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Modeling large electorates with Fourier series, with applications to Nash equilibria in proximity and directional models of spatial competition

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Abstract. In this paper we introduce harmonic analysis (Fourier series) as a tool for characterizing the existence of Nash equilibria in two-dimensional spatial majority rule voting games with large electorates. We apply our methods both to traditional proximity models and to directional models. In the latter voters exhibit preferences over directions rather than over alternatives, per se. A directional equilibrium can be characterized as a Condorcet direction, in analogy to the Condorcet (majority) winner in the usual voting models, i.e., a direction which is preferred by a majority to (or at least is not beaten by) any other direction. We provide a parallel treatment of the total median condition for equilibrium under proximity voting and equilibrium conditions for directional voting that shows that the former result is in terms of a strict equality (a knife-edge result very unlikely to hold) while the latter is in terms of an inequality which is relatively easy to satisfy. For the Matthews [3] directional model and a variant of the Rabinowitz and Macdonald [7] directional model, we present a sufficiency condition for the existence of a Condorcet directional vector in terms of the odd-numbered components of the Fourier series representing the density distribution of the voter points. We interpret our theoretical results by looking at real-world voter distributions and direction fields among voter points derived from U.S. and Norwegian survey data.

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1. Introduction

It is well known that, in two (or more) dimensions, equilibrium under the standard proximity model occurs only if all median lines (hyperplanes) pass through a single point (Davis et al. [2]). Such as point, called a *total median*, occurs only under very restrictive symmetry conditions on the set of voter ideal points. In particular, Plott [6] shows that an equilibrium (total median), **N**, occurs in an electorate consisting of a (odd) finite number of voters if and only if one voter is located at **N** and all other voters come in pairs diametrically opposed to **N**. For a continuous distribution representing a large electorate, McKelvey et al. [4] show that a condition weaker than radial symmetry of the probability density is necessary and sufficient for an equilibrium, namely that the measure of the voters in every cone originating from **N** is equal to the measure of the negative cone. We show that in two dimensions the existence of a total median can be specified by a simple condition on the Fourier coefficients of a function associated with the electorate.

Directional models have been developed by Matthews [3] and Rabinowitz and Macdonald [7]. Voters and candidates are associated with directions in which they wish policy to move from a neutral or status quo point. Distance from this point represents intensity in the Rabinowitz and Macdonald formulation but has no meaning in the Matthews model. Competition between candidates focuses on the choice of directions. Nash equilibria are defined by undominated directions, which we call Condorcet directional vectors. We describe such equilibria in terms of Fourier series and show that their existence is much more likely than under the proximity model.

In Sect. 2, we specify the conditions for a total median in terms of Fourier series, carrying forward the analysis as far as possible without introducing directional models. Equilibria under the latter are described in Sect. 3 and related to Fourier series in two-dimensional models. In Sect. 4 we determine and interpret Condorcet directional vectors and direction fields of vectors emanating from potential neutral points throughout the plane for both hypothetical and empirical distributions of voter points.

2. Using Fourier series to represent continuous distributions of voter ideal points in proximity models

2.1. The characteristic vector and support function

In this section we look at continuous distributions of voter ideal points – intended to model large electorates – in order to characterize the distributions which give rise to a total median in a proximity spatial model. We fix a point **N** in multidimensional space. **N** may be a neutral or status quo point but need not be interpreted as such. Each voter or candidate ideal point can be represented by a vector emanating from **N** to that ideal point. By change of coordinates, if necessary, we can take **N** to be the origin. We project each voter, **V**, onto the unit sphere about **N** by replacing **V** by $\mathbf{V}/|\mathbf{V}|$. Let *Pr* be the probability measure of voters on this sphere; we assume throughout that this probability distribution is continuous and represented by a density function, *f*. Any voters located at **N** have probability zero.

Each half-space, whose boundary passes through **N**, is of the form

$$S_{\mathbf{A}} = \{\mathbf{V}: \mathbf{V} \cdot \mathbf{A} \geq 0\} \tag{1}$$

where **A** is a vector perpendicular to the boundary and lying in the half-space. We will refer to **A** as the *characteristic vector* of the half-space and assume that **A** is normalized to be of length 1. Conversely, any vector, **A**, of length one defines a half-space of which it is the characteristic vector. We define the *support function*, g, as a function of the characteristic vector, **A**, specified by

$$\begin{aligned} g(\mathbf{A}) &= Pr[\mathbf{V}: \mathbf{V} \cdot \mathbf{A} \geq 0] \\ &= [\mathbf{V}: \mathbf{V} \in S_{\mathbf{A}}]. \end{aligned} \tag{2}$$

Thus, $g(\mathbf{A})$ represents the proportion of the electorate in the half-space determined by the characteristic vector, **A**. If **A** is normal to the indifference hyperplane between two candidates, then $g(\mathbf{A})$ and $1 - g(\mathbf{A})$ represent the proportions of the electorate supporting each of the two candidates. In this notation, a hyperplane is called a *median*, if for a normal, **A**, to the hyperplane, $g(\mathbf{A}) = 1/2$ and $g(-\mathbf{A}) = 1/2$. A point, **N**, is a *total median* if all hyperplanes through **N** are medians. The following lemma is obvious.

