Measures of Bias and Proportionality in Seats-Votes Relationships

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I. INTRODUCTION

There has been a good deal of recent interest in the functional relationship in a two-party, single-member district system between a party's aggregate vote share across all legislative districts (V) and the proportion of seats that it wins (S). While some early work simply regressed S on V (see, e.g., Dahl, 1956) and looked at the slope and intercept of the regression line, recent work has focused on nonlinear models of seats-votes relationships. Theil (1969) and Taagepera (1973) have proposed a general functional relationship of the form

$$\frac{S}{1 - S} = \left(\frac{V}{1 - V}\right)^{B_1} . \qquad (1)$$

Tufte (1973) fitted a logarithmic transformation of this relationship to data from elections in Britain, New Zealand, and the United States, of the form

$$\log \left(\frac{S}{1 - S}\right) = B_1 \log \left(\frac{V}{1 - V}\right) + B_0 , \qquad (2)$$

where B_0 is a stochastic error term.

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*This research was supported by NSF Grant #SES 81-07554, Political Science Program. I am indebted to Cheryl Larsson of the Word Processing Center, University of California, Irvine, for the preparation of the graphs and tables in this paper; to Kathy Alberti, Cheryl Larsson, Helen Wildman and Sue Pursche for typing and proofreading; to Laurel Eaton for bibliographic assistance; and to the anonymous referees for helpful suggestions. Any errors remaining are the sole responsibility of the author.

Linehan and Schrodt (1978) have proposed an alternative specification of the relationship in Equation (1):

$$\frac{S}{1 - S} = \left(\frac{V}{1 - V}\right)^{B_1} + \varepsilon \,, \qquad (3)$$

where ε is again a stochastic error term hypothesized to have zero mean, but with a normal distribution rather than the log normal distribution Tufte (1973) proposed.

At one time it was thought that three was the most likely value for B_1. This conjecture is known as the "cube law" of politics (Kendall and Stuart, 1950). With the exception of the estimates offered by Linehan and Schrodt (1978) and Schrodt (1981), recent work has found a wide variation in B_1, with only parliamentary elections in Great Britain closely approximating the magic number of three. Estimated values of B_1 (some from linear, some from non-linear models) ranged from .7 (U.S. Congressional elections in the period 1966-1970 [Tufte, 1973] to 4.4 (the U.S. Electoral College, 1928-1968 [Taagepera, 1973]). Most of the fitted values are, however, between 2 and 3.[1,2]

Our aim in this paper is limited to one aspect of the seats-votes relationship, specifying useful theoretical measures of the "bias" in seats-votes relationships for two-party, single-member district contests. We consider six definitions of "bias" offered in the literature, and propose a seventh and eighth definition of our own, inspired by the Gini index of inequality (see, e.g., Taagepera and Ray, 1977) and related to an index of maximum/minimum electoral bias (distortion) proposed by Grofman (1975). We also clarify the distinction between measures of "bias" and measures of "proportionality" in seats-votes relationships.

II. MEASURES OF PROPORTIONALITY

We propose two criteria that any measure of degree of proportionality of the seats-votes relationship ought to satisfy.

First Criterion of Measurement of Proportionality: If any set of election outcomes may be characterized by the function $S = V$, then any satisfactory measure of deviation from proportionality must assign a value of zero to that set. (Analogously, any satisfactory measure of degree of proportionality must assign a value of one to that set.)

Second Criterion of Measurement of Proportionality: If two sets of observations of seats-votes relationships are generated by the same functional relationship between seats and votes (including identical parameters of that function), then any satisfactory measure of deviation from proportionality (or degree of proportionality) must yield the same value for both sets of observations.

We henceforth denote these criteria as P1 and P2.

In a single-member district system of elections, we would never expect to find complete proportionality between a party's vote share and its seat share. In general, we would expect that the graph of the seats-votes relationship will be an S-shaped curve such as is generated by the power function in Equation (1).[3] (See Figure 1.) The parameter, B_1, which represents the slope of the seats-votes curve in the neighborhood of $V = .5$, has come to be known as the swing ratio (Tufte, 1973). It is an index of the proportionality of seats-votes relationships. Similarly $|B_1 - 1|$ is an indicator of deviation from proportionality in seats-votes relationships. For the functional relationship shown in Equation (2), if $B_0 = 0$, only for $B_1 = 1$ will the percentage of seats won equal the percentage of votes received. Note that $|B_1 - 1|$ satisfies both criterion P1 and criterion P2 as a measure of deviation from proportionality and that B_1 satisfies P1 and P2 as a measure of degree of proportionality. Any measure that varies with V_t (party I's vote share at a particular election t) will violate P2.

III. MEASURES OF BIAS

We follow the Niemi and Deegan (1978) definition of bias in a set of election outcomes. If the seat share S earned by party I for a given vote share V is the same as the seat share earned by party II for that identical vote share, then the election outcomes shall be said to be unbiased for that value of V. If an election system is unbiased for all values of V, we refer to it as completely unbiased. We propose the following criterion that any measure of bias ought to satisfy:

First Criterion of Measurement of Bias: If a set of election outcomes is unbiased for all elements of the set, then any satisfactory measure of bias must assign a value of zero to that set.

We propose a second criterion that any measure of bias in seats-votes relationships also ought to satisfy.

Second Criterion of Measurement of Bias: If two sets of observations of seats-votes relationships are generated by the same functional relationship between seats and votes (including identical parameters of that function), then any satisfactory measure of bias must yield the same value for both sets of observations.

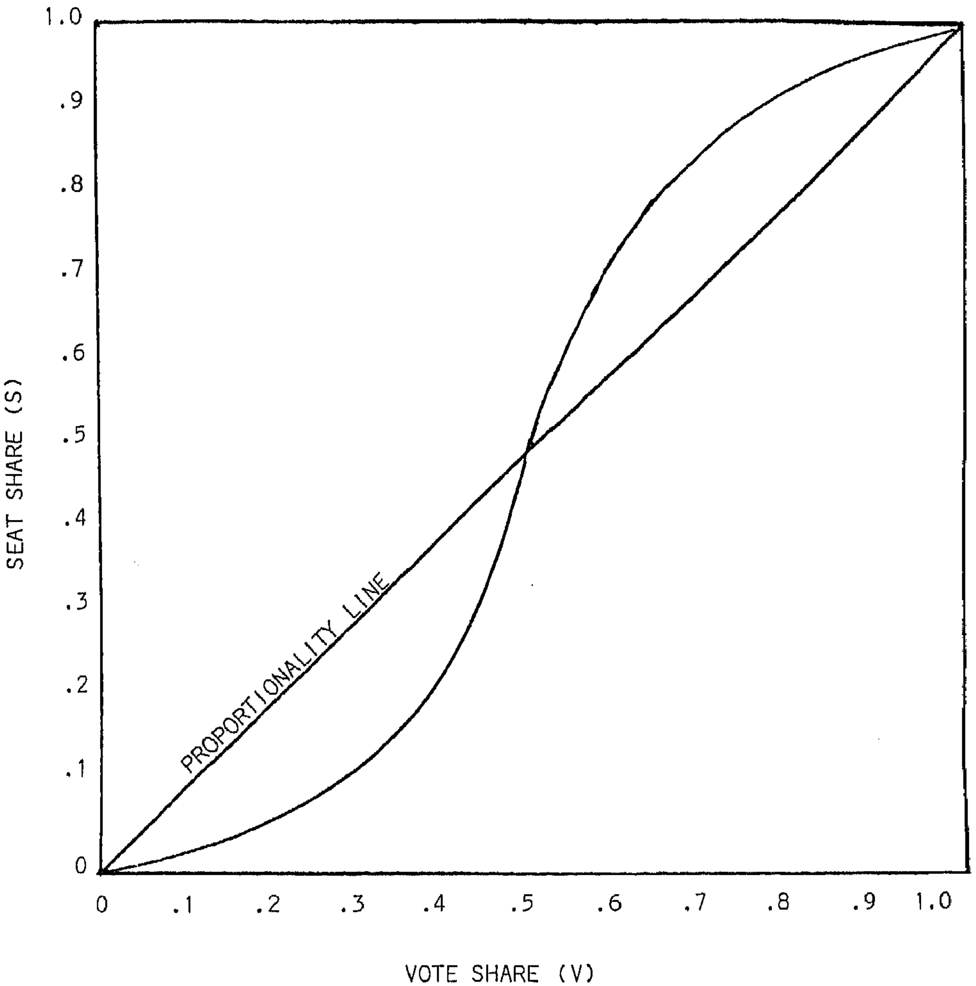

Figure 1. Relationship Between Vote Share (V) and Seat Share (S) for a Single-Member District Legislature for which $S/(1-S) = [V/(1-V)]^3$.

