Public Choice **98:** 369–383, 1999. 369

Nash equilibrium strategies in directional models of two-candidate spatial competition*

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Accepted 31 January 1997

Abstract. The standard approach to two-party political competition in a multi-dimensional issue space models voters as voting for the alternative that is located closest to their own most preferred location. Another approach to understanding voter choice is based on preferred direction of change with respect to some specified neutral point (e.g., an origin or status quo point).

For the two-dimensional Matthews directional model (Matthews, 1979), we provide geometric conditions in terms of the number of medians through the neutral point for there to be a Condorcet (undominated) direction. In this two-dimensional setting, the set of residual locations for which no Condorcet directions exist is identical to the null dual set (Schofield, 1978) and to the heart (Schofield, 1993). In two dimensions, for most locations of the origin/status quo point, a Condorcet direction exists and points toward the yolk, a geometric construct first identified by McKelvey (1986). We also provide some simulation results on the size of the null dual set in two dimensions when the underlying distribution of points is non-symmetric.

1. Introduction

There is a vast literature on spatial models of social choice. We may divide substantive applications into three subareas: (1) models of committee voting, (2) models of candidate competition, and (3) models of coalition formation. A number of the mathematical results in these literatures are equivalent or very similar, e.g., the search for the core of a spatial voting game is essentially equivalent to the search for an equilibrium location of candidates in a two-party competition and is closely related to the search for a stable coalition structure.

* We are indebted to Norman Schofield for helpful comments and to Dorothy Gormick and Chau Tran for library assistance. The paper was written while the first author was a Visiting Scholar in the Department of Biostatistics at the University of Washington. An earlier version of this paper was presented at the meeting of the Public Choice Society, 12–14 April 1996, in Houston, TX.

An important distinction in the recent political science literature on candidate competition has been between the standard (Downsian) models of voter choice and models that are based on a preferred *direction of change*. The former are expressed in *proximity terms*, where voters are posited to vote for the candidate who is closest to them in terms of issues – with issue preference represented as a location (voter ideal point) in n-dimensional issue space (Downs, 1957; Davis, Hinich, and Ordeshook, 1970; Enelow and Hinich, 1990). The latter are described in directional terms, with the work of Rabinowitz and Macdonald (1989) being best known. These ideas have been further developed by, e.g., Macdonald, Listhaug, and Rabinowitz, 1991; Iversen, 1994; Merrill, 1993; Merrill and Grofman, 1997a, 1997c; Westholm, 1997. Schofield (1978, 1983, 1985) has shown, however, that a sharp distinction between directional and proximity voting is misguided.

In this paper we will deal exclusively with the application of our model to two-party/two-candidate competition. We focus on a directional analogue to the Condorcet winner[1] we call the *Condorcet directional vector*. The set of Condorcet directional vectors at a point **N** is called the *directional core* in Cohen and Matthews (1980). Using results of Schofield (1977, 1978), Cohen and Matthews observe that the directional core is non-empty if and only if Schofield's (1978) *null dual condition* does not hold for **N**. We offer constructive, geometric conditions in terms of the number of medians through a point that are both necessary and sufficient for the existence of a directional equilibrium, i.e., the existence of a Condorcet directional vector.[2]

Whereas equilibrium conditions for proximity voting constitute a knife-edge result very unlikely to hold, the corresponding conditions for directional voting will be expressed in terms of an inequality which, in two dimensions, is relatively easy to satisfy. We also show in two dimensions, that all Condorcet directional vectors must pass through the "yolk", which is generally a small and centrally located disk (McKelvey, 1986; Tovey, 1992; Feld, Grofman, and Miller, 1988).[3]

2. Nash equilibria for the case of a finite number of voters

2.1. *Nash equilibria under the Downsian proximity model: The complete median*

The *Downsian proximity model* specifies that utility is a declining function of distance from voter to candidate in a finite-dimensional Euclidean space. It is well known that, in two (or more) dimensions, equilibrium under the standard proximity model requires existence of a *complete median*, i.e., all median lines pass through a single (voter ideal) point (Davis, DeGroot, and Hinich, 1972; Kramer, 1978).[4] In particular, Plott (1967) shows that an equilibrium (complete median), **N**, occurs in a two dimensional electorate consisting of a (odd) finite number of voters if and only if one voter is located at **N** and all other voters come in pairs with each pair on opposite sides of a line through **N**. For a continuous distribution representing a large electorate, McKelvey, Ordeshook, and Ungar (1980) show that a condition slightly weaker than radial symmetry of the probability density is necessary and sufficient for an equilibrium.[5]

