } k \in N \tag{1}$$

That is, (A1)–(A3) guarantee that each player's probability of success is governed by the same function as every other player's probability. Examples of CSF's satisfying (A1)–(A3) include: (i) $e^{ky_i}/(\sum_{j\in N} e^{ky_j})$ where $k > 0$, (ii) $y_i^m/(\sum_{j\in N} y_j^m)$ where $m > 0$, (iii) $y_i^m - (\sum_{j\neq i} y_j^m)/(n-1) + 1/n$ where $y_j^m \leq 1/n$ for all j, and (iv) $g[f(y_i) - f(\sum_{j\neq i} y_j^m)/(n-1)]$ for appropriately defined $g(\cdot)$ and $f(\cdot)$.[5]

2.2. *The CSF with fewer players*

Given the CSF that provides the probability of each player's success in the n-way contest, where every player acts independently of the others, what should be the more generalized CSF governing the interaction between coalitions of players or between players belonging to a subset of the original set of players? Specifically, we ask the following question: If a nonempty subset of the players, $M \subseteq N$, were to break off from the other players and engage in a contest amongst themselves, what would be the probability of success of each player in that subset? Denote by $p_m^i(y)$ the ith player's probability of success who participates in a contest among the members of the subset M which we assume to have at least two elements. We assume this to

[4] Thus this axiom excludes a CSF in which the player with the largest strategic endowment wins for sure, like in Baye et al. (1993). Such a CSF could be considered as the limiting case of, say, the CSF in (A6′) below as m approaches infinity.

[5] This last example (and the more special case in (iii)) is one possible generalization of the two-player CSF that Dixit (1987) calls "probit."

be as follows:

$$p^i_m(y) = p^i(y)/[\sum_{j\in M} p^j(y)]\ \forall i \in M \text{ and } \forall M \subseteq N \text{ with at least two elements.} \quad \text{(A4)}$$

This is a basic consistency property, implying that the contests among smaller numbers of players are qualitatively similar to those among a larger number of them. It is a non-trivial property as indicated by considering example (iii) above. If we originally had $n = 2$, the winning probability of player 1 would equal $y_1 - y_2 + 1/2$. If, however, n were set equal to 3 and we were to derive using (A4) the winning probability of player 1 if he were to enter into a contest against 2 (with player 3 staying in the sidelines), we would arrive at the number $[y_1 - (1/2)y_2 - (1/2)y_3 + 1/3]/[(1/2)y_1 + (1/2)y_2 - y_3 + 2/3]$ which is in general different from the one above. This last number also shows the possible dependence of the outcome of the contest among a subset of the players on the efforts or strategic endowments of other players not participating in that specific contests. To rule out such possibilities, in addition to (A4), (A5) below is adopted.

(A5) $p^i_m(y)$ is independent of the efforts of the players not included in the subset M ($\subset N$); or, $p^i_m(y)$ can be written as $p^i_m(y_m)$ where $y_m = (y_j; j \in M)$.

Thus, the outcome in the m-way contest among the members of M should not depend on the efforts of the players not participating in the contest and (A5) can be characterized as an "independence from irrelevant alternatives" property. That property has been criticized in the context of individual probabilistic choice; the choice in any subset of alternatives does not depend on the alternatives available outside the subset (see, for example, Samuelson, 1985). This criticism though does not have much force in our context where the choices are not consciously made by an individual.

The projection of the CSF into dimensions lower than n can be employeed not only in contests between a subset of players, but also in contests between coalitions of players. For example, with $n = 3$ any two players could pool their efforts and engage in a contest against the third player as if they were one player. That is, if the members of the two-player coalition had efforts of w and z, and the third player had x, the winning probability of the coalition would be represented by $p_2(w + z, x)[p_2(\cdot,\cdot)$ is the projection of the CSF in two dimensions] while the win probability of the third player would equal $p_2(x, w + z)\ [= 1 - p_2(w + z, x)]$.

Examples (iii) and (iv), mentioned at the end of section 2.1, do not satisfy (A5). Examples (i) and (ii), however, do satisfy (A5) and (A4); both of these examples belong to the following class of CSFs that satisfies (A1)–(A5):

(A5′) $p^i(y) = f(y_i)/(\sum_{j\in N} f(y_j))$ for all $i \in N$ and $p^i_m(y) = f(y_i)/(\sum_{j\in M} f(y_j))$ for all $i \in M(\subset N)$ where $f(\cdot)$ is a positive increasing function of its argument.

It is evident that $f(\cdot)$ is unique up to positive multiplicative transformations. The following result establishes that the class described in (A5′) is also the only class of CSF's satisfying (A1)–(A5).