We denote these criteria as B1 and B2, respectively. Any purported measure of bias that varies with V_t will violate B2.

B2 and P2 are examples of what Rae (1981) has called "lot-regarding" criteria of equality, in which all identically situated actors (in this case political parties) must be identically treated. Our four criteria for measures of bias and measures of proportionality may seem to be either trivial or tautological. As we shall see, the most common measures of bias/proportionality that have been proposed do not satisfy them!

Measure 1: Bias as the Simple Discrepancy Between Seats and Votes

Consider the hypothetical graph of seat share as a function of vote share (Figure 1). Consider an election time t that generates some point (V_t, S_t) on that graph (i.e., a hypothetical election outcome). A seemingly natural definition of the bias in that election outcome is the discrepancy between the observed (V_t, S_t) outcome and that obtained if votes were transformed into seats in a perfectly proportional manner--i.e., the outcome (V_t, V_t). We may specify this discrepancy, which we shall label D_1, in terms of a difference measure:

$$D_1 = V_t - S_t . \quad (4)$$

This measure will be positive or negative depending on whether party I or party II is favored (i.e., receives a greater seat share than vote share). This measure of bias is the one most commonly used in the political geography literature (see, e.g., Johnston, 1979:58-60). It is also the most common measure of "fairness" of election outcomes in the political science literature on the representation of racial minorities in ward vs. at-large elections (see the literature review in Grofman, 1981).

Imagine that the seats-votes relationship for a given legislature is being generated by the cubic relationship pictured in Figure 1. It is apparent from Figure 2 that bias, as defined by the D_1 measure, depends on V_t. If we happen to observe an election (or series of elections) in which V_t is close to .5, D_1 will be near zero. If we observe an election or elections in which V_t is around .7 (.3), we will find a very large positive (negative) bias. Some authors have compared D_1 values in different elections or sets of elections to measure differences in bias at different points in time between two different polities, or between two different types of election systems (see, e.g., Uslaner and Weber, 1979; Cole, 1974; MacManus, 1978; Taebel, 1978; Rabinowitz and Hamilton, 1980). However, unless the V_+ values across the different sets of elections are nearly identically distributed, such comparisons are not really meaningful, since the identical underlying functional relationship between seats and votes can give rise to very different D_1 values depending on the value(s) of V_+ in the election(s) sampled.[4] Unless $B_1 = 1$, if seats and votes are related as in Equation (1), the relationship between D_1 and V will be nonmonotonic.

If we look at a hypothesized direct linear relationship between seat share and vote share

$$S = B_1 V + \psi , \tag{5}$$

we see that if $V = 1 - V = .5$, then $S = 1 - S = .5$ only if $\psi = .5(1 - B_1)$. If $\psi = 0$, this requires $B_1 = 1$.[5]

If the relationship between S and V is as specified in Equation (5), then

$$\begin{aligned} D_1 &= V_+ - S_+ = V_+ - B_1 V_+ - \psi \\ &= V_+(1 - B_1) - \psi . \end{aligned} \tag{6}$$

Hence, estimated bias (D_1) will decrease (and at some point become negative) with increasing V_+ if $B_1 > 1$, while it will linearly increase with increasing V_+ if $B_1 < 1$.

Hence, if we posit a power relationship [as in Equation (1)], D_1 can be expected to vary with V_+ nonmonotonically; and even if we posit a linear seats-votes relationship [as in Equation (5)], then D_1 becomes a linear function of V_+. In neither case does D_1 offer a desirable measure of bias. Moreover, even if $B_0 = 0 (\psi = 0)$ and there is perfect symmetry in the seats-vote transformation rule for each of the parties, D_1 will still be nonzero. Hence, D_1 fails to satisfy either criterion B2 or criterion P2 and thus is not suitable as a measure of either bias or proportionality.

Measure 2: Bias as the Ratio of Seats to Votes

Some authors (e.g., Robinson and Dye, 1978) have looked at

$$D_1' = \frac{S_+}{V_+} \tag{7}$$

as their measure of bias. D_1' satisfies criterion B1 and criterion P1. If seats and votes are linearly related as in Equation (5), we have

$$D_1' = \frac{B_1 V_+ + \psi}{V_+} = B_1 + \frac{\psi}{V_+} . \tag{8}$$

Unless V_+ is much larger than ψ, D_1' appears as a linear function of V_+. Hence, in general, if we are comparing two different seats-votes graphs for bias, unless the two graphs have identical values of $\bar{V}_+$ with observations similarly distributed around that mean, D_1' will be very misleading for measurement of bias. In particular, D_1' fails to satisfy criterion B2. For analogous reasons, D_1' is also not a good measure of proportionality. For V_+ very large relative to ψ, it is not too bad, since in this case $D_1' \approx B_1$, but for values of $V_+ \geq \psi$, D_1' varies with V_+ and hence fails to satisfy criterion P2.

A very similar argument can be constructed to show that D_1' is unsatisfactory as a measure of either bias or proportionality if the seats-votes relationship is as specified in Equation (2). In both cases D_1' increases monotonically with V_+.

Measure 3: Bias as a Function of Vote Share Needed to Gain a Fifty Percent Seat Share

A number of authors (in particular, Tufte, 1973) have proposed to define electoral bias in two-party elections as the difference between .5 and the vote share a party needs to get a .5 fraction of the seats. Let $V_{(.5)}$ denote the vote share required to earn a 50 percent seat share. We may define our third measure of bias, D_2, as

$$D_2 \equiv V_{(.5)} - \frac{1}{2} . \tag{9}$$

If seats and votes are linearly related according to Equation (5), then at $S = .5$

$$.5 = B_1 V_{(.5)} + \psi ,$$

and hence

$$V_{(.5)} = \frac{.5 - \psi}{B_1} . \tag{10}$$

Thus, if seats and votes are linearly related as in Equation (5), then

$$D_2 = \frac{1 - 2\psi - B_1}{2B_1} . \tag{11}$$

For example, Dahl (1956) looked at U.S. Congressional elections (1928-1954) and U.S. Senate elections (1928-1952) and found best fitting regression lines of S = 2.5 V - .70 and S = 3.02 V - .95, respectively. Using Equation (9), we find that those give rise to bias measures (D_2) of -.02 for both House and Senate elections (a negative value indicates advantage for the Democrats as Dahl defined his variables).