2.2. *Nash equilibria under the Matthews directional model and the RM model with circle of acceptability: Condorcet directional vectors*

2.2.1. *Directional models*

In directional models, a neutral (or status quo) point, **N**, is assumed. Voters and candidates are represented by positions in m-dimensional space with $\mathbf{V} = (V_1, \ldots, V_m)$ and $\mathbf{C} = (C_1, \ldots, C_m)$ denoting the vectors of spatial locations of a voter and a candidate, respectively, relative to **N**. The *Matthews directional model* is defined (Matthews, 1979) by the utility function[6]

$$U(\mathbf{V}, \mathbf{C}) = \frac{\mathbf{V} \bullet \mathbf{C}}{|\mathbf{V}||\mathbf{C}|} \tag{1}$$

where $\mathbf{V} \bullet \mathbf{C} = \sum_{i=1}^{m} V_i C_i$ is the scalar product of **V** and **C**, and |**V**| and |**C**| are the lengths of the vectors **V** and **C**, respectively. If either **V** or **C** is **0**, the utility is defined to be 0. Voter utility reflects only the direction and not the intensity of voter and candidate positions.

The *RM directional model* is defined (Rabinowitz and Macdonald, 1989) via the utility function

$$U(\mathbf{V}, \mathbf{C}) = \mathbf{V} \bullet \mathbf{C} \tag{2}$$

where **V** and **C** are defined as above.[7] Voter utility reflects both direction and intensity of voter and candidate positions.

2.2.2. *The Condorcet directional vector*

Our analysis for the Matthews model also applies to the RM model with circle of acceptability.[8] We use the term directional model for either. For the moment, we assume that **N** is fixed and that no voter is located at **N**.

For a proximity or directional model and a two-candidate contest we may define any half-space whose boundary passes through **N** by $\mathbf{S}(\mathbf{A}|\mathbf{N}) = \{\mathbf{V} : \mathbf{V} \bullet \mathbf{A} \geq \mathbf{N} \bullet \mathbf{A}\}$ where **A** is a vector perpendicular to the boundary. We will refer to **A** as the *characteristic vector* of the half-space and assume that **A** is normalized to be of length 1. Conversely, any vector, **A**, of length one defines a half-space of which it is the characteristic vector. Denote by n the total number of voter points by $n_{\mathbf{A}}$ the number of points in **S**(**A**|**N**). We define the *support function*, g, as a function of the characteristic vector, **A**, specified by g(**A**) =n_A.

In this notation, a hyperplane is called a *median*, if for a normal, **A**, to the hyperplane, $g(\mathbf{A}) \geq n/2$ and $g(-\mathbf{A}) \geq n/2$. A point, **N**, is a complete median if and only if all hyperplanes through **N** are medians, i.e., if

$$g(\mathbf{A}) \geq n/2 \text{ for } all\ \mathbf{A}.^9 \tag{3}$$

Under the directional models, the boundary of the support set for a candidate, $\mathbf{C}^*$, with opponent, **C**, passes through the origin, is perpendicular to the line segment joining the unit-vector normalizations of $\mathbf{C}^*$ and **C** relative to **N** and represents the indifference plane between the candidates.

Definition. A vector (candidate) $\mathbf{C}^*$ relative to the neutral point **N** is a *Condorcet directional vector* if for any other vector, **C**, relative to **N**, the proportion of the voters favoring $\mathbf{C}^*$ is greater than or equal to the proportion favoring **C**.

The following theorem is proved in Matthews (1979).

Theorem 1. A vector, $\mathbf{C}^*$, is a Condorcet directional vector if and only if

$$g(\mathbf{A}) \geq g(-\mathbf{A}) \text{ for all } \mathbf{A} \text{ with } \mathbf{A} \bullet \mathbf{C}^* \geq 0. \tag{4}$$

Note that $\mathbf{A} \bullet \mathbf{C}^* \geq 0$ if and only if A lies in the half-space of which $\mathbf{C}^*$ is the characteristic vector. Thus, the theorem states that $\mathbf{C}^*$ is a Condorcet directional vector if and only if the proportion of voters in any half-space whose characteristic vector lies within 90 degrees of $\mathbf{C}^*$ is a majority.