Theorem 1: *(A1)–(A5) are satisfied if and only if the Contest Success Function satisfies (A5′).*

Proof: It is straightforward to show that a CSF satisfying (A5′) satisfies (A1)–(A5). Thus, we only show that (A1)–(A5) imply a CSF of the form described in (A5′). By (A4) and (A5) we have

$$\frac{p^i(y)}{p^j(y)} = \frac{p^i_m(y_m)}{p^j_m(y_m)} \tag{2}$$

for all subsets M of N. By (A1)–(A3) (which imply the representation in (1)), along with (A4) and (A5), for every $M \subset N$ there is a function of $|M|$ variables such that

$$p^i_m(y_m) = p_m(y_i, y_{-i})$$

where y_{-i} represents the vector efforts of the other players who belong to M and $p_m(\cdot)$ is the projection of the function $p(\cdot)$ in $|M|$ dimensions. In addition, by (A1) and (A4)

$$\sum_{i \in M} P_m(y_i, y_{-i}) = 1$$

Thus, we have [$p_2(\cdot,\cdot)$ denotes the two-player CSF]

$$\begin{aligned}
1 &= \left[p^i_m(y_m) \Big/ \sum_{j \neq i} p^j_m(y_m) \right] [p^k_m(y_m)/p^i_m(y_m)] \left[\sum_{j \neq i} p^j_m(y_m)/p^k_m(y_m) \right] \\
&= [p_m(y_i, y_{-i})/(1 - p_m(y_i, y_{-i}))][p_2(y_k, y_i)/p_2(y_i, y_k)] \left[\sum_{\substack{j \neq i \\ j \in M}} p_2(y_j, y_k)/p_2(y_k, y_j) \right] \\
&= [p_m(y_i, y_{-i})/(1 - p_m(y_i, y_{-i}))][(1 - p_2(y_i, y_k))/p_2(y_i, y_k)] \\
&\quad \times \left[\sum_{\substack{j \neq i \\ j \in M}} p_2(y_j, y_k)/(1 - p_2(y_j, y_k)) \right]
\end{aligned} \tag{3}$$

Fix $y_k = \alpha > 0$ and let $f(y_i) = p_2(y_i, \alpha)/(1 - p_2(y_i, \alpha))$ which is positive (since, by (A1), $0 < p_2(y_i, \alpha) < 1$) and, by (A2), increasing in y_i. Then (3) becomes

$$1 = [p_m(y_i, y_{-i})/(1 - p_m(y_i, y_{-i}))][1/f(y_i)] \left[\sum_{\substack{j \neq i \\ j \in M}} f(y_j) \right]$$

which implies

$$p_m(y_i, y_{-i}) = f(y_i) \Big/ \left[\sum_{j \in M} f(y_j) \right]$$

Therefore, for all $M \subset N$ the CSF has the additive representation described in (A5′). ■

2.3. *Two Functional Forms: Power and Logit*

The next task is to axiomatize the two functional forms that have been employeed in applications of contests; both are special cases of (A5′) and therefore satisfy (A1)–(A5).

First, let $\lambda y = (\lambda y_1, \lambda y_2, \ldots, \lambda y_n)$ and consider the following homogeneity axiom.

$$p^i(\lambda y) = p^i(y) \text{ for all } \lambda > 0 \text{ and for all } i \in N. \tag{A6}$$

According to this property an equiproportionate change in the efforts of all players would leave the winning probability of every player unaffected. The implication is that the ratio of winning probabilities of any two players depends on the ratio of their efforts. The following functional form, mentioned earlier and which is widely employed in the rent seeking literature, satisfies (A6) [in addition to (A1)–(A5)]:

$$f(y_i) = \alpha y_i^m \text{ for some } \alpha > 0 \text{ and } m > 0. \tag{A6'}$$

Moreoever, as shown below this is the only continuous functional form satisfying all six axioms.

Theorem 2: *(A6′) is the only continuous functional form satisfying (A1)–(A6).*

Proof: By Theorem 1, (A1)–(A5) imply the additive representation in (A5′) which along with (A6) yields

$$f(\lambda x)/f(x) = f(\lambda z)/f(z) \text{ for any } x, z, \lambda > 0 \tag{4}$$

By setting $z = 1$, (4) implies:

$$\frac{f(\lambda x)}{f(1)} = \frac{f(\lambda)}{f(1)} \frac{f(x)}{f(1)} \tag{4'}$$

