Public Choice **97:** 23–34, 1998.

A neo-Downsian model of group-oriented voting and racial backlash*

AMIHAI GLAZER, BERNARD GROFMAN & GUILLERMO OWEN
School of Social Sciences, University of California, Irvine, Irvine, CA 92697, U.S.A.

Accepted 27 March 1996

Abstract. We extend the standard Downsian framework to suppose that voters consider the identity of each candidate's supporters when deciding whom to support, rather than considering only the announced policy positions of the candidates. In particular we posit the existence of a class of voters whose support for a candidate reduces support by some other voters for that candidate. Our most important result concerns the conditions under which the addition to the electorate of new voters on one side of the policy spectrum shifts the equilibrium toward the opposite direction. The model can explain why enfranchisement of blacks did not immediately help the election of liberal candidates.

1. Introduction

In choosing between candidates (or parties), voters, at least implicitly, join an electoral coalition; this coalition may be a more reliable cue about the policies that will be implemented by the candidate than the candidate's own proclaimed policy positions. Alternatively, a voter may base his support or opposition to candidates on his loyalties or antipathies to the groups that support the candidates.[1] We refer to voting based in whole or in part on the nature of a candidate's expected support coalition as group-oriented voting. Adding group-oriented voting to the standard Downsian model (Downs, 1957) leads to a better understanding of contemporary American politics, especially the policy significance of racial cleavages.[2] While the model we offer is in no way limited to that particular construction, the reader may find it useful for purposes of empirical application in the U.S. context to think of the disliked voters as racial minorities.

* We are indebted to the staff of the World Processing Center, School of Social Sciences, UCI, for initial manuscript typing, and to Dorothy Green for bibliographic assistance. We are grateful to Robert Erikson and two anonymous referees for helpful suggestions. This research was partially supported by the UCI Interdisciplinary Focused Research Program in Public Choice. The listing of authors is alphabetical. Theorem proofs are due to the third-named author.

The model we offer builds on work by Grofman, Griffin, and Glazer (1992), which in turn follows the ideas of Key (1949) and Keech (1968). These authors find that passage of the Voting Rights Act of 1965, which greatly increased black enfranchisement in the South, did not immediately lead to Democratic congressional candidates shifting to the left as liberal black voters were added to their constituencies and to their electoral support coalitions. Rather, in the deep South, electorates that became heavily (but not majority) black were frequently represented in the 1970s and on into the early 1980s by Democrats who were more conservative than the Democrats who represented districts with relatively few black voters. Such a finding is inconsistent with the usual Downsian framework in which the addition of liberal (black) voters to the electorate (which moves the median voter to the left) should also shift the location of the winning candidate (here, in the time period in question, almost certainly a Democrat) to the left. Moreover, even if we follow authors such as Aldrich (1983) or Baron (1994) to view each party as choosing candidates located close(r) to the median voter in that party, the puzzle remains: since black voters are much more liberal than most southern white Democrats, we would expect that, in general, the more black Democratic voters in a constituency, the more liberal should any Democratic representative be. But, until the 1990s, this expectation is not confirmed by the data.

Several authors consider racial backlash in electoral models (see especially Huckfeldt and Kohfeld, 1989). Our aim is to meld racial backlash ideas (especially those of Glazer, 1993), with the standard Downsian framework of voter choice based on policy proximity. As far as we are aware, we are the first authors to attempt such a melding.

Now, we turn to the exposition of the formal model.

2. Group-oriented voting within a one-dimensional issue space

There are two candidates, I and II. Candidate I's position is at x, candidate II's position at y. We follow Downs (1957) in assuming that each candidate adopts the position that will maximize his share of the vote. Each candidate can adopt any position he wishes; he knows the position adopted by the other candidate and also knows the preferences of all voters. The focus of the analysis is to determine the positions that candidates will adopt, given that the choices of some voters depend in part upon their expectations of each candidate's support coalition.

We distinguish between three mutually exclusive types of voters: conventional voters, disliked voters, and group-oriented (or expressive) voters. Members of the first two classes of voters choose between candidates based solely on ideological proximity. The third type of voter makes choices based both on the policy positions of the candidates, and on his expectations about what proportion of the support coalition of each candidate comes from members of the disliked class of voters. More specifically, we have m voters, divided into three subsets of size c, d, and e respectively, such that $c + d + e = m$.