Lemma 1. *For a continuous electorate, a point,* **N**, *is a total median if and only if*

$$g(\mathbf{A}) = 1/2 \quad \text{for } all\ \mathbf{A}.[1] \tag{3}$$

For a *two* dimensional spatial model, the characteristic vector, **A**, of any half-space of support uniquely specifies an angle, γ, in the direction of **A**. For each γ, $-\pi < \gamma \leq \pi$, the value of the support function, $g(\gamma)$, is the proportion of voters in the associated half-plane. With each two-dimensional electorate and center point, **N**, we associate a density function, $f = f(\gamma)$, on the unit circle which represents the density of projections of the electorate on the circle.

Definition. An electorate is said to be *directionally symmetric* with respect to a point, **N**, if the associated density function, f, satisfies the condition

$$f(\gamma + \pi) = f(\gamma)$$

for all γ. Note that a directionally symmetric electorate need not be symmetric in the ordinary sense, because diametrically opposed voters need not be at the same distance from **N**.

2.2. Necessary and sufficient conditions for the existence of a complete median expressed in terms of a Fourier series

2.2.1. Harmonic analysis

In this section we decompose a two-dimensional electorate into "harmonic" components, allowing us to simultaneously address questions of the existence of equilibria in proximity and, later, directional models. In the

[1] For a finite electorate, the condition is $g(\mathbf{A}) \geq 1/2$ for *all* **A**. If **N** is a total median in a finite electorate, the value, $g(\mathbf{A})$, can exceed 1/2 only if voter points lie on the boundary of the halfspace of which **A** is the characteristic vector.

two-dimensional case, we will investigate those probability densities on the unit circle for which such equilibria exist in terms of characteristics of the harmonic components. Each harmonic represents a regular pattern of concentration of voters into a single hump, two diametrically opposed humps, three equally spaced humps, etc. The density function, f, associated with the electorate and the center point, **N**, is expressed as a Fourier series, each term of which represents a harmonic of frequency n of the form $\sin n\gamma$ and $\cos n\gamma$[2].

By definition of the support function, g,

$$g(\gamma) = \int_{\gamma-\pi/2}^{\gamma+\pi/2} f(\theta)d\theta,$$

so that

$$g'(\gamma) = f(\gamma + \pi/2) - f(\gamma - \pi/2), \quad -\pi < \gamma \leq \pi \tag{4}$$

(except possibly for a set of measure zero) where arguments are interpreted modulo 2π. If f is of the form $f(\gamma) = b_0 + b_1 \sin(\gamma)$, then $b_0 = 1/2\pi$, since the integral of f is 1 and $b_1 \leq 1/2\pi$ since f is non-negative. Calculation from (4) shows that $g(\gamma) = 1/2 + 2b_1 \sin(\gamma)$, so that g is also a sine function, and a similar formula holds for $\cos(\gamma)$. More generally, if $f(\gamma) = 1/2\pi + b_n \sin n\gamma$, g becomes

$$g(\gamma) = 1/2 + \frac{2(-1)^k}{n} b_n \sin n\gamma$$

where n is odd of the form $n = 2k + 1$, but

$$g(\gamma) = 1/2$$

when n is even and similarly for $\cos n\gamma$. In particular, *even* sine and cosine terms are not reflected in the support function at all, but *odd* sine and cosine terms are reflected by like terms with modified coefficients.

2.2.2. Characterization of a total median for proximity models

We show below that the existence of a total median can be specified either by a simple condition of the Fourier coefficients of the density function, f, associated with the electorate, or in terms of directional symmetry.

Theorem 1. *The following properties for a point* **N** *and bounded density function, f, in two-dimensional space are equivalent.*

(i) **N** is a total median.
(ii) The Fourier series of the associated density function, f, on the unit circle about **N** as neutral point has only even terms.
(iii) The electorate is directionally symmetric with respect to **N**.

[2] For large electorates, Fourier series simplifies a complicated situation; for small electorates, it makes a simple situation complicated. For an analysis of Condorcet directional vectors for a small, finite electorate, see Merrill and Grofman [5]. The reader is referred to Zygmund [8] for basic results on Fourier series.

Proof. If **N** is considered as the neutral point, with g as support function for f, then by Lemma 1, **N** is a total median if and only if $g(\gamma) = 1/2$ for all γ. If the probability density, f, of the electorate is bounded and has Fourier series

$$f(\gamma) \sim (1/2\pi) + \sum_{n=1}^{\infty} a_n \cos n\gamma + \sum_{n=1}^{\infty} b_n \sin n\gamma \tag{5}$$

then the support function, g, has Fourier series

$$g(\gamma) \sim 1/2 + \sum_{k=0}^{\infty} \frac{2(-1)^k}{n} a_n \cos n\gamma + \sum_{k=0}^{\infty} \frac{2(-1)^k}{n} b_n \sin n\gamma, \tag{6}$$

where $n = 2k + 1$, $k = 0, 1, 2, \ldots$, i.e., the support function, g, can have only *odd* Fourier coefficients. Clearly (6) holds for finite series by calculations similar to those above. Since f is bounded, the Fourier series for g converges uniformly to g, and is thus the Fourier series of its sum. By the uniqueness of Fourier series, $g(\gamma) = 1/2$ for all γ (except for a set of measure zero) precisely when all Fourier coefficients of g, except the constant term, vanish. But this occurs exactly when the coefficients a_n and b_n of the density function, f, vanish, where $n = 2k + 1$, $k = 0, 1, 2, \ldots$, i.e., f has only even Fourier coefficients. This establishes the equivalence of (i) and (ii). To see that (iii) is equivalent to (i) and (ii), one may either check by direct computation that a function on the unit circle is symmetric if and only if it has only even Fourier coefficients, or one may use the definition of the support function, g.[3]

Under reasonable regularity conditions (e.g., piece-wise smoothness), the density function, f, is in fact the sum of its Fourier series. Even harmonics, those terms of the form $\cos n\gamma$ and $\sin n\gamma$ for even n, each have $n/2$ pairs of diametrically opposed humps (modes), i.e., concentrations of voters are paired on either side of the neutral point. Thus the condition in Theorem 1 about f having only even components of its Fourier series if there is to be a total median is directly analogous to Plott's [6] result concerning diametrically opposed individual voters and the result of McKelvey et al. [4] about diametrically opposed cones of voters.