Let $F(z) \equiv f(z)/f(1)$. Then (4′) becomes:

$$F(\lambda x) = F(\lambda)F(x)(\lambda > 0 \text{ and } x > 0) \tag{4''}$$

The rest of the proof consists of transforming (4″) to a *Cauchy* equation (see (5′) below).[6] Let $\lambda = e^t$, $x = e^s$, and denote $F(e^t)$ by $g(t)$. By substitution into (4″) we obtain:

$$g(t + s) = g(t)g(s) \tag{5}$$

Finally, letting $h(u) = \ln g(u)$, (5) implies:

$$h(t + s) = h(t) + h(s) \tag{5'}$$

The only continuous solution of (5′) [see, e.g., Aczel (1969)] is $h(u) = mu$ for some m, or that $g(u) = e^{mu}$. Substituting back into $F(\cdot)$ we have $F(e^u) = (e^u)^m$ and $z = e^u$ yields $F(z) = z^m$. Finally, we have $f(z) = az^m$ where $f(1) = a > 0$ and, given (A2), $m > 0$ which satisfies (A6′). □

To axiomatize the second functional form, let $c \in R^n$ denote the constant vector with all its components equal to c and consider the following axiom which implies that the winning probability of each player depends only on the difference in the efforts among all players.

$p^i(y) = p^i(y + c)$ $\forall c \in R^n$ such that $y_i + c \geq 0, \forall i \in N$; in addition, assume $f(\cdot)$ is defined for $y_i = 0$. (A7)

[6] See Aczel (1969) which the rest of the proof closely follows (pp. 51–53).

Clearly, this is a strong property since it requires the winning probabilities of, say, $y_1 = 1$ and $y_2 = 2$ to be the same as when $y_1 = 10001$ and $y_2 = 10002$. The main objective of an axiomatization, however, is a better understanding of the representation that is being axiomatized, not necessarily the advocacy of a certain axiom. And, as we will mention at the end of this section, the companion functional form to (A7) is not completely devoid of merit. This companion functional form is the logit function:

$$f(y_i) = e^{ky_i} \text{ for some } k > 0. \tag{A7'}$$

Suppose (A7′). Clearly such a functional form satisfies (A1)–(A5). To show that it satisfies (A7) as well, divide both the numerator and the denominator of $p^i(y + c)$ by $e^{k(y_i + c)}$, so that $p^i(y + c) = 1/[1 + \sum_{j \neq i} e^{k(y_j - y_i)}] = p^i(y)$. Thus, (A7′) implies (A1)–(A5) and (A7). As shown below, the direction of implication essentially goes the other way as well.

Theorem 3: *(A7′) is the only continuous functional form satisfying (A1)–(A5) and (A7).*

Proof: Suppose (A1)–(A6) and (A7) are satisfied. By Theorem 1, (A1)–(A5) imply the additive representation in (A5′). Then, (A5′) along with (A7) implies that for all $x, z \geqslant 0$ and for all c such that $x + c \geq 0$ and $z + c \geq 0$ we have

$$f(x)/f(z) = f(x + c)/f(z + c)$$

By setting $z = 0$ (and thus $c \geq 0$) and letting $F(w) = f(w)/f(0)$ (> 0 since, by (A6′) $f(\cdot)$ is positive), the equation above implies

$$F(x + c) = F(x)F(c)$$

which has the same form as (5) in the proof of Theorem 2. Since $F(\cdot)$ is positive, its only continuous solution is $F(w) = e^{kw}$ for some $k > 0$ and for all $w \geq 0$. Consequently, $f(w) = f(0)e^{kw} = e^{kw}$ with $f(0) = 1$, which satisfies (A7′). □

Although, as indicated earlier, (A7) can be criticized for its treatment of the outcome of contests when c is large, its associated logit function in (A7′) has some merits. First, it can be derived from primitive distributional assumptions as in McFadden (1984); the same cannot be said of the power or "ratio" form in (A6′). Second, as Hirshleifer (1989) has suggested using data from battles in Dupuy (1987), the outcomes of at least some conflict situations appear to fit the logit function well, especially in the neighborhood where the players have equal winning probabilities. Although this evidence is far from being conclusive and it concerns only battles, not necessarily other kinds of contests, this is the only empirical evidence on CSFs that we are aware of. Thus, the criticisms that can be levelled against (A7) represent one side of the coin and should not be the only criterion in judging the relevance of the logit function as a CSF – perhaps it holds locally in some contest situations when c cannot vary much.

3. Concluding comment

The axiomatizations in this paper are meant to provide a better understanding of the properties and a grounding of the functional forms of Contest Success Functions used in different applications of contests. An independence from irrelevant alternatives axiom is mainly responsible for generating the class of additive functional forms. Within that class, the power or "ratio" form, used in the rent-seeking literature, and the logit function were also axiomatized. Although helpful, axiomatizations by themselves are unlikely to settle the issue of appropriateness of a CSF for any particular contest situation. As with production functions (and to a lesser degree, with utility functions), finding ways to discriminate among functional forms empirically would be a complentary and welcome endeavor.

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