The three types are

(a) The *disliked* voters (D voters). The number of such voters is d. Without loss of generality we may suppose that all of them vote for the candidate furthest to the left. That is, all disliked voters vote for I if $x < y$; all vote for II if $y < x$.

(b) The *conventional* voters (C voters). The number of such voters is c. Their ideal points are distributed along the interval $(0, 1)$ according to a distribution function $F(t)$. The corresponding probability density function is $f(t)$.

Each conventional voter votes for the candidate with the position closest to him. That is, if $x < y$, then all conventional voters with ideal points to the left of $t = (x + y)/2$ vote for I, all others vote for II. This means that for $x < y$, candidate I receives $cF((x + y)/2)$ votes from conventional voters, while II receives $c - cF((x + y)/2)$ votes.

(c) The *group-oriented* voters (E, or expressive, voters). The number of such voters is e. These voters dislike supporting a candidate supported by disliked voters.[3]

Let s_i be the number of voters the left-hand candidate is expected to get from type-I voters. Then a group-oriented voter supports this candidate with probability

$$P(s_e + s_c, s_d). \tag{1}$$

We assume that this probability function P is defined for all non-negative values of its two arguments; moreover, it is non-decreasing with respect to the first argument, and non-increasing with respect to the second argument. For simplicity, we assume differentiability with

$$\frac{\partial P}{\partial (s_e + s_c)} = P_1 \geq 0$$

and

$$\frac{\partial P}{\partial s_d} = P_2 \leq 0$$

Thus, the expressive voters are more likely to vote for a candidate of the left the larger the fraction of that candidate's support from the conventional and expressive voters, i.e., the lower the support from the disliked voters.

In this model, the position a candidate takes can affect his share of the vote in two ways. The direct effect arises from voters' concern about the candidate's policy position. The indirect effect arises when some voters care about the identity of a candidate's supporters. A change in a candidate's announced position changes the identity of his supporters, which has a further, indirect, effect on the choice of voters, and therefore on the identity of his supporters.

If voters' expectations about the proportion of disliked voters who support a given candidate are accurate, then, for a given numbers (s_d and s_c) of disliked and conventional voters who support the left candidate, the number of expressive voters supporting him must satisfy the following condition.[4]

$$s_e = eP(s_e + s_c, s_d). \tag{2}$$

Consider next the equilibrium strategies of the candidate.

Let A be the number of votes won by the candidate on the left, when that candidate locates at position t:

$$A(t; c, d, e) = d + cF(t) + \varphi(t; c, d, e), \tag{3}$$

where $\varphi(t; c, d, e)$ is the number of votes the candidate on the left receives from expressive voters, or $\varphi = eP(cF(t) + \varphi, d)$.

As shown in the theorem whose proof is given in the Appendix, there exists an equilibrium position, t^* at which both candidates will locate such that each wins half the vote. In general t^* will not be the overall voter median.

Our main interest lies in how t^* varies with the number of voters of the different types. It is straightforward to differentiate t as a function of the numbers c, d, and e. In particular, from the definitions of P_1 and P_2 given above, and from equation (15) in the Appendix (which describes $cF(t^*)$) we find that

$$\frac{\partial t^*}{\partial d} = \frac{eP_1 - 1 + 2eP_2}{2cf(t^*)} \tag{4}$$

The denominator, $2cf(t^*)$, is positive. As for the numerator, we note that $eP_1 - 1$ is negative (see Eq. (4) in the Appendix). The last term, eP_2, is positive, and may be large enough to make the numerator positive. But remember that, in general, P_2 is roughly inversely proportional to the number of voters supporting the candidate, and therefore decreases in magnitude as d increases.

We conclude that if the number of group-oriented voters, e, is large, and if they are really group-oriented (so that the partial derivatives of P with respect to s_e is not too small), and if there is a reasonably large number of non-disliked voters at equilibrium (so that $(s_e + s_c)/s_d$ is large) then $\partial t^*/\partial d > 0$. That is, increasing the number of disliked voters (which would normally strengthen the left) will shift the equilibrium position to the right. In the opposite case, e.g., if e is very small, or if the derivatives of P are small, then $\partial t^*/\partial d$ is negative. In this case we obtain the 'commonsense' result that increasing the number of disliked voters will move the equilibrium to the left.