If the relationship between seats and votes is of the nonlinear form specified in Equation (2), for S = .5 we have

$$B_1 \log_e \left(\frac{V_{.5}}{1 - V_{.5}}\right) = \log_e 1 - B_0 = -B_0 . \tag{12}$$

Analogously, taking logarithms on both sides of Equation (3), we obtain

$$B_1 \log_e \left(\frac{V_{.5}}{1 - V_{.5}}\right) = \log_e (1 - \varepsilon) . \tag{13}$$

While we could use (12) or (13) to obtain a value for $\log_e V_{(.5)}/1 - V_{(.5)}$ and then solve for $V_{(.5)}$, it is easy to use a well-known approximation to log (1 + x) (see, e.g., Feller, 1957)[6] to reexpress Equation (13) as

$$B_1 \log_e \left(\frac{V_{(.5)}}{1 - V_{(.5)}}\right) \approx -\varepsilon . \tag{14}$$

Thus, Equation (2) and Equation (3) have essentially identical approximations. Henceforth, we shall use Equation (2) to estimate our logit model. After taking antilogarithms and performing some simple algebraic manipulations on Equation (12), we obtain a convenient expression for $V_{(.5)}$,

$$V_{(.5)} = \frac{1}{e^{B_0/B_1} + 1} . \tag{15}$$

Hence, where seats and votes are related as in Equation (2), we have

$$D_2 = \frac{1}{e^{B_0/B_1} + 1} - .5 . \tag{16}$$

D_2 is in several ways an admirable measure of bias. D_2 satisfies both criterion B1 and criterion B2 and permits meaningful comparisons. Also, D_2 focuses attention on the crucial point in a two-party competition, the point at which control of the legislature changes hands. Moreover, the estimates of D_2 do not appear to be substantially affected by the choice of Equation (2) or Equation (5). Tufte (1973: 546, Table 2) fits the logit model [Equation (2)] to data from Great Britain ($\hat{B}_0$ = -.02, $\hat{B}_1$ = 2.88), New Zealand ($\hat{B}_0$ = -.12, $\hat{B}_1$ = 2.31), and the U.S. 1868-1970 ($\hat{B}_0$ = .09, $\hat{B}_1$ = 2.52). Using these values to estimate D_2 from Equation (15), we obtain values of .002, .013, and -.009 for Great Britain, New Zealand, and the U.S., respectively. Tufte (1973:543, Table 1) fitted a regression line to the same data. Using the linear model [Equation (5)], he obtained D_2 values of .002, .014, and -.009, respectively.[7]

A slightly different way to approximate Equation (2) fits well for S and V values reasonably near .5 (say between .3 and .7), and fits quite well for S and V values between .45 and .55.[8] This method can be used to derive a linearized logit-based estimate for D_2. We use a Taylor expansion around .5 (see Feller, 1957:49) to obtain

$$\log_e \left(\frac{p}{1 - p}\right) \approx 4p - \frac{1}{2} . \tag{17}$$

Hence, from Equation (2)

$$4S - \frac{1}{2} \approx 4B_1 \left(V - \frac{1}{2}\right) + B_0 , \tag{18}$$

which may be reexpressed as

$$S \approx B_1 V + \frac{1}{2} - \frac{B_1}{2} + \frac{B_0}{4} . \tag{19}$$

This is not a bad approximation. Consider, for example, Tufte's (1973) linear estimate of the data on the British parliament. He found S = 2.83 V - .921 to be the best fitting regression line. His logit estimates for the same data were B_1 = 2.88 and B_0 = 0.02. Substituting these values into Equation (19), we obtain a linear regression estimate of $S = 2.88\ V - .935$, which matches very closely the result obtained directly from a regression model, especially when we take into account the standard errors of the various parameter estimates. (Of course if $B_0 = 0$, Equation (19) reduces to $S = 3x - 1$ for $B_1 = 3$, and we have a linear version of the "cube law.")

Substituting the value of ψ obtained from Equation (5) into Equation (19), we obtain for the linear approximation to the logit, the nice approximation for D_2

$$D_2 \approx \frac{B_0}{4B_1} . \qquad (20)$$

This is a good approximation: we get values of +.002, +.013, and -.009 for the three cases previously considered--estimates of D_2 virtually identical to those obtained directly from the fitted regression line.[9] Since $B_1 \approx 2.5$ for the three cases considered, a rough and ready approximation of D_2 for these data sets in Tufte (1973) is $D_2 = B_0/10$.

Measure 4: Bias as Seat Share Needed to Gain a Fifty Percent Vote Share

By looking at the .5 vote share rather than the .5 seat share, we can define a measure of bias directly analogous to that of D_2 (cf. Tufte, 1973:543, n. 4). Let $S_{(.5)}$ denote the seat share obtained when party I gets a 50 percent share of the votes. We define D_3 as the difference between .5 and the seat share obtained when party I gets a .5 fraction of the vote, i.e.,

$$D_3 = S_{(.5)} - \frac{1}{2} . \qquad (21)$$

For the linear model of Equation (5) for $V = 1/2$ we have

$$S_{(.5)} = .5B_1 + \psi . \qquad (22)$$

Hence

$$D_3 = .5(B_1 - 1) + \psi . \qquad (22)$$

For the logit model of Equation (2) for $V = .5$ we have

$$\log_e \left(\frac{S_{(.5)}}{1 - S_{(.5)}} \right) = B_0 . \qquad (24)$$

Hence

$$S_{(.5)} = \frac{e^{B_0}}{1 + e^{B_0}} . \qquad (25)$$

Thus, for the logit model

$$D_3 = \left(\frac{e^{B_0}}{1 + e^{B_0}} \right) - \frac{1}{2} . \qquad (26)$$

It is interesting to see that, for the logit model, B_1 does not enter into the specification of bias as measured by D_3.

We may reanalyze Tufte's (1973) estimates of linear and logit models for Great Britain, New Zealand, and the U.S. to obtain estimates of D_3 for those three countries. Using the linear model, we obtain D_3 estimates of -.006, -.032, and .022, respectively. Using Tufte's logit model estimates for the same data, we obtain D_3 estimates of -.005, -.030, and .023. Using our linear approximation to the logit model (and Tufte's logit estimates), we obtain essentially identical values. Again, as with D_2, logit and linear estimates of bias (D_3) in the three cases are virtually identical.[10]

Clearly D_3 has much the same strength as D_2. It satisfies criterion B1 and criterion B2, permits direct comparisons of bias across different sets of elections, is straightforwardly defined, and focuses on a "natural" point on the seats-vote graph for two-party competition, $V = .5$, where both parties receive the same vote shares.

D_2 and D_3 are in fact closely related. For the linear model from Equation (10) we have

$$-D_2 = \frac{.5(B_1 - 1) + \psi}{B_1} . \qquad (27)$$

Substituting in Equation (10) we obtain

$$D_3 = -B_1 D_2 \ . \qquad (28)$$

This relationship has been noted by Tufte (1973:543, n. 4). For the linear approximation of D_3 derived from a logit estimate, it follows that

$$D_3 \approx \frac{B_0}{4} \ . \qquad (29)$$[11]

The only real problem with D_2 and D_3 appears to be that each focuses exclusively on one point on the seats-votes graph, the point that corresponds to a seat (vote) share of .5. While it is clearly natural to focus on such points (especially the former), it may be that different estimates of bias would be generated were we to look elsewhere on the graph. Of course, if we pick a particular estimating technique (say the logit model), then whether we measure bias at $V_{(.5)}$ ($S_{(.5)}$) or at some other V value (S value) might appear arbitrary, as long as we are always consistent in our choice. This is, however, too simplistic a view.

For D_2, both the logit and the linear model imply (at least in some range around V = .5) a consistency in the direction of bias; i.e., if party I is advantaged (disadvantaged) when S = .5, it will also be advantaged (disadvantaged) when S = .5 ± d (see Figure 1). However, it is, empirically, not true that a districting system (and distribution of partisan strength and differential turnout) that favors one party for certain values of V will necessarily favor that same party for all values of V, even those close to $V_{(.5)}$. The same features of a districting system that are advantageous to a party at one level of overall vote strength (e.g., winning a number of districts by bare majorities or having a larger number of "safe" seats than its opponent) may become disadvantageous (relative to the proportionality norm) if its vote strength changes. Exactly analogous remarks apply if we use D_3. This potential difficulty with D_2 or D_3 has led several authors to a somewhat more general approach to measuring bias that looks at bias at points other than S = .5 or V = .5.