The following corollaries characterize total medians in a proximity model and are not tied to particular points as neutral points.

Corollary 1. *In a two-dimensional spatial model, there exists a total median for the electorate if and only if the Fourier series for f has only even coefficients where f is the density associated with the point defined by the coordinate medians.*

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[3] The equivalence of (i) and (iii) extends to greater than two dimensions, if f is defined to be symmetric if $f(-\mathbf{A}) = f(\mathbf{A})$ for any non-zero vector, **A**, where $f(\mathbf{A})$ is interpreted as the density on the hypersphere centered at the origin in the direction of **A** (without loss of generality, **N** can be taken to be the origin). Then, for any **A**,

$$g(\mathbf{A}) = Pr[\mathbf{V}: \mathbf{V} \cdot \mathbf{A} \geq 0] = Pr[\mathbf{V}: \mathbf{V} \cdot (-\mathbf{A}) \geq 0] = g(-\mathbf{A}).$$

Thus, $g(\mathbf{A}) = 1/2$ since $g(\mathbf{A}) + g(-\mathbf{A}) = 1$, so that **N** is a total median. This argument can be reversed to show that (i) implies (iii). This is a special case of the result of McKelvey et al. [4].

Proof. If a total median exists, it must be the point, **N**, defined by the coordinate medians. By Theorem 1, the density associated with **N** has only even Fourier coefficients. Conversely, if the density associated with a point, **N**, has only even Fourier coefficients, **N** must be a total median. □

Corollary 2. *In a two-dimensional spatial model, there exists a total median for the electorate if and only if it is directionally symmetric with respect to the point defined by the coordinate medians.*

Proof. The proof follows directly from Theorem 1, as in Corollary 1. □

3. Use of Fourier series in directional models

3.1. Directional models

We consider directional models in which we again assume a center or neutral point, **N**, but with all voters and candidates projected onto the (hyper)sphere of unit radius centered at **N**. Unit vectors with endpoints on the unit sphere about **N** represent directions in which a candidate may propose to move policy or a voter may desire change. Thus, each voter or candidate can be represented by a vector of unit length emanating from **N**. By change of coordinates, if necessary, we may take **N** to be the origin.

In the *Matthews* directional model,[4] either all candidates can be assumed to lie on the sphere of radius one, or can be normalized to do so, since only the normalized candidate vector enters the definition of unity. For the *Rabinowitz–Macdonald* (RM) directional model with circle of acceptability[5]

[4] The *Matthews directional model* is defined (Matthews [3]) by the utility function

$$U(\mathbf{V}, \mathbf{C}) = \frac{\mathbf{V} \cdot \mathbf{C}}{|\mathbf{V}||\mathbf{C}|}$$

where **V** and **C** are the vectors of spatial locations of voter and candidate, respectively and $\mathbf{V} \cdot \mathbf{C} = \sum_{i=1}^{n} V_i C_i$ is the scalar product of **V** and **C**, and $|\mathbf{V}|$ and $|\mathbf{C}|$ are the lengths of the vectors **V** and **C**, respectively. If either **V** or **C** is **0**, the utility is defined to be 0. Voter utility reflects only the direction and not the intensity of voter and candidate positions. See also Cohen and Matthews [1].

[5] The *Rabinowitz–Macdonald directional model* is defined (Rabinowitz and Macdonald [7]) via the utility function

$$U(\mathbf{V}, \mathbf{C}) = \mathbf{V} \cdot \mathbf{C} = \sum_{i=1}^{n} V_i C_i.$$

The utility function is further modified so that for a candidate outside of a fixed "circle of acceptability," the utility of any voter for that candidate declines. Thus, any candidate, **A**, lying in the interior (or the exterior) of the circle but not at the neutral point can be dominated by a second candidate, **B**, in the same or opposite direction but on the circle. (In the "unusual" case that **N** is a total median, all positions within or on the circle are equally strong.) Thus, all candidates can be expected to move to the circle of acceptability. Under this interpretation, voters should behave as they would under the Matthews model.

undominated positions lie on the circle of acceptability (which can be taken to be of radius one) or are tied with ones that do. In each of these cases, a voter's choice depends on direction alone, so for equilibrium analysis, each voter's directional vector, too, can be assumed to have length one. Under these interpretations, both these models are subsumed under the assumptions indicated above.

3.2. Condorcet directional vectors

Definition. A vector (candidate) $\mathbf{C}^*$ on the unit circle centered at $\mathbf{N}$ is a *Condorcet directional vector if* for any other vector, $\mathbf{C}$, on the unit circle, the proportion of the voters favoring $\mathbf{C}^*$ over $\mathbf{C}$ is greater than or equal to the proportion favoring $\mathbf{C}$ over $\mathbf{C}^*$.

The condition that a given vector be a Condorcet directional vector is considerably weaker than the condition that $\mathbf{N}$ be a total median, as the following theorem indicates. Theorem 2 is adapted from Matthews [3]; a proof is given in Merrill and Grofman [5].