Table 1.

d	t^*
50	0.050
100	0.125
150	0.185
200	0.205
250	0.190
300	0.145
350	0.108

We now consider some numerical examples of this non-monotonic relationship between the proportion of disliked voters in the electorate and the ideological location of the winner.

2.1. *Examples*

Consider a situation with 200 conventional and 200 group-oriented voters and an analytically convenient probability function that satisfies our previously stated requirement. Let $F(t) = t$, and let

$$P(s_e + s_c, s_d) = \frac{\tau}{2} \tan^{-1} \frac{s_e + s_c}{s_d}$$

We shall look at several values of d, the number of disliked voters. Since F is given, we use the method given in the Appendix to solve for t^*.

(a) Let $d = 50$. Then $(c + e - d)/2 = 175$, and so $\varphi = 200P(175, 50) = 165$. Hence, $cF(t^*) = 175 - 165 = 10$. Since $c = 200$, and $F(t) = t$, this gives us $200t^* = 10$, or $t^* = 0.05$. Thus the equilibrium position is at $t^* = 0.05$. If one candidate chooses this, while the other candidate moves ever so slightly to the right, then the first candidate will receive support from the 50 disliked voters, 10 of the conventional voters, and 165 of the group-oriented voters, for a total of 225 votes out of 450.

(b) Several other values for d, and the corresponding of t^*, are shown in Table 1 above.

The effect of increasing d is easily seen here. An increase in d can scare the group-oriented voters away from the left candidate. For small values of d, the candidates will want to reassure their voters and therefore shift their positions to the right.

Beyond a certain point, however (in our illustrative data, at $d = 203$), the disliked voters become an important part of the electorate and the candidates no longer need to shift to the right. Thus this equilibrium position moves to the left. Beyond $d = 400$, of course, the disliked voters are a majority of the electorate and so the equilibrium position then stay at $t = 0$.

3. Discussion

One important way to apply the theory to U.S. electoral politics is to think of Disliked persons as blacks, Conventional persons as non-racist whites, and Group-oriented persons as racist whites. Under this interpretation racist white voters will be less likely to support a candidate the more black voters support that candidate. Consider the deep South. It is well known that white southern support for the Democratic presidential ticket declined after 1960, as black enfranchisement and voter participation grew. Of course, factors other than racial backlash also operated. Within the Downsian framework, Aldrich (1983), Aranson and Ordeshook (1972), Baron (1994) and Wittman (1977, 1983) consider divergence in party positions due to the roles of ideologically driven party activists, interest groups, policy-oriented candidates, or a two-stage election process with both a primary and a general election. In these neo-Downsian models, as liberal (black) voters are added to the Democratic electorate the Democratic party median shifts to the left, leading to the nomination of candidates by the Democratic party who are more liberal, causing centrist voters to shift to the Republican party.

Nonetheless, certain electoral phenomena in the South are difficult to explain without positing some form of racial backlash. For example, in line with what we might expect if group-oriented voting were taking place, the willingness of white voters to support the Democratic nominee falls directly with the proportion black in the state. Using National Election Study data, Black and Black (1992: 291) show that the mean White vote in the core and peripheral south for Democratic presidential candidates was strongly inverse to the state's black population in elections from 1972 through 1988.[5] Similar effects appeared in the 1968 presidential general election between Hubert Humphrey, George Wallace, and Richard Nixon. Using a large sample survey, Wright (1977) finds that the probability a white in a southern state votes for Wallace increased with the proportion of blacks in the voter's county and state.[6] This follows on Key's (1949) demonstratin that racial concerns are overwhelmingly important in explaining southern politics, e.g., black population concentrations help explain differences in the intensity of Jim Crow laws among deep South and border South states, and black concentrations predict which counties were pro-secession at the time of the Civil War (see also Keech, 1968; Act, 1994).

But a peculiar feature of southern politics is even harder to account for under either the standard Downsian model of convergence toward the overall median or the neo-Downsian models of party divergence in which each party nominates candidates that represent the median voter in that party. The feature of southern politics we have in mind is empirically investigated by Grofman, Griffin, and Glazer (1992): in the 1970s and early 1980s, southern areas with significant but not overwhelmingly large black populations were represented in the House of Representatives by Democrats (or in a handful of instances, Republicans) who were, on balance, more conservative than the Democratic representatives elected from districts with fewer blacks.