The condition that a given vector be a Condorcet directional vector is considerably weaker than the condition that **N** be a complete median, as the inequality in Eq. (4) is far easier to satisfy than the much more demanding equality in Eq. (3). In fact, the set of values, **N**, from which no such directional vector exists will be shown to be, in a certain sense, small.

3. Characterization of Condorcet directional vectors in two dimensions

3.1. *The yolk*

In two dimensions, Theorem 1 has the following interpretation. The characteristic vector, **A**, of any half-space of support uniquely specifies an angle,

γ. For each $\gamma, -\pi < \gamma \leq \pi$, the value of the support function, g(γ), is the number of voters in the associated half-plane. The necessary and sufficient condition of Theorem 1 for $\mathbf{C}^*$ to be a Condorcet directional vector becomes

$$g(\gamma) \geq g(\gamma + \pi) \text{ for } \gamma_0 - \pi/2 \leq \gamma \leq \gamma_0 + \pi/2 \qquad (4')$$

where γ_0 is the directional angle of $\mathbf{C}^*$.

Corollary 1. Let **N** be the neutral point and fix a direction from it, γ_0, which defines a median through **N** and suppose that for each interval, I, such that $\gamma_0 \in \mathrm{I} \subseteq (\gamma_0 - \pi/2, \gamma_0 + \pi/2)$, the number of voter points in I is at least as great as the number in the antipodal interval, $\bar{\mathrm{I}}$. Then the direction, γ_0, defines a Condorcet directional vector.

Henceforth we allow the location of the neutral point to vary.

Definition. The *yolk* is the smaller hypersphere that is tangent to or intersects all median hyperplanes (McKelvey, 1986). In two dimensions, the yolk is the smallest disc that is tangent to or intersects all median lines.

Definition. A line is called a *Condorcet line* if it contains a Condorcet directional vector for some neutral point, **N**. The *directional yolk* is the smallest hypersphere that is tangent to or intersects all Condorcet lines.

In fact, in two dimensions, the two definitions of yolk are the same.

Lemma 1. In two dimensions, a line is a median if and only if it is a Condorcet line. Thus, the yolk and the directional yolk are the same.

Proof. Suppose a line l contains a Condorcet directional vector in the direction γ_0 from some **N**. Then $g(\gamma_0 + \pi/2) \geq n/2$ and $g(\gamma_0 - \pi/2 \geq n/2$. Thus, l is a median. Conversely, suppose l is a median. Choose a point, **N**, on l outside the convex hull, **H**, of the voter set and let γ_0 be the direction from **N** along l into **H**. Then, for $\gamma_0 - \pi/2 \leq \gamma \leq \gamma_0 + \pi/2$, either $g(\gamma) \geq g(\gamma_0 - \pi/2) \geq n/2$ or $g(\gamma) \geq g(\gamma_0 + \pi/2) \geq n/2$ since **N** is outside **H**. Thus γ_0 specifies a Condorcet directional vector.

Corollary 2. In two dimensions, the line containing any Condorcet directional vector must pass through the yolk.

Corollary 3. In two dimensions, if **M** is a complete median, then unless **M** = **N**, there exists a Condorcet directional vector from any neutral point, **N**, and it lies in the direction from **N** to **M**.

Proof. If **M** is a complete median then this means that the yolk has radius zero and is centered at **M**. But then the only point from which there is no Condorcet direction is **M** itself and all Condorcet directional vectors point toward **M**.

Under the proximity model, a voter point is preferred to another if the latter is at least 2r further from the center of the yolk where r is the radius of the yolk (McKelvey, 1986; Feld, Grofman, Hartley, Kilgour, and Miller, 1987). An analogous result holds in the directional setting.

Lemma 2. Given a neutral point, **N**, in two dimensions, if voters choose between alternatives, **B** and **C**, according to their directional preference then **B** defeats **C** if the yolk lies entirely on the **B** side of the line that contains the bisector of the angle **BNC** and **B** loses to **C** if the yolk lies on the **C** side of this line. If the line passes through the yolk, no conclusion can be drawn without further information.

Proof. If the yolk lies on the **B** side of the bisector line, l, then a median line, m, parallel to but distinct from l passes through the yolk and hence lies on the **B** side of l. Since m is a median, but l is not, **B** beats **C**.