Measure 5: Bias as the Difference Between the Seat Shares Gained by Party I and by Party II when Each Obtains an Identical Share of the Vote

The fifth measure we look at is closely related to (indeed can be thought of as a natural generalization of) D_3. Let us look at what happens when party I receives a 50 percent vote share. If seats-votes are linearly related as in Equation (5), then party I will receive $.5B_1 + \psi$ seats, while with a vote share of .5, party II will receive $1 - .5B_1 - \psi$ seats. The difference in seats received by the two parties is given by

$$B_1 + 2\psi - 1 = -2B_1 D_2 = 2D_3 \ . \qquad (30)$$

Consider any other value of V, which we denote $V_{(x)}$.

Let us define

$$D_4(x) = \text{seat share of party I if its vote share is } V_{(x)} - \text{seat share of party II if its vote share is } V_{(x)} \ . \qquad (31)$$

Note that, in this measure, bias is independent of V_+. Hence, whatever values of V_+ we actually observe should not affect the amount of bias we "detect," and thus $D_4(x)$ satisfies criterion B2. It also satisfies criterion B1.

For the logit case of Equation (2), for B_1 unknown, it is difficult to solve directly for the required expression. Using Equation (19), we find that, for the linear approximation to the logit model,

$$D_4 \approx \frac{B_0}{2} \ . \qquad (32)$$

This is a rather nice result. Note that for the linearized logit estimates D_4 is independent of B_1 and of x, and hence we may drop the x-subscript.

Just as $D_4(x)$ is a natural generalization of D_3, we may readily generate a measure that is a natural generalization of D_2.[12]

Measure 6: Bias as the Difference Between the Vote Shares Obtained by Party I and by Party II when Each Obtains an Identical Share of the Seats

We may define $D_5(x)$:

$$D_5(x) = \text{vote share of party I if its seat share is } S_{(x)} \quad \text{vote share of party II if its seat share is } S_{(x)} . \qquad (33)$$

If seats and votes are linearly related as in Equation (5), we have

$$D_5 = \frac{-D_4}{B_1} = \frac{1 - 2\psi - B_1}{B_1} = 2D_2 = \frac{-2D_3}{B_1} . \qquad (34)$$

Since exploration of the logit model for this case adds little new, we omit it.

We shall not deal with the properties of D_5 since we wish to turn to a still further generalization of D_5 (and D_4).

Measure 7: Bias as a Gini-Index-Like Measure of the Area Under Seats-Votes Discrepancy Curves

If party I receives a given share of the vote, there is a minimum share of the seats that it could win. This minimum share would occur if party II received a majority of the votes in as many districts as possible and party I as far as possible had its votes concentrated into a handful of districts that it carried unanimously. We denote this minimum as $S_{min}(x)$. Similarly, if party I receives a vote share of x, there is maximum seat share it could win. This maximum share would occur if its votes were spread so as to give it a bare majority in as many districts as possible. We denote this maximum as $S_{max_I}(x)$. Figure 2 shows minimum and maximum seats curves for a legislature with a very large number of districts, all of which are contested and all of which are of equal size.

Until now, we have implicitly assumed that data on the seats-votes relationship were generated across a series of elections, with each election specifying one point on the seats-votes graph. If district level data are available, an alternative method exists for generating a seats-votes curve. As far as we are aware, Butler (1951) was the first to suggest this procedure. Tufte (1973) also makes use of

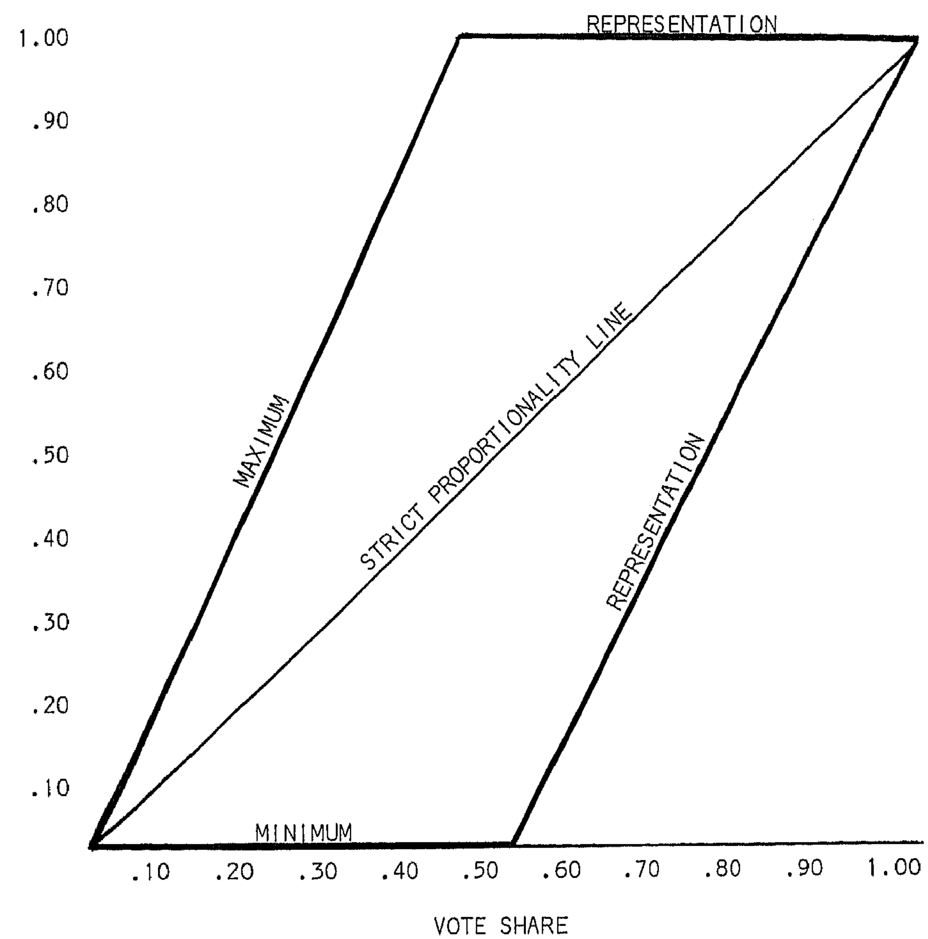

Figure 2. Graph of Theoretical Maximum/Minimum Seat Shares in a Two-Party, Single-Member District Election (Source: Grofman, 1975:318, Figure 6).

it, as does Scarrow (1981, 1982). The idea is a quite simple one. Any given election gives rise to a particular outcome--i.e., a point in seats-vote space. Imagine that, in a given election, in every district election, party I received one percentage point less and party II received one percentage point more. This would give rise to a new "hypothetical" election outcome--the outcome of an election that would have resulted if party I's strength had uniformly dropped one percentage point in all districts.[13] In like manner we can generate hypothetical election outcomes for all possible decrements (increments) in party I's vote share. These will give us a seats-votes graph.[14] This graph will be monotonic in V, though there can be "flat spots" (because of small number "lumpiness" effects).

Scarrow (1981, 1982) used this method to generate seats-votes curves for state legislative elections in New York and Connecticut. He combined this simulation technique with use of the D_4 measure of bias to examine bias in the .45 - .55 vote share range in Connecticut and New York State Assembly and Senate elections in the past two decades, including data prior and subsequent to early 1970s apportionments. If seats-votes could be <u>perfectly</u> fit by a linear model (or for that matter a logit model), we should observe a perfectly or nearly perfectly constant bias as measured by D_4 (or D_5). In the real world constant bias should be rare. We have reproduced a portion of Scarrow's (1981) data for the Connecticut Assembly elections in 1970 and 1972 (Table 1). These two elections illustrate most of the points we wish to stress.