If Democratic candidates shift to the right in a general election to attract (conventional) white voters, then the shift should be less in heavily (but not overwhelmingly) black areas than in areas that are less heavily black: because *ceteris paribus*, the more blacks in the constituency the further left will the median voter be in the general election. On the other hand, in the neo-Downsian models of party divergence, since blacks are overwhelmingly Democratic, *ceteris paribus*, the more blacks in the constituency the further left will the median voter be in the Democratic primary. Yet, it is not until the late 1980s and thereafter that this monotonic pattern clearly emerges. Our model of group-oriented voting is consistent with these data.

We believe our paper usefully combines Downsian ideas with ideas about expressive voting such as are found in Glazer (1993) and Brennan and Lomasky (1994). Nevertheless, our model is far from the last word. In particular, as in the standard Downsian unidimensional model, the equilibrium we find is unrealistic in predicting that the two candidates choose the same positions (see Poole and Rosenthal, 1984; Grofman, Griffin, and Glazer, 1990; Grofman, 1993; Alesina and Rosenthal, 1993). Thus it would be desirable to combine the racial backlash/group-oriented voting features of our model with ideas drawn from the neo-Downsian literature on party divergence.

Notes

1. Fenno (1978) calls attention to the potential divergence among a candidate's geographic constituency, his electoral constituency (including primary constituency), and his "inner" campaign constituency. Aldrich (1983) revives older ideas of parties as coalitions of interests, and posits that each party's policy position is located at the center of gravity of its support coalition (cf. Grofman, 1982; Wittman, 1983).
2. Carmines and Stimson (1989) show that voter images of the national political parties appear to track changes in the (racial) issue positions of party activists, and that racial attitudes appear to be responsible for much of the observed greater ideological "issue constraint" of recent decades. Huckfeldt and Kohfeld (1989) show that as blacks became a larger part of the Democratic national party's electoral support coalition, white support for that party's presidential candidate declined. Lodge et al. (1985, 1986) show that party images have changed so that the terms "Democrat" and "liberal" in part connote "pro-black." Other authors develop the thesis that even though white support for overt forms of segregation has declined dramatically, white behavior still reflects what Sears

and Donald call "symbolic racism" (McConahay, 1982; Sears and Kinder, 1971; Sears, Hensler, and Speer, 1979).

3. Alternatively, we may think of the disliked voters as merely "distinguishable voters," i.e., voters whose presence in a candidate's support coalition sends a signal to the group-oriented voters about the candidate's probable policies, based on the notion that the candidate will respond to his *electoral* constituency.
4. We show in the Appendix that a unique solution to Equation (2) exists.
5. Similar findings are generated with other data sources. For example, for the 1988 presidential election, the CBS/New York *Times* poll had election poll data on white voters in 23 states. Using these data we estimated a linear regression in which the dependent variable is the percentage of whites who voted for Dukakis in a particular state, and the explanatory variable is the percentage of blacks in that state. We find

   %Dukakis = 48.7 - 0.78 %Black

   The correlation coefficient is 0.59, the t-statistic is a highly significant 3.3; a dummy variable for the South is not statistically significant. Throughout the country, a one percent increase in black population is associated with a drop of over three-quarters of a percentage point in white support for Dukakis in 1988.

   Further support for the idea of group-based voting comes from an examination of the 1988 Democratic presidential primary. In that year Jesse Jackson, a black, made a strong appeal to black voters, and obtained about 90% of the black vote. The theory developed here predicts that the fraction of whites who supported Jackson would be greater the smaller the fraction of blacks in that state. The data bear this out. The CBS/New York *Times* poll had 1988 primary exit poll data on white voters for 22 states broken down by race. Using these data we estimated a linear regression in which the dependent variable is the percentage of *whites* who voted for Jackson in a particular state, and the explanatory variable is the percentage of blacks in that state. We find

   %Jackson = 16.65 - 0.41 %Black

   The t-statistic on the percentage black variable is a highly significant 3.64; the correlation coefficient for the regression is 0.62. This regression says that, on average, a one percentage point increase in the proportion of blacks in a state's population is associated with a drop of nearly one-half percentage point in the support of white voters for Jesse Jackson. Essentially the same relationship holds even when we confine ourselves exclusively to southern states.
6. Similar results for the link between the size of the Wallace vote and percent black are found for the South as a whole and for counties in North Carolina using ecological regression (Black and Black, 1992: 170–171; Grofman and Handley, 1995). Evidence of a similar sort is found in Huckfeldt and Kohfeld (1989).