3.2. *Condorcet directional vectors in two dimensions for an odd number of voter points*

Schofield (1978) defined the *null dual condition* for a point **N**, showing that it holds if and only if there is a local cycle at **N**, or equivalently if there is no Condorcet directional vector at **N**. In subsequent work, Schofield further showed that in two dimensions, the null dual set (i.e., the set of points satisfying the null dual condition) forms a star-shaped region of the plane, and provided a number of illustrations (Schofield, 1985).

Consider a two dimensional configuration of n voter points, $P_1, \ldots, P_n$. Direct derivations – based on the number of medians through **N** – of these geometrical conditions for there to be a Condorcet directional vector and the direction in which that vector will point are presented in Theorems 2 and 3 and its corollaries below. The necessary and sufficient conditions depend on the location of the neutral point and the parity of the number of voter points. First we assume n is odd.

Definition. The *star angle* (see Grofman, Owen, Noviello, and Glazer, 1987;

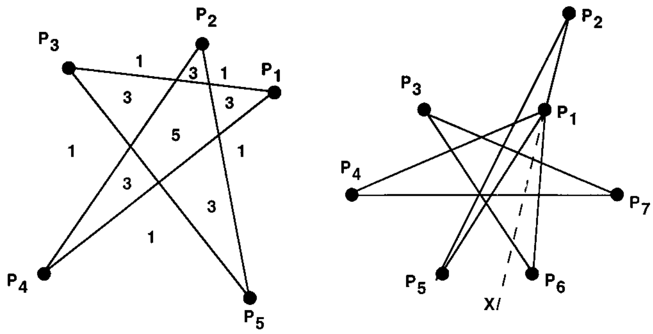

Figure 1. Star angles for small electorates.

Shapley and Owen, 1989) with respect to a neutral point, **N**, with vertex at a voter point, P_i, is the angle of the form $P_jP_iP_k$ (or the union of such angles) such that all lines through P_i and within the angle are medians (see Figure 1). If none of the P_i are interior to the convex hull of $P_1,\ldots,P_n$, then the line segments of the form $\overline{P_jP_i}$ and $\overline{P_kP_i}$ defining the star angles form a star with points at the P_i. The star angle for an interior point may, however, be composed of disjoint angles (see, e.g., the star angle at point, P_1, in Figure 1B, which is composed of the two angles $P_4P_1P_5$ and XP_1P_6 where the segment, $\overline{XP_1}$, is an extension of the segment, $\overline{P_2P_1}$).

We define a *residual median* through **N** as a median through **N** that remains a median after all possible pairs of voter points which are collinear with **N** and on opposite sides of **N** have been deleted. Note that if no two points are collinear with **N** (the "usual" situation), then a residual median is the same as the median. Furthermore, define

$$h(\gamma) = g(\gamma) - g(\gamma + \pi),^{10}$$

where $g(\gamma)$ is the support function with respect to **N**. Note first that if **N** is one of the voter points, then either the line through **N** and some voter point is the only median through **N** or there are infinitely many. These line(s) pointing into the star angle of **N** are all Condorcet directional vectors.

Lemma 3. If the number, n, of voters is odd, then, for any neutral point **N**, other than one of the voter points, the number of residual medians through **N** is odd.

Proof. Let **N** not be a voter point. If there are more than one voter point on some line through **N**, delete pairs of voter points – one from each side of **N** – until all remaining voter points (if any) lie on one side of **N**. As the indifference line through **N** rotates, $h(\gamma)$ changes sign every time the indifference line coincides with a residual median, changing from negative to positive when the leading edge crosses a voter and reversing each time an antipode of a voter is passed. If for some $\gamma_0, h(\gamma_0) < 0$, then $h(\gamma_0 + \pi) > 0$. The number of sign changes in this interval from γ_0 to $\gamma_0 + \pi$ must be odd so that the sign changes overall. Each sign change occurs exactly when the indifference line crosses a residual median and the number of sign changes must be odd.

Corollary 4. If the number, n, of voters is odd and no line through N contains more than one voter point, then the number of median lines through **N** is odd.

A Condorcet directional vector for **N** exists if and only if as γ increases, h(γ) changes from negative to positive (which only occurs when the indifference line crosses a voter point which lies on a residual median line) only once. The following theorem is immediate.

Theorem 2. Given any configuration of n voters, where n is odd, and a neutral point **N** other than one of the voter points, the following are equivalent.