Theorem 2. *A vector, $\mathbf{C}^*$, is a Condorcet directional vector if and only if*

$$g(\mathbf{A}) \geq 1/2 \quad \text{for all } \mathbf{A} \text{ with } \mathbf{A}\cdot\mathbf{C}^* \geq 0. \tag{7}$$

Note that $\mathbf{A}\cdot\mathbf{C}^* \geq 0$ if and only if $\mathbf{A}$ lies in the half-space of which $\mathbf{C}^*$ is the characteristic vector. Thus, the theorem states that $\mathbf{C}^*$ is a Condorcet directional vector if and only if the proportion of voters in any half-space whose characteristic vector lies within 90 degrees of $\mathbf{C}^*$ is a majority.

In two dimensions, Theorem 2 takes the following form.

Theorem 2′. *In two dimensions, for a continuous distribution, a necessary and sufficient condition for $\mathbf{C}^*$ to be a Condorcet directional vector is*

$$g(\gamma) \geq 1/2 \quad \text{for } \gamma_0 - \pi/2 < \gamma \leq \gamma_0 + \pi/2 \tag{8}$$

where γ_0 is the directional angle of $\mathbf{C}^*$.

Corollary 3. *In a two-dimensional directional model, a Condorcet directional vector exists if and only if there exists an interval of length π for which $g(\gamma) \geq 1/2$.*[6]

Proof. If such an interval exists, let γ_0 be its midpoint. The corresponding vector, $\mathbf{C}^*$, is a Condorcet directional vector by Theorem 2′. □

Generally speaking, the conditions of Theorems 2 and 2′ intuitively mean that the electorate is more concentrated in the general direction of the Condorcet directional vector than in the opposite direction. This idea is made precise in the following theorem.

[6] Note that a Condorcet directional vector, if one exists, need not be unique. For a non-trivial example, if $f(\gamma) = 1/\pi$ on $(\varepsilon, \pi/2)$, 0 on $(\pi/2, \pi - \varepsilon)$, $1/2\pi$ on $(0, \varepsilon)$ and $(\pi - \varepsilon, \pi)$ and $f(-\gamma) = f(\gamma)$, then all $\gamma \in [-\varepsilon, \varepsilon]$ are Condorcet directional vectors. The set of Condorcet directions must, however, form an interval; any two candidates choosing directions in this interval will receive an equal proportion of the voters.

Theorem 3. *Let* **N** *be a fixed neutral point and f the associated probability density of the electorate. If there exists* γ_0 *on a median through* **N** *such that for any arc I with* $\gamma_0 \in I \subseteq [\gamma_0 - \pi/2, \gamma_0 + \pi/2]$,

$$\int_I f(\gamma)d\gamma \geq \int_{\bar{I}} f(\gamma)\,d\gamma \tag{9}$$

holds, when $\bar{I}$ *denotes the arc antipodal to I, then*

$$g(\gamma) \geq 1/2 \quad \textit{for } \gamma_0 - \pi/2 \leq \gamma \leq \gamma_0 + \pi/2$$

so that γ_0 *is a Condorcet directional vector. Conversely, if* γ_0 *is a Condorcet directional vector, then* γ_0 *is on a median by Theorem* 2 *and* (9) *holds.*

Proof. For any γ, $\gamma_0 \leq \gamma \leq \gamma_0 + \pi/2$,

$$g(\gamma) = \int_{\gamma-\pi/2}^{\gamma+\pi/2} f(\tau)d\tau = \int_{\gamma_0}^{\gamma_0+\pi} f(\tau)d\tau - \int_{\gamma+\pi/2}^{\gamma_0+\pi} f(\tau)d\tau + \int_{\gamma-\pi/2}^{\gamma_0} f(\tau)d\tau$$
$$\geq \int_{\gamma_0}^{\gamma_0+\pi} f(\tau)d\tau = 1/2$$

where the inequality follows from the hypothesis. A dual argument holds for $\gamma_0 - \pi/2 \leq \gamma \leq \gamma_0$. This shows that γ_0 is a Condorcet directional vector. Similar arguments in reverse prove the converse.

3.3. Relation of Fourier series to Condorcet directional vectors

In the two-dimensional case, we investigate the nature of the probability densities on the unit circle for which a Condorcet directional vector exists in terms of Fourier series in order to provide results that parallel those given earlier for the total median. We have seen that the conditions for there to be a Condorcet direction are much easier to satisfy than the requirement for a total median given in Theorem 1 that the Fourier series of f have only even terms.

However, before we can link Fourier series to directional equilibria, we need to introduce the amplitude/phase shift form of Fourier series. We write the nth harmonic term of the Fourier series for f in the form $c_n \cos n(\gamma - \phi_n)$ where

$$c_n = \sqrt{a_n^2 + b_n^2} \tag{10}$$

is the amplitude and

$$\phi_n = (1/n)\arctan(b_n/a_n) \tag{11}$$

is the phase shift. This harmonic corresponds to the term

$$\frac{2(-1)^k}{n} c_n \cos n(\gamma - \phi_n)$$

of the support function, g, for odd n, where $n = 2k + 1$, $k = 0, 1, 2, \ldots$.

If the associated density is composed entirely of even terms, the support function, g, is identically equal to 1/2 and a total median exists, as demonstrated in Theorem 1. Insofar as the density includes odd harmonics, g

diverges from the value, 1/2, and certain lines through the center will partition the electorate into unequal parts and the total median is thus destroyed. However, even though the distribution may lack a total median, there may be many possible status quo points from which there will exist Condorcet directional vectors.