References

Aldrich, J. H. (1983). A Downsian spatial model with party activism. *American Political Science Review* 77: 974–990.

Alesina, A. and Rosenthal, H. (1993). *Partisan politics: Divided government*. New York: Cambridge University Press.

Alt, J. (1994). Race and voter registration in the South. In C. Davidson and B. Grofman, *Quiet revolution in the South: The impact of the Voting Rights Act.* Princeton: Princeton University Press.

Aranson, P. and Ordeshook, P. C. (1972). Spatial strategy for sequential elections. In R.G. Niemi and H.F. Weisberg (Eds.), *Probability models of collective decision making*. Columbus OH: Merrill.

Baron, D. D. (1994). Electoral competition with informed and uninformed voters. *American Political Science Review* 88: 33–47.

Black, E. and Black, M. (1992). *The vital South: How presidents are elected*. Cambridge, MA: Harvard University Press.

Brennan, G. and Lomasky, L. (1994). *Democracy and decision: The pure theory of electoral preference*. New York: Cambridge University Press.

Carmines, E. G. and Stimson, J. A. (1989). *Issue evolution: Race and the transformation of American politics*. Princeton: Princeton University Press.

Downs, A. (1957). *An economic theory of democracy*. New York: Harper.

Fenno, R. F. (1978). *Home style: House members in the their districts*. Boston: Little Brown.

Glazer, A. (1993). Political equilibrium under group identification. In B. Grofman (Ed.), *Information, participation and choice*. Ann Arbor: University of Michigan Press.

Grofman, B. (1982). A dynamic model of protocoalition formation in ideological N-space. *Behavioral Science* 27: 77–90.

Grofman, B. (1993). Toward an institution-rich theory of political competition with a supply side component. In B. Grofman (Ed.), *Information, participation and choice*. Ann Arbor: University of Michigan Press.

Grofman, B. and Handley, L. (1995). Racial context, the 1968 Wallace vote, and southern presidential dealignment: Evidence from North Carolina and elsewhere. In M. Eagles (Ed.), *Spatial and contextual models in political research*. London: Taylor and Francis.

Grofman, B., Griffin, R. and Glazer, A. (1990). Identical geography, different constituencies, see what a difference party makes. In R.J. Johnston, F. Shelley, and P. Taylor (Eds.), *Developments in electoral geography*, 207–217. London: Croom Helm.

Grofman, B., Griffin, R. and Glazer, A. 91992). The effects of black population on electing Democrats and liberals to the House of Representatives. *Legislative Studies Quarterly* 17: 365–379.

Huckfeldt, R. and Kohfeld, C. (1989). *Race and the decline of class in American politics*. Champaign-Urbana: University of Illinois Press.

Keech, W. R. (1968). *The impact of Negro voting: The vote in the quest for equality*. Chicago: Rand McNally.

Key, V. O. Jr. (1949). *Southern politics*. New York: Knopf.

Lodge, M., Moskowitz, D., Pfau, M. and Ramachandran, S. (1985). Discriminating tweedledum from tweedledee: Perceived party preferences as a function of social class and racial differences. Prepared for delivery at the Annual Meeting of the American Political Science Association, New Orleans, August.

Lodge, M., Moskowitz, D., Pfau, M. and Wahlke, J. (1986). Race and class components of party differences: An experiment in political information processing. Unpublished manuscript. Laboratory for Behavioral Research, State University of New York at Stony Brook.

McConahay, J. B. (1982). Self-interest versus racial attitudes as correlates of anti-busing attitudes in Louisville: Is it the buses or the blacks? *Journal of Politics* 44: 692–720.

Poole, K. and Rosenthal, H. (1984). A spatial model for legislative roll-call analysis. *American Journal of Political Science* 29: 357–384.

Sears, D. O., Hensler, C. P. and Speer, L. K. (1979). White's opposition to busing: Self-interest or symbolic politics? *American Political Science Review* 73: 369–384.

Sears, D. O. and Kinder, D. (1971). Racial tensions and voting in Los Angeles. In W.Z. Hirsch (Ed.), *Los Angeles: Viability and prospects for metropolitan leadership*. New York: Praeger.

Wittman, D. (1977). Candidates with policy preferences: A dynamic model. *Journal of Economic Theory* 14: 180–189.