(i) There exists no Condorcet directional vector for **N**.
(ii) **N** lies on more than one residual median line.
(iii) **N** lies in more than one open star angle.
(iv) **N** lies inside the interior of the star.

If these conditions fail, a Condorcet directional vector does exist and lies in the direction of the vertex of the star angle containing **N** (or whose vertical angle contains **N**), i.e., the Condorcet directional vector must point toward some particular voter. Since Condorcet directional vectors must pass through the yolk, if all neutral points have a Condorcet directional vector, the latter must all pass through the (non-empty) core.[11]

The illustrative configuration of five voter points depicted in Figure 1A shows the star angles and labels the number of median lines passing through a point in each open region. **N** has no Condorcet directional vector if and only if **N** lies inside the open star.

For sufficiently large electorates sampled from an underlying symmetric distribution, as Schofield and Tovey (1992) have shown, the measure of the null dual set shrinks to zero. However, even if the distribution is not symmetric about each neutral point, our simulation results suggest that, in two dimensions, Condorcet directional vectors will exist for **N** located at most points inside the Pareto set.[12] Moreover, for even moderately large odd n, the star angles will be narrow (see Figure 2 which presents a star for 201 voters generated uniformly on the unit disc). If we reject the very thin lines radiating from the central portion of the star as being implausible locations

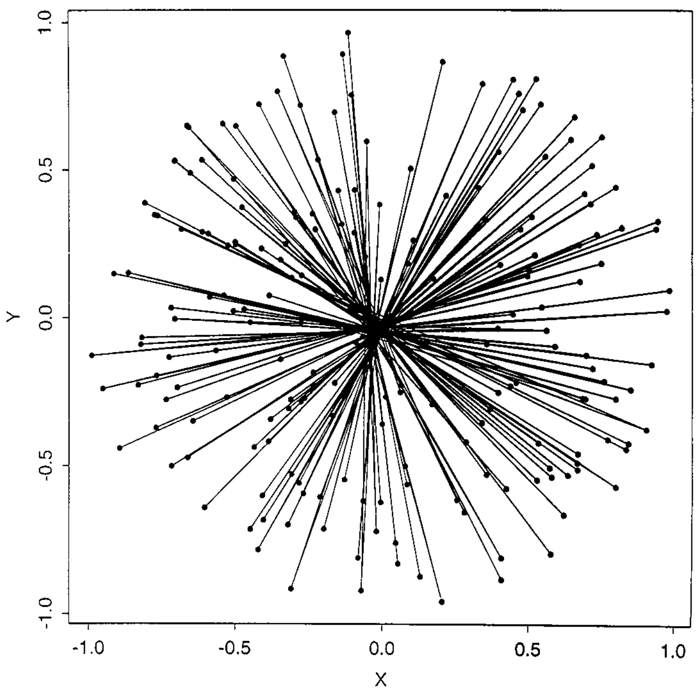

Figure 2. Star for 201 voters uniformly distributed on the unit disc.

for candidates, there exists a rather small and quite centrally located zone of the space in which we expect candidates to locate themselves when voters use a directional rule. We conjecture that this zone will correspond closely to the yolk.

3.3. *Condorcet directional vectors in two dimensions for an even number of voter points*

If n is even and no two voter points are collinear with **N**, **N** may lie on two median lines even when a Condorcet directional vector exists. For example, suppose that four voters are located at $\pm$ 30 degrees and $\pm$ 120 degrees. The coordinate axes are both medians but γ = 0 degrees defines a Condorcet directional vector pointing due east. With n even, the support function can dip to exactly n/2, then rise again.

In fact, for n even, the condition for existence of a Condorcet directional vector is quite different from that for odd n. Let n = 2k. If no three voter points are collinear, through each voter point there passes one (or more) median, called an *opposing median*, which partitions the voter points into two exactly equal sets (see Figure 3). Suppose that **N** does not lie on any opposing medi-

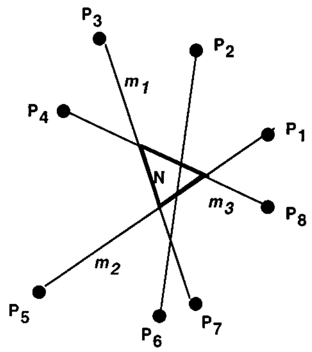

Figure 3. Neutral point with no Condorcet directional vector for electorate of size eight.

an. The details of the proof of the following theorem are available from the authors upon request.