The first harmonic, $c_1 \cos(\gamma - \phi_1)$, represents a hump or concentration of voters in the direction ϕ_1 of size proportional to c_1. For example, if $\phi_1 = 0$, then the first harmonic peaks at 0, and the Condorcet directional vector is in the direction of the positive X-axis (see Fig. 1a).

The third harmonic, $c_3 \cos 3(\gamma - \phi_3)$, represents a three-hump concentration of voters, at regular intervals 120° apart. If $\phi_3 = 0$, this harmonic peaks at 0° but also at $\pm$ 120° (see Fig. 1c). The sign of the corresponding harmonic for the support function, g, is, however, flipped reflecting the fact that two-thirds of the voters represented by the third harmonic lie on the other side of the circle than the positive X-axis. As indicated by Fig. 1c, the support function, g, oscillates above and below the value 1/2, without remaining above it for an interval of length, π, so that no Condorcet directional vector exists.

The fifth harmonic represents a five-hump concentration of voters at regular intervals, the seventh harmonic represents a seven-hump concentration, etc., each reflected as a similar or reversed alternation of humps in the support function. As with the third harmonic, g does not remain above 1/2 on an interval of length, π, and again no Condorcet directional vector exists. Thus the first harmonic is the only odd pure harmonic for which a Condorcet directional vector exists.

Typically, of course, a function is a combination of many harmonics. Note that the smoothing process implicit in the integral definition of g which leads to the factor n in the denominator of its Fourier coefficients, causes the harmonics of g to attenuate rapidly, suggesting that in practice only the first few (odd) harmonics of f may be significant in determining majority winners and hence Condorcet directional vectors (see the empirical analysis below). Because of this attenuation and especially if the coefficient of the first harmonic is relatively large, the support function, g, is likely to remain greater than or equal to 1/2 on an interval (of length π) in a direction related to that on which the first harmonic peaks. In particular we can show:

Theorem 4. *A sufficient condition on the probability density, f, that there exist a Condorcet directional vector is that f be of the form*

$$f(\gamma) = [1 + c_1 \cos \gamma + c_n \cos n(\gamma - \phi_n)]/2\pi$$

when n is odd and $c_n \le c_1$. *(We assume that* ϕ_n *is chosen so that* c_n *is non-negative and that* c_1 *and* c_n *are sufficiently restricted so that f is non-negative.)*

Proof. See Appendix. □

We conjecture that the conclusion of the theorem remains true for any density, f, for which

$$c_1 \ge c_3 + c_5 + c_7 + \cdots$$

where, again, the ϕ_n are chosen so that the c_n are non-negative. Intuitively, the theorem suggests that if the amplitude of a single hump dominates the amplitudes of the tripolar component, the quintapolar component, etc, then