While 1966-1976 projections for Connecticut Assembly and Senate roles are roughly consistent with a constant bias for almost all elections (see Scarrow, 1981), in the 1972 Connecticut House we have a nonmonotonic relationship, i.e., a bias reversal--below 53 percent of the votes D_4 indicates a Republican advantage, above 53 percent of the vote D_4 indicates a Democratic advantage (this advantage continues past V = .55).[15] Figures 3A and 3B graphically represent the data in Table 1, a clear visual display of the patterns noted above.

As far as we are aware, the particular form of graphic display of the bias in seats-votes relations over a range of aggregate vote outcomes shown in Figure 3 has never before been used. With the bias data in this graphic form, a "natural" measure of bias suggests itself, the <u>area between</u> the party I and party II curves (the solid and the dotted lines in Figure 3). Moreover, if we use a positive sign for the area where the curve for party I is above that for party II, we have a natural way of capturing in a <u>single number</u> the <u>net</u> bias over a range of election outcomes.

TABLE 1

DEMOCRATIC AND REPUBLICAN SEAT SHARES IN THE CONNECTICUT ASSEMBLY AT SELECTED PROPORTIONS OF THE STATEWIDE (TWO-PARTY) VOTE IN 1970 AND 1972 FOR HYPOTHETICAL ELECTIONS BASED ON UNIFORM SWINGS ACROSS ALL DISTRICTS FROM THE OBSERVED SEAT-VOTE VALUE IN THAT YEAR

Proportion of Statewide Vote*		45±	46±	47±	48±	49±	50±	51±	52±	53±	54±	55±	Swing Ratio
1970	Dem	42.9	45.8	48.0	49.7	**52.0** →	55.9	55.9	59.3	62.1	64.4	65.5	
	Rep	34.5	35.6	37.9	40.7	44.1	44.1	**48.0** →	50.3	52.0	54.2	57.1	
	Bias	+8.4	+10.2	+10.1	+9.0	+7.9	+11.8	+7.9	+9.0	+10.1	+10.2	+8.4	2.26
1972	Dem	36.4	38.4	40.4 →	41.1	42.4	**48.3**	51.0	55.0	58.3	62.9	67.5	
	Rep	32.5	37.1	41.7	45.0	49.0	51.7	57.6	58.9	59.6 →	61.6	63.6	
	Bias	+3.9	+1.3	-1.3	-3.9	-6.6	-3.4	-6.6	-3.9	-1.3	+1.3	+3.9	3.11

Source: Scarrow (1981, Table III). Cell entries indicate seat percentages that a party would have achieved at the (column) specified vote share. Arrows indicate actual election outcomes. Boxed outcomes represent situations where a party with a vote share less (more) than .5 would achieve a projected seat share greater (less) than .5.

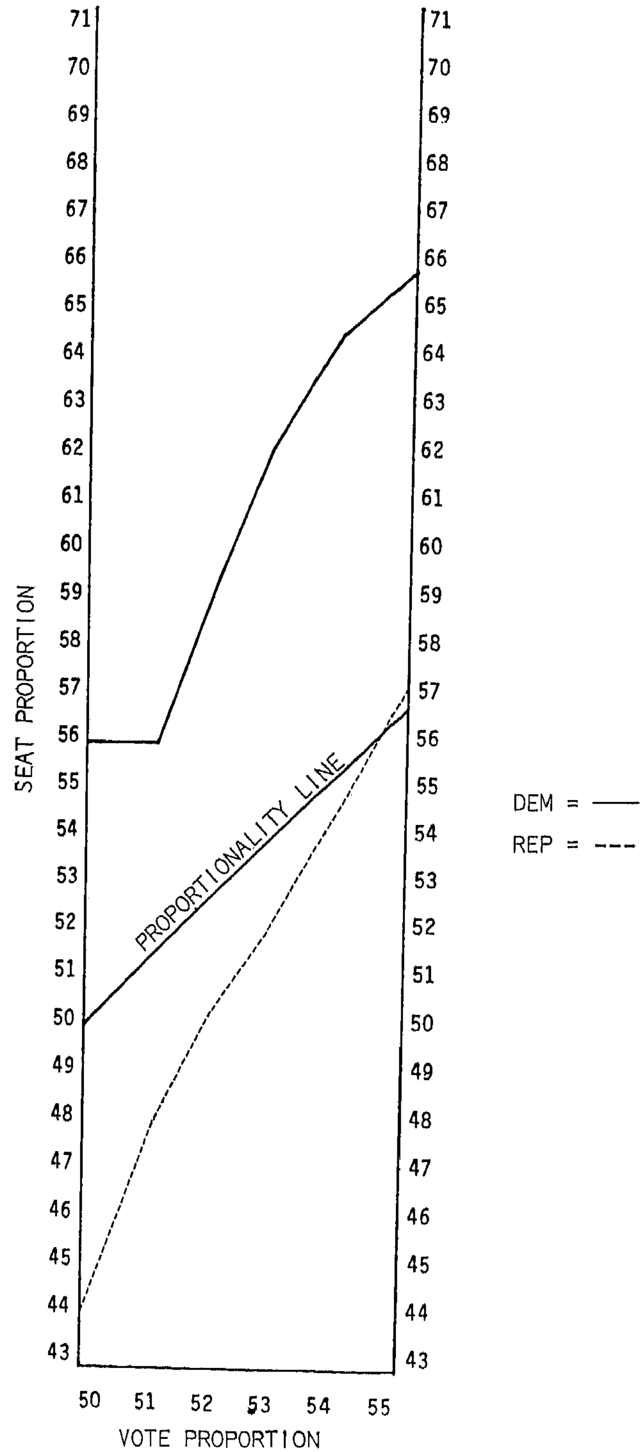

Figure 3A. Graph of Projected Seats Votes Discrepancies in the Connecticut Assembly, 1970 (Data source:

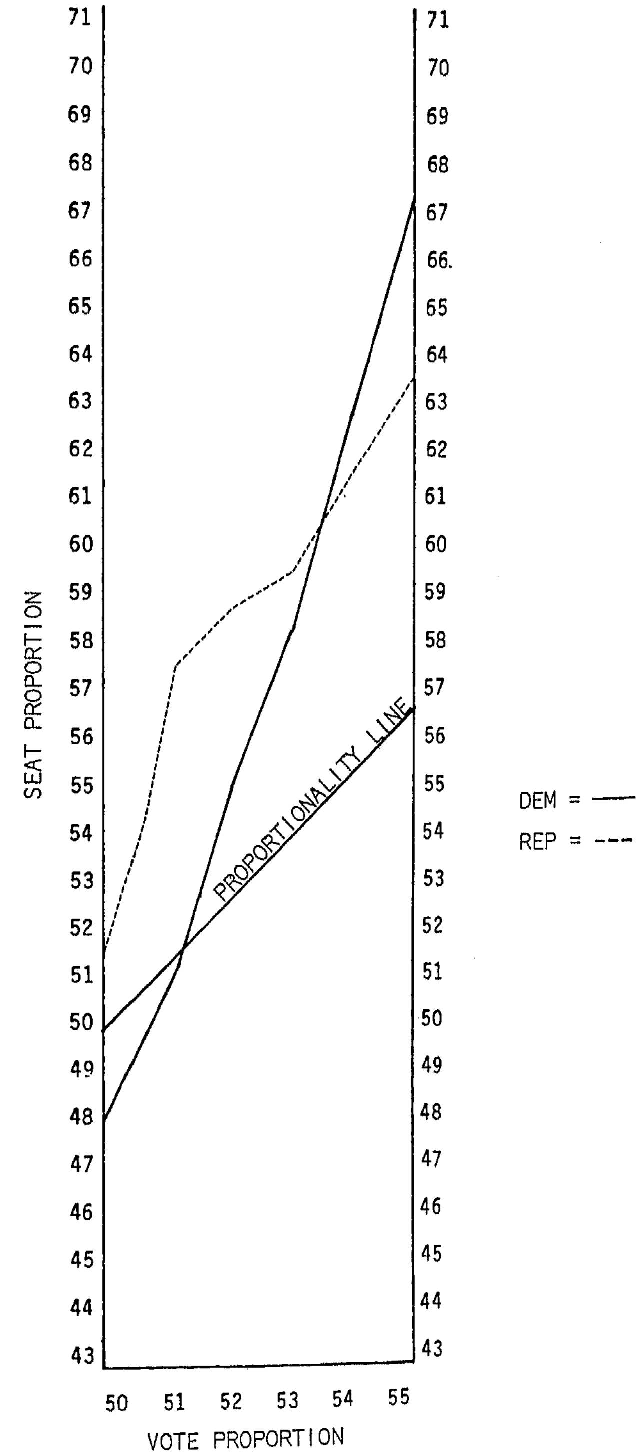

Figure 3B. Graph of Projected Seats Votes Discrepancies in the Connecticut Assembly, 1972 (Data source:

Let $S_I(x)$ be the seat share for party I corresponding to a vote share of x, and similarly define $S_{II}(x)$ for party II. We define D_6 as follows.

$$D_6 = \int_{.5}^{x} S_I(x) - S_{II}(x)$$

$$= \int_{.5}^{x} D_4(x) \ . \qquad (35)$$

Hence, if S and V are linearly related as in Equation (5), we have

$$D_6 = \int_{.5}^{x} (B_1 x + \psi) - (1 - B_1(1 - x) + \psi)$$

$$= - \int_{.5}^{x} 1 - B_1 - 2\psi = \int_{.5}^{x} D_4$$

$$= (.5 - x)(1 - B_1 - 2\psi) \ . \qquad (36)$$

We shall not bother to work out the implications for D_6 of a nonlinear seats-votes relationship such as in Equation (2).

Measure 8: A Normalized Measure of Bias in the Interval (.5, x)

Grofman (1975) has proposed to measure the maximum possible bias in seats votes relationships (over the vote range [0, 1]) for different types of election systems by looking at

$$D_{max-min} = \int_0^1 S_{max_I} - S_{min_I} \ . \qquad (37)$$

For two-party single-member-district elections under plurality, $D_{max-min} = 1/2$. We can generate a value of $D_{max-min}$ for the vote range (.5, x) by defining

$$D_{max-min}(x) = \int_{.5}^{x} S_{max_I}(x) - S_{min_I}(x) \ . \qquad (38)$$

For two-party single-member-district plurality contests,

$$D_{max-min} = x - \frac{1}{2} - \int_{.5}^{x} 2x - 1 = 2x - x^2 - \frac{3}{4} \ . \qquad (39)$$

It would be desirable to have D_6 range between -1 and +1 (D_1 through D_5 vary over that range). We may accomplish this by normalizing $D_6(x)$ by $D_{max-min}(x)$; i.e., we look at

$$D_6'(x) = \frac{D_6(x)}{2x - x^2 - \frac{3}{4}} \qquad (40)$$

For x = .55, the value used by Scarrow, $D_{max-min}(x) = .0475$. For the 1970 and 1972 Connecticut Assembly elections we show values for D_1, D_1', D_2, D_3, $D_4(.50)$, $D_5(.50)$, $D_6(.55)$ and $D_6'(.55)$ in Table 2.

If we compare the 1970 and 1972 elections according to our various measures, we find no agreement among them (although disparities among D_2 and D_3; $D_4(.5)$ and $D_5(.5)$; and $D_6(.55)$ and $D_6'(.55)$ are more apparent than real, since these measures are functionally related to another). In 1970 V_T was close to S_T (.49 v. .52), and we obtain a value of D_1' slightly over 1. In 1972 V_T was reasonably close to S_T (.47 v. 40) but now smaller than S_T rather than larger. This gives rise to an anti-Democratic bias, as shown by a D_1' value of less than one (.85). Both D_1 and D_1' are quite misleading, as visual inspection of Figures 3A and 3B suggest. Both D_1 and D_1' show that 1972 has more bias than 1970. This is not what we observe in Figure 3A. The error arises because in 1970 we had S_T reasonably close to V_T, but yet the result turned less than a majority of the votes into a majority of the seats; in 1972 S_T was further from V_T but the differences might reasonably be expected, given a nonlinear (and roughly symmetric) transformation of votes into seats. D_1 not only gets the direction wrong but also exaggerates the magnitude of the differences between 1970 and 1972, in that D_1 for the latter year is more than twice D_1 for the former year in absolute value (.030 v. -.066). Measures D_2 through D_5 get the directionality right but overestimate, in our view, the magnitude of the bias in 1972 relative to that in 1970 by looking only at bias at the point V = .5. As can be seen from Figures 3A and 3B, at V = .5 the difference between S_I and S_{II} is about .12 for 1970 and about .04 for 1972, a ratio of about 4 to 1. This ratio squares roughly with what we find in comparing the D_2 through D_5 values for the two years. However, in

TABLE 2

COMPARISON OF SEVEN INDICES OF BIAS IN SEATS-VOTES RELATIONSHIPS FOR TWO CONNECTICUT ASSEMBLY ELECTIONS, 1970 AND 1972

	D_1	D_1'	D_2	D_3	$D_4(.5)$	$D_5(.5)$	$D_6(.55)$	$D_6'(.55)$
1970	.030	1.06	-.018	.059	.118	.036	.0048	.101
1972	-.066	.85	+.004	-.017	-0.34	-.008	.0118	-.017

Source: Scarrow (1981).

1972, while bias stays roughly constant from V = .50 through V = .52, at V = .53 there is a bias reversal and, thus, over the entire range (.5, .55) the net bias in favor of the Republican gets significantly reduced from its value at V = .5.

III. CONCLUSIONS

Our concern has been with developing appropriate measures of bias and proportionality. We regard B_1 as the most appropriate measure of degree of proportionality, with $|B_1 - 1|$ indicating deviation from proportionality. We have demonstrated (a) that the two most common measures of bias (D_1 and D_1') are inappropriate and (b) that most of the remaining measures previously proposed in the literature are, in fact, simple transformations of one another.[16]

Although D_2 through D_5 are reasonable measures, and D_2 and D_3, in particular, have the advantages both of ease of calculation and interpretation, the measure of bias best able to deal with properties of the seats-votes relationship over the entire range of V is D_6.[17] Once we opt for D_6, however, it makes sense to use D_6', since the normalization used gives us a measure that will range between +1 and -1.

NOTES

1. For empirical data on elections in New Zealand, see, e.g., Johnston, 1976a; Brookes, 1959; Schrodt, 1981; for Canada see, e.g., Qualter, 1968; Spafford, 1970; for Australia see, e.g., Rydon, 1957; and Soper and Rydon, 1958; Schrodt, 1981; for England see, e.g., Johnston, 1977; and Gudgin and Taylor, 1979. For work on seats-votes relationships in the U.S. (including U.S. state legislatures) see, e.g., Dahl, 1956; Tufte, 1973, 1975; Collins, 1978; Backstrom, Robins and Eller, 1978; and Scarrow, 1981, 1982. Recent theoretical work on the topic of seats-votes in single-member-district systems includes March, 1957; Brookes, 1960; Coleman, 1963; Theil, 1969, 1970; Sankoff and Mellos, 1972; Musgrove, 1973; Tufte, 1973; Taylor, 1973; Spafford, 1973; Quandt, 1974; Johnston, 1976a; Engstrom and Wildgen, 1977; Niemi and Deegan, 1978; Gudgin and Taylor, 1979; Grofman, 1981; Owen and Grofman, 1981; O'Loughlin, 1982; Schrodt, 1981.