Theorem 3. Given any configuration of n voters, where n is even, then a sufficient condition that a point, **N**, not on an opposing median, not have a Condorcet directional vector is that **N** lie in some triangle formed by the intersection of three opposing medians, say m_1, m_2, and m_3. If all voter points lie on a circle centered on **N**, this condition is also necessary that **N** not be a Condorcet directional vector.

A plot of opposing medians for 200 voters generated uniformly on the unit disc shows a patterns (Figure 4) remarkably similar to the star for a comparable odd number of voters.

For two dimensions, the regions specified by Theorems 2 and 3 within which no Condorcet directional vectors exist is the same as the *electoral heart* defined by Schofield (1993a, 1993b) as the union of the core and the cycle set.[13] Schofield observes that around any point within the triangle formed by three intersecting medians there will exist a cycle. In the directional context, from such a point no direction is undominated.

The situation changes abruptly when we move to three or more dimensions. In three dimensions, a three-voter example[14] is sufficient to show that the set of neutral points without Condorcet directional vectors may be dense in the space. In fact, this set is dense under rather general conditions (Schofield, 1978), although the heart is not. Similar examples for larger voter sets and/or more than three dimensions indicate that Condorcet directional vectors are not common in more than two dimensions.

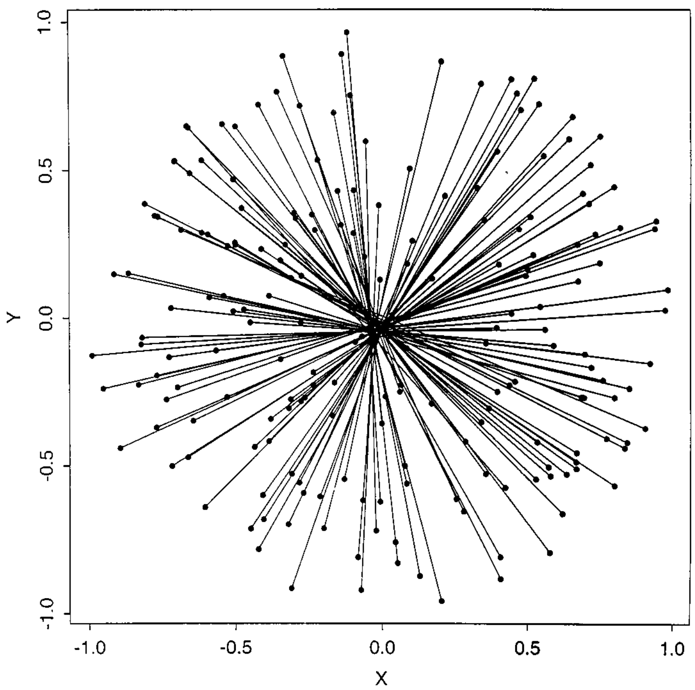

Figure 4. Opposing medians for 200 voters uniformly distributed on the unit disc.

Schofield (1993a) proffers the heart as a solution set for a class of voting games including the directional ones considered here. For two dimensions, from any point outside the heart the Condorcet directional vector exists and points toward the yolk. Once inside instability occurs.

4. Discussion

For the important class of directional models of voting that has recently been applied to model voting processes in majority-rule polities, we have specified in terms of median lines the geometry that defines the region of the space in which Condorcet directional vectors will exist for the two-dimensional case. Hypothetical data sets (Figures 2 and 4) suggest that, for large electorates,[15] for two dimensions, the regions for which no Condorcet directional vector exist is small relative to the Pareto set when the number of voter points is on the order of tens or hundreds, even where there is considerable asymmetry in the underlying distribution. By contrast, no Condorcet location exists at all when we look for an equilibrium in terms of proximity based voting.

Any realistic model of candidate strategy involves both centripetal and centrifugal components. We have attempted to model one type of centripetal force which has its origin in directional movement from the status quo toward the center of the voters. In a more fully realistic model of politics, e.g., one with further institutional details, such a centripetal pull will be counteracted by a host of centrifugal effects including the influence of activists, intraparty primaries, and the desire of candidates to implement policy as opposed to winning per se.