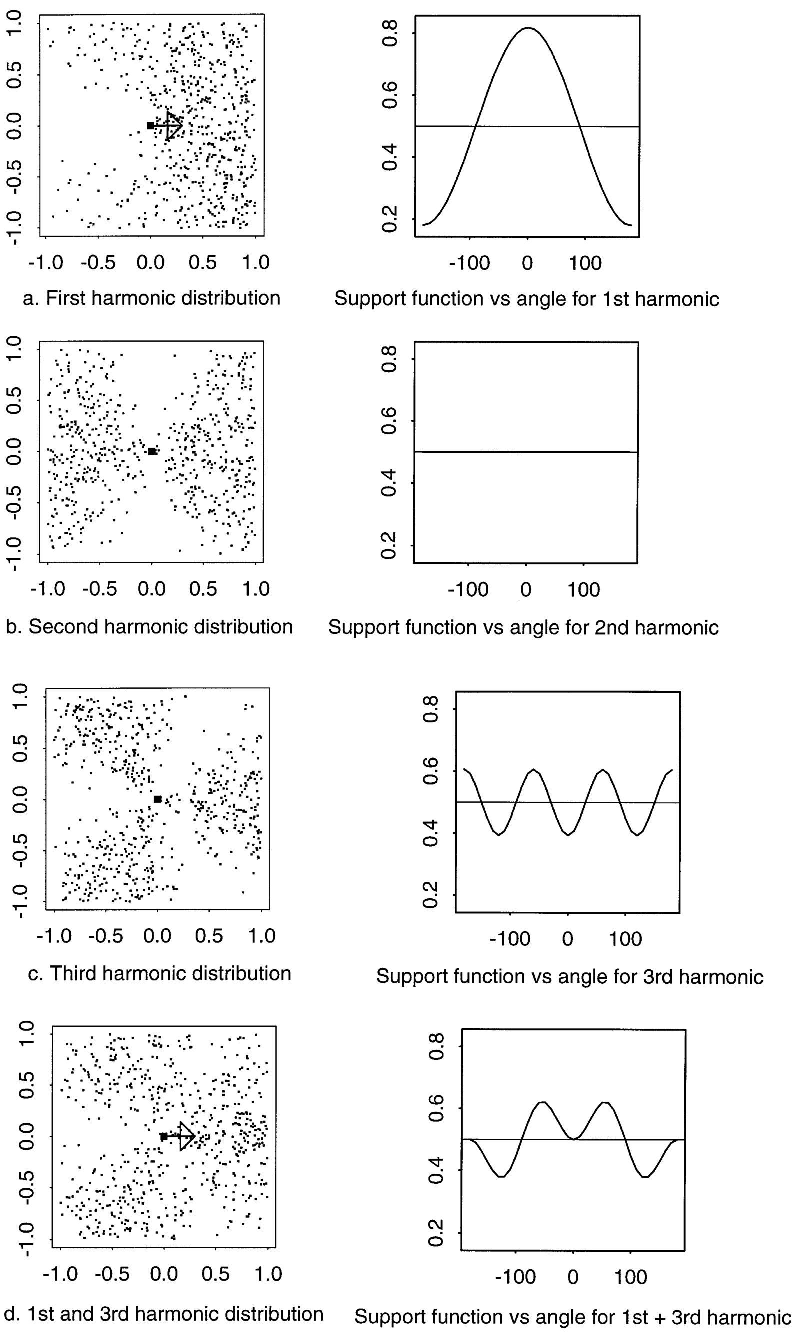

Fig. 1. Voter distributions and support functions for pure harmonics

a Condorcet directional vector exists. Thus, dominance of the unimodal harmonic appears a sufficient condition for existence of a Condorcet directional vector, but not a necessary one, as can be seen from the support function in Fig. 1d.

In summary, we have decomposed the electorate into (1) a symmetric portion (represented by even harmonics) which has no bearing whatever on the existence of either a total median or a Condorcet directional vector and (2) a non-symmetric portion. The latter is further decomposed into odd harmonics each of which is reflected by the support function in similar (or reversed) patterns of humps at equally-space directions and of attenuated amplitude, as n increases.

In our later discussion of empirical results, we will see that the one-hump, three-hump, and five-hump patterns appear generally sufficient to represent real electorates. Moreover, even a preponderantly trimodal component to the distribution may be compatible with an equilibrium in the directional sense.

4. Locating Condorcet directional vectors: evidence from the United States and Norway

4.1. The 1992 American National Election Study

Figs. 2a and 3a give scatter plots of respondent self-placements for the U.S. electorate based on data from the 1992 American National Election Study (NES). Respondents were asked to place themselves on a scale for each of a number of issues. The vertical coordinate for a respondent in two-dimensional space was determined in Fig. 2a from the seven-point scale for the government services question and in Fig. 3a from the four-point scale for the abortion question. The horizontal coordinate for each plot was based on

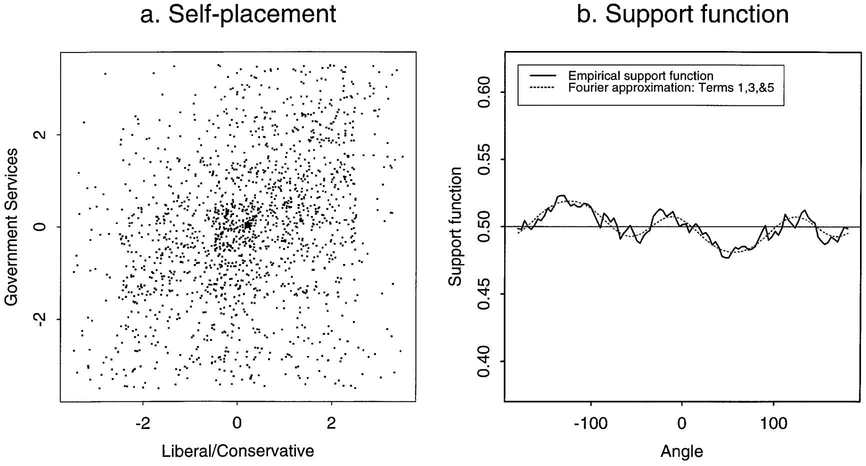

Fig. 2. United States, 1992. Government services versus liberal/conservative

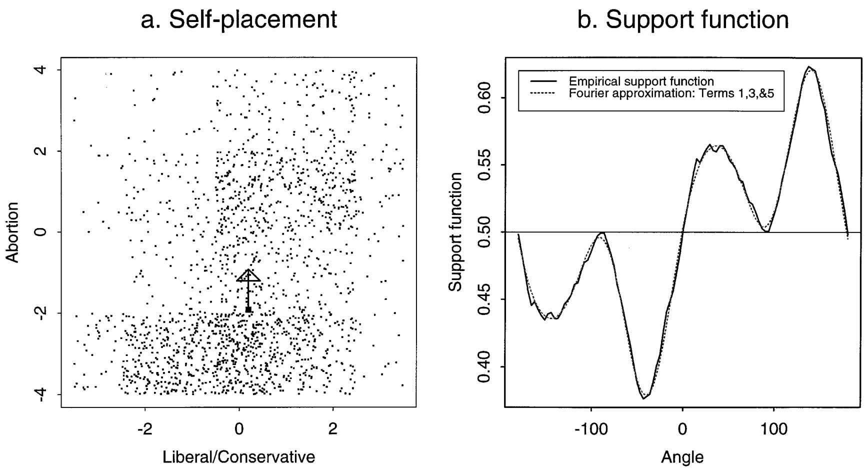

Fig. 3. United States, 1992. Abortion versus liberal/conservative

the seven-point liberal/conservative scale.[7] For each issue, to the left and down represents a liberal position; to the right and up represents a conservative position. On each plot the neutral point, **N**, was determined by the coordinate medians, and is marked by a small square.

Figs. 2b and 3b present plots of the support function, g, for these two spatial models, respectively. Amplitudes and phase angles for the harmonics of these and other distributions are provided in Table 1, along with coordinate medians and Condorcet directional vectors, when they exist.