2. In any actual election system, the value of B_1 (and of B_0) will depend on the spatial distribution of partisan/group support across districts. Very roughly speaking, the more the distribution of partisan/group strength is similar

in all districts, the higher will be B_1. In general, we would expect $B_1 \geq 1$. See Tufte, 1973; Linehan and Schrodt, 1978; Wildgen and Engstrom, 1980; Musgrove, 1973; Niemi and Deegan, 1978; Johnston, 1976a, 1979; and Gudgin and Taylor, 1979; and Schrodt, 1981, for more on this point. Schrodt (1981) has shown that in Equation (2), parameter estimates are sensitive to which party is treated as party I and which as party II and has reestimated some of the equations in Tufte (1973), finding values considerably closer to 3.

3. An S-shaped curve will also be generated under various other plausible assumptions about the underlying functional relationship between seats and votes. See Gudgin and Taylor, 1979:18-19, and Owen and Grofman, 1981. Values of S for specified values of V and B_1 for the seats-votes relationship defined by Equation (1) are given in Table 1 in Grofman (1982b).

Many politicians and lawyers falsely assume that "fair" single-member districting yields proportionality between seats and votes and that absence of proportionality is proof of bias; e.g., "A(n) . . . indicator of racial discrimination in the drawing of congressional district lines is that the percentage of Blacks in a state's congressional districts is usually much less than the percentage of Blacks in the state" (Smith, 1975). In fairness to Smith, he also lists other indicators of discrimination including "the division of substantial minorities of Blacks into several contiguous districts so that they are unable to elect a Black in one or more of those districts" (Smith, 1975:671).

4. Even identical V_+ values in the sets of elections being compared would not alleviate the problem. The relationship specified in Equation (1) [or Equations (2) or (3)] between S_+ and V_+ is nonlinear and thus is not expectation-preserving. It should be apparent that the same problem will manifest itself whatever values of B_1 we pick, although it will be less severe if B_1 is close to 1.

5. We are deliberately using the same symbol, B_1, in Equation (5) as in Equation (2), since in both cases B_1 is taken to be a measure of the <u>swing ratio</u>, even though the value of B_1 estimated from a linear function as in Equation (5) is unlikely to be identical to that obtained by fitting the power function of Equation (1). March (1957) has shown that, for $B_1 = 3$, Equation (1) in the range V = .4 to .6 can be approximated by the straight line S = 2.808 V - .904. We show below that in the range V = .45 to .55, Equation (1) can be well approximated by the straight line S = 3 V - 1.

6. We are indebted to Scott Feld for calling this approximation to our attention (cf. March, 1957; Theil, 1969; also see Feld and Grofman, 1980).

7. In these three cases, the differences between linear and logit estimates are minimal. Tufte (1973:543, n. 4), who looks at several other cases in addition to the three we reported, remarks that the linear and the logit method (and two other methods he discusses) "revealed small differences in most estimates (of D_2) when the bias was less than 5 percent and the correspondence between seats and votes was fairly high (usually the case); otherwise the estimates diverged."

8. As with D_2, this need not be true when bias is large. A glance at Figure 1 reveals that linear and logit models are unlikely to yield similar estimates of bias (defined here as the difference between the point on the estimated seat-vote graph and the corresponding point on the proportionality line) if we look at S values (V values) away from .5.

9. We might also note that if $S_+ = 1/2$, then $D_2 = D_1$; if $V_+ = 1/2$, then $D_3 = D_1$.

10. The relationship between D_2 and D_3 is considerably more complex when each is estimated from the logit model of Equation (2). Combining Equations (15) and (21) we have

$$D_2 = D_3 + \frac{1}{1 + e^{B_0/B_1}} - \frac{e^{B_0}}{1 + e^{B_0}} ;$$

i.e.,

$$D_2 = D_3 + \frac{e^{-(B_0/B_1)} - e^{B_0}}{\left[1 + e^{-(B_0/B_1)}\right]\left[1 + e^{B_0}\right]} .$$

I do not find this expression especially enlightening.

11. Even when the regression estimate obtained by substituting the logit estimate into Equation (28) does not correspond perfectly to the best linear fit, it is likely to give a regression line nearly as good (in terms of R^2).

12. A measure of bias that is a variant of D_4 has been offered by Brookes (1959). We shall not, however, discuss this measure since it adds little or nothing new.

13. If we neglect differences in constituency size and constituency turnout, we are, in effect, looking at what happens when party I's aggregate vote share goes down one percentage point, with the decrease uniform across districts.

14. It might appear that it ought to be harder for party II to gain strength in a district in which it was already strong than in one where it was weak. Except for extreme cases (e.g., districts that are nearly unanimous for a given party), the available statistical evidence seems to support the notion of a swing across districts based on changes in "percentage points and not percentages." According to Tufte (1973:545) "percentages swings are relatively independent of the starting point and are therefore best assessed in terms of untransformed percentages differences." This has been called the "paradox of swing." (See Butler, 1953, for a full discussion of this point; see also Scarrow, 1981, and Taylor and Johnston, 1979, especially Chapter 3.)

15. The pro-Democratic bias appears to be decreasing with increasing V in the 1974 Assembly election, but the effect is slight.

16. We have not attempted to identify the source of bias. Roughly speaking, bias arises when the mean value of overall party strength does not coincide with the median value of party strength across districts (see Soper and Rydon, 1958:97; Johnston, 1979:63-67). Such a discrepancy can occur for a number of reasons. Using a linear approach to estimation, a number of geographers (e.g., Brookes, 1959, 1960; Soper and Rydon, 1958; Gudgin and Taylor, 1979; Taylor and Johnston, 1979) have looked at how D_2 (or D_3) might be decomposed into components reflecting (a) inequality in the number of voters in the seats won by each of the parties (which in turn can be divided into inequality caused by unequal district size and inequality caused by differential turnout of partisan supporters); (b) differential geographic concentration of partisan support across districts (which in turn can be divided into "natural" differential concentration and that aggravated by the way in which the district lines have been drawn: i.e., intentional or unintentional gerrymandering); and (c) the differential impact of minor parties and the distribution of their vote strength. We shall not, however, pursue these issues further here.

Unfortunately, the work on the political geography of electoral relationships done by geographers (primarily British ones) is not familiar to most American political scientists. This work is of very high methodological sophistication and deserves to be far better known. We would especially like to call to the attention of American political scientists Gudgin and Taylor (1979), Johnston (1979), and Taylor and Johnston (1979).

17. A limitation of measures that look only at V = .5 is that they are plausible only if elections are fully competitive, with outcomes consistently near an equal vote division.

REFERENCES

Backstrom, Charles, Leonard Robins, and Scott Eller.
1978 "Issues in Gerrymandering: An Exploratory Measure of Partisan Gerrymandering Applied in Minnesota." Minnesota Law Review 62:1121-1159.

Brookes, R. H.
1959 "Electoral Distortion in New Zealand." Australian Journal of Politics and History 5: 218-233.

1960 "The Analysis of Distorted Representation in Two Party, Single Member Elections." Political Science 12:158-167.

Butler, D. E.
1951 "Appendix." In H. G. Nicholas, The British General Election of 1950. London: Macmillan: 306-333.

1953 The Electoral System in Britain 1918-1951. London: Oxford University Press.

Cole, Leonard A.
1974 "Electing Blacks to Municipal Office: Structural and Social Determinants." Urban Affairs Quarterly 10:17-39.

Coleman, James S.
1963 Introduction to Mathematical Sociology. New York: Free Press.

Collins, W. P.
1978 "The Georgia County-Unit System." Paper delivered at the Annual Meeting of the Public Choice Society.