Wittman, D. (1983). Candidate motivation: A synthesis of alternative theories. *American Political Science Review* 72: 78–90.

Wright, G. C. (1977). Contextual models of electoral behavior: The southern Wallace vote. *American Political Science Review* 71: 497–508.

Appendix

Proof of the existence and uniqueness of t^* (the optimal candidate location when some voters are group-oriented)

We first determine whether a number s_e, satisfying Eq. (2) in the text exists, and, if so, whether it is unique. Recall that

$$s_e = eP(s_e + s_c, s_d). \tag{2}$$

To demonstrate existence, note that $s_d = d$ since we are talking about the candidate on the left. Now, for a fixed s_c, equation (2) takes the form

$$s_e = eP(s_e + s_c, d) \tag{5}$$

For $s_e = 0$, the left-hand side of this equation is clearly smaller than or equal to the right-hand side. For $s_e = e$, the opposite inequality holds, since $0 \leq P \leq 1$ for all values of the variables. Now P is continuous, and so at least one value of s_e makes equation (2) valid.

To prove uniqueness, we suppose that voters are more interested in the percentage makeup of the candidate's supporters (i.e., what percentage of them are members of the disliked class) than in their actual numbers. Now, an increase of one voter of any one of the three types will change these percentages by an amount which is roughly inversely proportion to the total number of voters supporting him. Thus, we would expect that both partial derivatives, P_1 and P_2, are of the order of magnitude of the reciprocal of the number of voters. We therefore assume that

$$\frac{\partial P}{\partial s_e + s_c} \equiv P_1 < \frac{d}{e(s_c + s_c + s_d)}. \tag{6}$$

Suppose, now, that there were two solutions to equation (5), say s and s'. We would then have $s = eP(s + s_c, d)$ and $s' = eP(s' + s_c, d)$. Subtracting, we have $s - s' = eP(s + s_c, d) - eP(s' + s_c, d)$ or

$$\frac{P(s + s_c, d) - P(s' + s_c, d)}{s - s'} = \frac{1}{e}$$

By the mean value theorem, there would be some value of s for which $P_1(s + s_c, d)$ is equal to $1/e$. But, by (6), this is not possible. Thus s_e is unique.

Condition (6) is interesting. It effectively says that if the disliked voters are too small a fraction of the population (more exactly if the ratio d/e is too small), then our conclusions below will not hold. Only when d/e passes a critical value (the maximum of qP_1 where q is the number of voters supporting the candidate) do we begin to get results in which the presence of group-oriented voters changes the nature of the outcome. This seems reasonable: it is only when the group-oriented voters begin to see a large number of disliked voters that they behave in a way that a standard model would consider irrational.

Consider next the equilibrium strategies of the candidate. Recall the number of votes the candidate to the left receives:

$$A(t; c, d, e) = d + cF(t) + \varphi(t; c, d, e). \tag{3}$$

where $\varphi(t; c, d, e)$ is the solution of the equation

$$\varphi = cP(cF(t) + \varphi, d). \tag{7}$$

If Eq. (6) above holds, then the solution $\varphi(t; c, d, e)$ exists and is unique. Thus the quantity A is well-defined.

Differentiating (7) with respect to t gives

$$\frac{\partial \varphi}{\partial t} = \left(cf(t) + \frac{\partial \varphi}{\partial t} \right) eP_1$$

which simplifies to

$$\frac{\partial \varphi}{\partial t} = \frac{ecf(t)P_1}{1 - eP_1} \tag{8}$$

where P_1 is evaluated at $(cf + s_e, d)$. We note that $c, e, P;$, and f are all non-negative, and, by Eq. (5) the denominator in (8) is strictly positive. Thus φ is a monotone non-decreasing function of t. Since, moreover, F is monotone non-decreasing in t, we see that A is monotone non-decreasing in t.

Let candidate I choose position x, $0 \leq x \leq 1$, and let II choose y, $x < y \leq 1$. In this case, since Candidate I is on the left, he will receive:

(I) $d$votes from the disliked voters.
(II) $cf(t)$votes from ordinary voters, where $t = (x + y)/2$.
(III) $\varphi(t; c, d, e)$votes from group-oriented voters.

Thus I wins $A(t)$ votes, while II wins the remaining $m - A(t)$ votes.