As indicated by the large value of c_2, the distribution for the government services versus liberal/conservative plot is heavily dominated by the symmetric component with a phase angle of 46°, giving an oblong scatter with axis from the lower left to upper right (see Fig. 2a). In fact, this component of the scatter plot obscures the effects of odd harmonics and the fact that c_3 is much larger than c_1, resulting in no Condorcet directional vector as Fig. 2b shows. The support function is, however, nearly constant, so that the conditions for either a Condorcet directional vector or a total median are only slightly violated.

In the abortion versus liberal/conservative scatter plot the coordinate median for abortion is substantially negative, so that the neutral point is shifted down (see Fig. 3a). The unimodal harmonic has a phase angle of 105°, representing the direction up toward the more distant anti-abortion position. As expected from Theorem 3, this harmonic dominates the tripolar harmonic in its influence on g, despite the latter's larger amplitude, because it is directionally more focused.

[7] To smooth the graininess of these scales, respondent positions were subjected to random scatter, e.g., individuals giving a response of 2 were scattered uniformly between 1.5 and 2.5. Scales where centered at 0 by subtracting 4. The abortion scale was first expanded to one to seven.

Table 1. Amplitudes, phase angles, and Condorcet directional vectors for the Fourier series of electorate densities[a]

	United States 1992		Norway 1989	
	L/C-GovSer	L/C-Abortion	L/R-Alcohol	L/R-Agriculture
Sample size	1659	1776	2018	1864
Coordinate medians				
X-coordinate[b]	0.21	0.20	– 0.04	0.01
Y-coordinate	0.03	– 1.93	– 0.04	– 1.34
Amplitude				
c_1	0.005	0.034	0.025	0.007
c_2	0.066	0.098	0.034	0.058
c_3	0.016	0.086	0.058	0.006
c_4	0.017	0.013	0.022	0.012
c_5	0.004	0.026	0.008	0.024
Phase angle (degrees)[c]				
ϕ_1	– 124	105	– 69	58
ϕ_2	46	73	60	77
ϕ_3	56	– 35	26	– 50
ϕ_4	– 44	– 13	8	38
ϕ_5	– 17	– 6	24	– 17
Condorcet directional vector (degrees)	none	90	– 88	56

[a] For each electorate, the neutral point, **N**, is placed at the coordinate medians.
[b] In each case, the X-coordinate is L/C or L/R. The observed coordinate median for the variable may vary slightly because of pairwise deletion of cases.
[c] The phase angles in the table are the smallest of the n equally spaced possible phase angles.

The support function, g, for the abortion issue remains above 0.5 throughout the interval from 0° to 180°, indicating that the midpoint ($\gamma = 90°$) of this interval represents a Condorcet directional vector emanating from **N** (and is shown in Fig. 3a). This example illustrates that the Condorcet directional vector need not be the point at which the support function assumes its maximum, which for this data is approximately 135°.

For each issue-pair in the U.S. survey (and for the Norwegian survey described below), a truncated support function

$$g_5(\gamma) = 1/2 + 2c_1 \cos(\gamma - \phi_1) - 2\frac{c_3}{3}\cos(\gamma - \phi_3) + 2\frac{c_5}{5}\cos(\gamma - \phi_5)$$

determined by the first three odd Fourier coefficients of the density, f, is plotted (using a dashed line) on the same axes with g. The agreement of these curves suggests that three harmonics may be sufficient to describe the size of majorities in real electorates.[8]

[8] Note that g_5 is constructed directly from the Fourier coefficients of f, and makes no direct use of the values of g, itself.

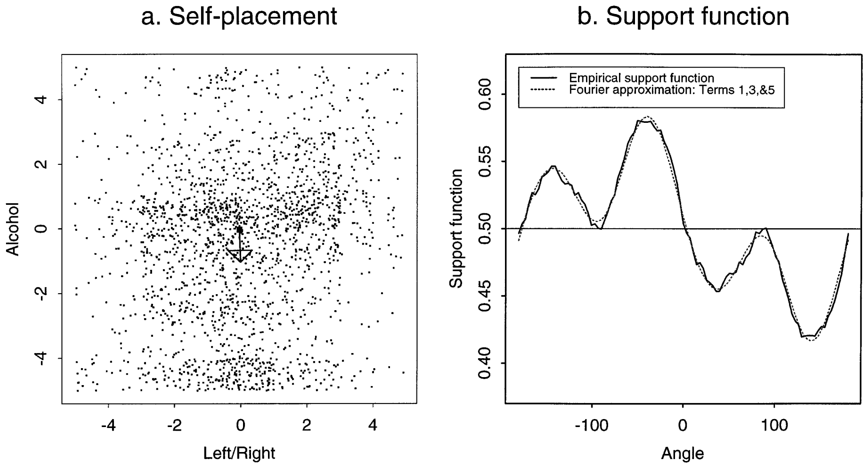

Fig. 4. Norway, 1989. Alcohol versus left/right

4.2. The 1989 Norwegian Electron Study

Data from the 1989 Norwegian Election Study show us what appears to be best characterized as a tripolar electorate, although we deliberately stacked the cards in choosing items from this data set in favor of what we expected would be a tripolar pattern to see whether the Fourier series approach would give us the results that correspond to the intuitions of knowledgeable observers. We chose an issue (alcohol policy) on which there is a constituency which takes an extreme view while maintaining a centrist view on other issues. Traditional (economic) liberals and conservatives (measured on the left/right scale) who take centrist views on alcohol constitute the other two poles of the tripolar distribution (see Fig. 4a).[9] Again, a Condorcet directional vector does exist, in the direction of $-88°$, i.e., directly down toward the extreme anti-alcohol position.