The basic result of this paper is that, for two dimensions, from most possible locations of the status quo, there is a clear directional consensus which necessarily points toward a centrally located portion of the space, the yolk. Thus, in two-candidate contests, it would appear to be relatively easy to move outcomes in a "centrist" direction when voters vote directionally.[16]

Notes

1. For majority rule voting games an equilibrium (core) location is also known as a Condorcet winner, i.e., a position that defeats (or at least is not defeated by) any other alternative in a paired contest.
2. There has been work on the conditions for equilibrium for various specific directional models. Coughlin and Nitzan (1983) show, under probabilistic assumptions, that at every status quo point there is a directional equilibrium. Rabinowitz and Macdonald (1989) show that – given a circle of acceptability – a dominant position exists if the electorate is symmetrically distributed about a voter point other than the neutral point. Macdonald and Rabinowitz (1993) – under the assumption that voters are restricted to a square (or hypercube) – prove that a dominant strategy exists if there is a quadrant such that every half-space containing that quadrant has more voters than the complementary half-space. Matthews (1979) obtains a necessary and sufficient condition (see Theorem 1 above) which is an important step in deriving our conditions.
3. The ideas we draw on, in addition to being identical to Schofield's game-theoretically inspired concept of the "heart" (Schofield, 1993a, 1993b) in the two-dimensional case to which we confine ourselves, also can be linked to mathematical ideas in Kramer's (1977) work on the "minmax set", and to the Feld and Grofman (1988) concept of the "Schattschneider set".
4. See Riker (1982) for a nontechnical view.
5. For a large continuous electorate Merrill and Grofman (1997v) present a way to specify this equilibrium condition in terms of odd and even terms of a Fourier series.
6. Matthews (1979) assumed directly that all voters and candidates are restricted to the unit sphere (with the exception of totally indifferent voters at the origin).
7. For $i = 1, \ldots n$, the absolute values of the coordinates, V_i and C_i, are interpreted as intensities with which a voter and a candidate hold positions on dichotomous issues. The signs (+ or –) of these coordinates reflect the positions taken. The neutral point, **N**, is interpreted as the point for which the voter (or candidate) is indifferent between the two positions on each issue.
8. Since under the Matthews model for a fixed neutral point, utility depends only on direction, each voter or candidate location can be replaced by a vector of unit length in the same direction from **N**. For the RM model with circle of acceptability, undominated positions lie on the circle of acceptability (which can be taken to be of radius one). In the RM model

(with circle of acceptability), any candidate, **A**, lying in the interior (or the exterior) of the circle but not at the neutral point can be dominated by a second candidate, **B**, in the same (or opposite) direction but on the circle. Thus, all candidates can be expected to move to the circle of acceptability. Under this interpretation, voters should behave as they would under the Matthews model.

9. The value, g(**A**), can exceed n/2 only if voter points lie on the boundary of the halfspace of which **A** is the characteristic vector.
10. This function is called the "gap function" by Schofield and Tovey (1992).
11. We thank N. Schofield for pointing out this conclusion when the null dual set is empty.
12. For a uniform distribution on a disc, this probability was found to vary from 8% (for n = 3) to 11% (for n near a dozen) and back to 9% (for n = 101 and n = 201). Standard errors in these simulations drop from about 0.7% to 0.3% over this range of n. For a normal distribution, probabilities are about 2 percentage points higher uniformly over n. For a tripolar distribution on the disc, they are about 4 percentage points higher, for virtually all n, than for a uniform distribution. Thus, for n up to at least 201, if the neutral points are drawn from the same distribution as that from which the voter points are drawn, simulation suggests that the probability that a neutral point lie in the portion of the space from which no Condorcet vectors exist is about 10%.
13. We thank N. Schofield for pointing out the mathematical identity of these two concepts.
14. Suppose, for example, there are three dimensions and three voter points all in the X-Y plane. If **N** is not in the X-Y plane, then no vector C^* is a Condorcet directional vector, because there will always be (indifference) planes through **N** which separate C^* from two (or more) of the voter points. If, on the other hand, **N** lies in the X-Y plane, there is a Condorcet directional vector through **N** if and only if **N** lies outside the interior of the triangle (Pareto set) formed by the voter points.
15. For large electorates, as noted earlier, the yolk can be expected to be very small (Tovey, 1992).
16. We would emphasize that there is a considerable body of empirical work on the dimensionality of voting in the U.S. Congress and the U.S. Supreme Court that suggests that most political conflict takes place in only one or two dimensions. There is a considerable literature as to why this might be the case (see e.g., Hinich and Munger, 1994; Poole and Rosenthal, 1996; Glazer and Grofman, 1989).

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