Dahl, Robert A.
1956 A Preface to Democratic Theory. Chicago: University of Chicago Press.

Engstrom, Richard L. and John K. Wildgen.
1977 "Pruning Thorns from the Thicket: An Empirical Test of the Existence of Racial Gerrymandering." Legislative Studies Quarterly 2:465-479.

Feld, Scott and Bernard N. Grofman.
1980 "Conflict of Interest between Faculty, Students and Administrators: Consequences of the Class Size Paradox." In Gordon Tullock, ed., Frontiers of Economics 3.

Feller, Williams.
1957 Introduction to Probability and Statistics. New York: Wiley.

Grofman, Bernard.
1975 "A Review of Macro-Election Systems." In Rudolf Wildenmann, ed., German Political Yearbook (Sozialwissenschaftliches Jahrbuch fur Politik), Vol. 4. Munich, Germany: Verlag: 303-352.

1981 "Ward vs. At-Large Elections: A Critique of 21 Recent Empirical Studies of the Effect of Election Structures on Minority Representation in U.S. Local Elections." Unpublished manuscript.

1982a "Alternatives to Single-Member District Plurality Elections." In B. Grofman, A. Lijphard, R. McKay and H. Scarrow, eds., Representation and Redistricting Issues. Lexington, Massachusetts: Lexington Books.

1982b "For Single-Member Districts Random Is Not Equal." In B. Grofman, A. Lijphart, R. McKay and H. Scarrow, eds., Representation and Redistricting Issues. Lexington, Massachusetts: Lexington Books.

Gudgin, Graham and Peter J. Taylor.
1974 "Electoral Bias and the Distribution of Party Voters." Transactions, Institute of British Geographers 63:53-74.

1979 Seats, Votes and the Spatial Organization of Elections. London: Pion.

Johnston, Ronald J.
1976a "Spatial Structure, Plurality Systems and Electoral Bias." Canadian Geographer 8:310-325.

1976b "Parliamentary Seat Redistribution: More Opinions of the Theme." Area 8:30-34.

1977 "The Electoral Geography of an Election Campaign." Scottish Geographical Magazine 93:98-108.

1979 Political, Electoral, and Spatial Systems: An Essay in Political Geography. Oxford: Clarendon Press.

Kendall, M. G. and A. Stuart.
1950 "The Law of Cubic Proportions in Election Results." British Journal of Sociology 1:183.

Lijphart, A. and R. W. Gibberd.
1977 "Thresholds and Payoffs in List Systems on Proportional Representation." European Journal of Political Research 5:219-244.

Linehan, William J. and Philip A. Schrodt.
1978 "A New Test of the Cube Law." Political Methodology 4:353-367.

MacManus, Susan A.
1978 "City Council Election Procedures and Minority Representation: Are They Related?" Social Science Quarterly 59:153-161.

March, James.
1957-59 "Party Legislative Representation as a Function of Election Results." Public Opinion Quarterly 21:521-542.

Musgrove, Philip.
1973, 1977 The General Theory of Gerrymandering. Sage Professional Papers in American Politics 3:04-034. Beverly Hills: Sage Publications.

Niemi, Richard and John Deegan, Jr.
1978 "Competition, Responsiveness and the Swing Ratio." American Political Science Review 72:1304-1323.

O'Loughlin, J.
1982 "The Identification and Evaluation of Racial Gerrymandering." Annals, Association of American Geographers 72.

Owen, Guillermo and Bernard Grofman.
1981 "Collective Representation and the Seats-Votes Swing Relationship." Unpublished manuscript.

1984 "Optimal Partisan Gerrymandering." Political Geography Quarterly, forthcoming.

Qualter, T. H.
1968 "Seats and Votes: An Application of the Cube Law to the Canadian Electoral System." Canadian Journal of Political Science 1

Quandt, Richard.
1974 "A Stochastic Model of Elections in Two-Party Systems." Journal of the American Statistical Association:315-324.

Rabinowitz, F. and E. K. Hamilton.
1980 "Alternative Electoral Structures and Responsiveness to Minorities." National Civic Review 69,7:371-385, 401.

Rae, Douglas.
1981 "Two Contradictory Ideas of (Political) Equality." Ethics 91:451-456.

Robinson, Theodore P. and Thomas R. Dye.
1978 "Reformism and Black Representation on City Councils." Social Science Quarterly 59:133-141.

Rydon, J.
1957 "The Relation of Votes to Seats in Elections for the Australian House of Representatives, 1949-54." Political Science 9:49-61.

Sankoff, David and Koulla Mellos.
1972 "The Swing Ratio and Game Theory." American Political Science Review 66:551-554.

1973 "La Regionalisation Electoral et l'Amplification des Proportions." Canadian Journal of Political Science 6:380-398.

Scarrow, Howard.
1981 "Partisan Gerrymandering--Insidious or Benevolent? Gaffney v. Cummings and Its Aftermath." Presented at the Annual Meeting of the Midwest Political Science Association, Cincinnati.

1982 "The Impact of Reapportionment on Party Representations in the State of New York." In B. Grofman, A. Lijphart, R. McKay and H. Scarrow, eds., Representation and Redistricting Issues. Lexington, Massachusetts: Lexington Books.

Schofield, Norman.
1982 "The Relationship Between Voting and Party Strength in an Electronic System." Prepared for delivery at the World Congress of the International Political Science Association, Rio de Janeiro.

Schrodt, Philip A.
1981 "A Statistical Study of the Cube Law in Five Electoral Systems." Political Methodology 7:31-54.

Silva, Ruth C.
1964 "Compared Values of the Single and the Multi-Member Legislative District." Western Political Quarterly 17:504-516.

Smith, George Bundy.
1975 "The Failure of Reapportionment: The Effect of Reapportionment on the Election of Blacks to Legislative Bodies." Harvard Law Journal 18:639-684.

Soper, C. S. and Joan Rydon.
1958 "Underrepresentation and Electoral Prediction." Australian Journal of Politics and History 4:94-106.

Spafford, Duff.
1970 "The Electoral System of Canada." American Political Science Review 64:168-176.

1973 "Seats and Votes in Plurality Elections When More Than Two Parties Contend." Presented at the Annual Meeting of the Public Choice Society.

Taagepera, Rein.
1973 "Seats and Votes: A Generalization of the Cube Law of Elections." Social Science Research 2: 257-275.

Taagepera, Rein and J. L. Ray.
1977 "A Generalized Index of Concentration." Sociological Methods and Research 5:367-384.

Taebel, Delbert.
1978 "Minority Representation on City Councils: The Impact of Electoral Structure on Blacks and Hispanics." Social Science Quarterly 59:142-152.

Taylor, Peter J.
1973 "A New Shape Measure for Evaluating Electoral District Patterns." American Political Science Review 67:947-950.

Taylor, Peter J. and R. J. Johnston.
1978 "Population Distributions and Political Power in the European Parliament." Regional Studies 12:61-68.

1979 Geography of Elections. Harmondsworth: Penguin Books.

Theil, H.
1969 "The Desire for Political Entropy." American Political Science Review 63:521-525.

1970 "The Cube Law Revisited." Journal of the American Statistical Association 65:1213.

Tufte, Edward R.
1973 "The Relationship Between Seats and Votes in Two-Party Systems." American Political Science Review 67:540-547.

1975 "Determinants of Midterm Congressional Elections." American Political Science Review 69: 812-826.

Uslaner, Eric M. and Ronald E. Weber.
1979 "Policy Congruence in the American States: Descriptive Representation Versus Electoral Accountability." Prepared for delivery at the 1979 Annual Meeting of the Midwest Science Association, Chicago.

Wildgen, John K. and Richard L. Engstrom.
1980 "Spatial Distribution of Partisan Support and the Seats/Votes Relationship." Legislative Studies Quarterly 5:423-435.