If instead $x > y$, then II wins $A(t)$ votes while I wins $m - A(t)$ votes. Finally, if $x = y$, then both candidates win $m/2$ votes.

We can now express this as a 2-person game between the candidates, where each player has the pure strategy space $[0, 1]$, and the payoff function $K(x, y)$ is given by the difference in the number of votes received, i.e.

$$K(x, y) = \begin{cases} 2A\frac{s+y}{2} - m, & x < y \\ m - 2A\frac{x+y}{2}, & x > y \\ 0, & x = y \end{cases} \tag{9}$$

This is clearly a symmetric game, i.e., $K(x, y) = -K(y, x)$. So, if a value for the game exists, it must be zero. Of course, not all infinite games have values, even allowing for mixed strategies. However, in this case, we find that optimal pure strategies exist.

Theorem. The game given by Eq. (9) has a solution in pure strategies. In this solution, both players choose $x = y = t^*$, where t^* satisfies

$$A(t^*) = m/2. \tag{10}$$

This means that each candidate gets half the votes. It does not mean that the candidates choose a position so that half of the voters have ideal points to the left and half to the right.

Proof. Suppose Candidate I chooses $x^* = t^*$, where t^* satisfies (9). Then, for $y > x^*$, we have $(x^* + y)/2 > t^*$, and so, remembering that A increases with t,

$$K(x^*, y) = 2A\frac{x^* + y}{2} - m \geq 2A(t^*) - m = 0.$$

On the other hand, for $y < x^*$ we will have $(x^* + y)/2 + t^*$, and so

$$K(x^*, y) = m - 2A\frac{x^* + y}{2} \geq m - 2A(t^*) = 0.$$

Of course, if $y = x$, then $K(x, y) = 0$. Thus, for all y, $K(x^*, y) \geq 0$ and so $x^* = t^*$ is optimal for Candidate I. By symmetry, $y^* = t^*$ is also optimal for II.

Suppose, however, that no t^* satisfies (10). Since A is continuous in t, this must mean that either $A(0) > m/2$ or that $A(1) < m/2$.

Then, if $A(0) > m/2$, the optimal strategy for both players is $x^* = y^* = 0$. If $A(1) < m/2$, the optimal strategies are $x^* = y^* = 1$. The proof, not detailed here, depends, of course, on the monotonicity of $A(t)$.

It might be noted that we assumed that both candidates can occupy the same position, i.e., $x = y$ is allowed. In some cases, this may be forbidden: a constraint of the type $\mid x - y \mid \geq \delta$ may be included. If so, whichever candidate moves first will have an advantage: he can choose $x = t^*$ and will almost certainly have $K(x^*, y) > 0$ for all y outside the interval $(x - \delta, x + \delta)$.

For small values of δ, however, this advantage will not be great: if Candidate I chooses $x = t^*$ then II should choose y to be either $t^* + \delta$ or $T^* - \delta$; the payoff will be $K(t^*, t^* + \delta) = 2A(t^* + \delta/2) - m$, or $K(t^*, t^*\delta) \approx \delta[\partial A(t^*)/\partial t]$.

A similar result holds for $K(t^*, t^* - \delta)$. For small δ, this number will be positive but small. (The first move gives a large advantage only if $A(0)$ is much more than $m/2$, or $A(1)$ is much less than $m/2$. Then Candidate I would choose $x^* = 0$ or 1 in the two cases; II would choose $y = \delta$ or $1 - \delta$ respectively, with a substantial advantage to I.)

Locating t^*

We now provide an alternative way to determine t^* from that given in the text. It is easy to see from the above results that t^* can be obtained as a solution of the two equations

$$d + cF(t) + \varphi - \frac{c + d + e}{2} = 0 \tag{11}$$

From (11) we have combining equation (12) with equation (11), we obtain

$$\varphi - eP(cF(t) + \varphi, d) = 0. \tag{12}$$

$$cF(t^*) + \varphi = \frac{c + e - d}{2}. \tag{13}$$

$$\varphi = eP\left(\frac{c + e - d}{2}, d\right) \tag{14}$$

Combining the latter two equations gives

$$cF(t^*) = \frac{c + e - d}{2} - eP\left(\frac{c + d - d}{2}, d\right). \tag{15}$$

This gives us $F(t^*)$ directly. Computation of t^* is then relatively easy (depending, of course, on the form of function F).