In the alcohol versus left/right scatter plot for Norway (Fig. 4a), the tripolar nature of the distribution is reflected in the high amplitude, c_3, for the third harmonic, with modes centering in directions 26° (ϕ_3), 146° ($\phi_3 + 120°$), and $-94°$ ($\phi_3 - 120°$). The concentrated peak at the bottom of the plot (near the mode at $-94°$) is further reflected in the first harmonic (with plase angle somewhat offset at $-69°$) via a vis the more diffuse scatter just above the center line.

Ceteris paribus, a concentration in the distribution tends to offset a more diffuse scatter in the opposite direction. Because a distant (more extreme) pole

[9] In Figs. 4a and 5a, the horizontal and vertical axes represent self-placement – on a scale of 1 to 10 – on the left/right scale and on the alcohol policy or agricultural policy scale, respectively. Placements are subjected to random scatter, as described above.

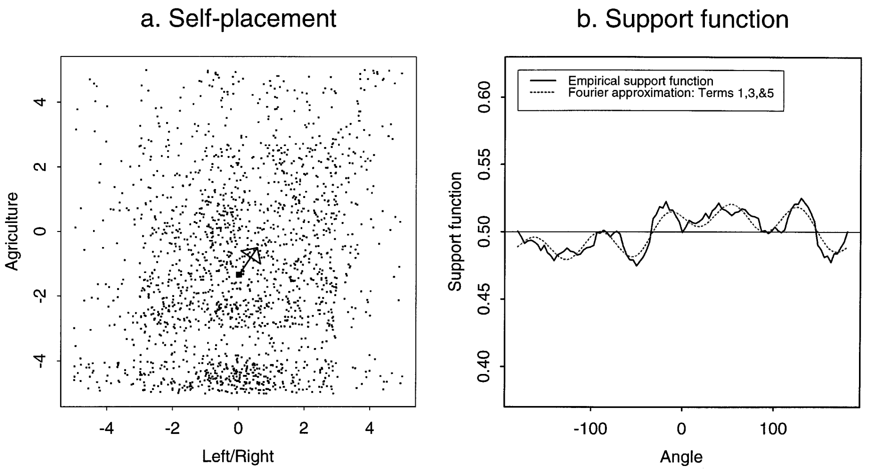

Fig. 5. Norway, 1989. Agriculture versus left/right

is likely to be concentrated directionally (i.e., subtend a smaller central angle), it tends to dominate a nearby pole, and may do so even if the nearby pole involves a larger number of voters, as is the case for the Norwegian alcohol data. This illustrates the tendency that a Condorcet directional vector points in the general direction of a concentration of voters, as made specific in Theorem 3 above.

A different situation arises when alcohol policy is replaced by agricultural policy (see Fig. 5a); here the symmetric second harmonic dominates (see Table 1). A Condorcet directional vector exists at 56°, roughly in the direction of the second harmonic. Plots of the support functions, g, (presented in Figs. 4b and 5b) for alcohol and agricultural policy satisfy the condition for a Condorcet directional vector when **N** lies at the coordinate medians.

4.3. Condorcet direction fields

Plotting the Condorcet directional vectors from each node of a lattice of possible neutral points provides a direction field with a residual blank region representing positions from which no Condorcet directional vector exists, which we call the *Condorcet vacuum.* Fig. 6 presents blow-ups of the central region of such direction fields for two hypothetical voter distributions centered at the coordinate medians, the first in which the probability distribution of voters is described by a pure first harmonic, $\cos(\gamma)$, and the second by a pure third harmonic $\cos(3\gamma)$.[10] In both cases all arrows point inward toward a central area. But the first harmonic satisfies Corollary 3, i.e., the support

[10] In determining the number of times that $g(\gamma)$ crossed 0.5, a tolerance of 0.005 was permitted to avoid counting inconsequential crossings including those due to numerical roundoff.

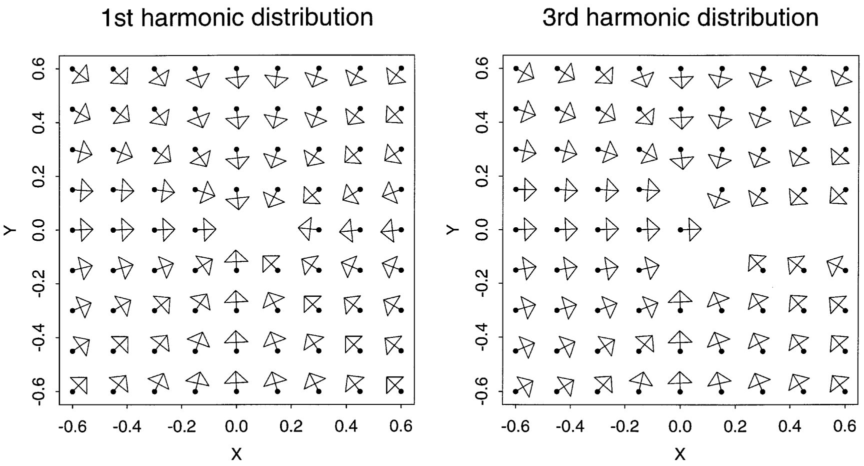

Fig. 6. Condorcet direction fields for pure harmonic distributions

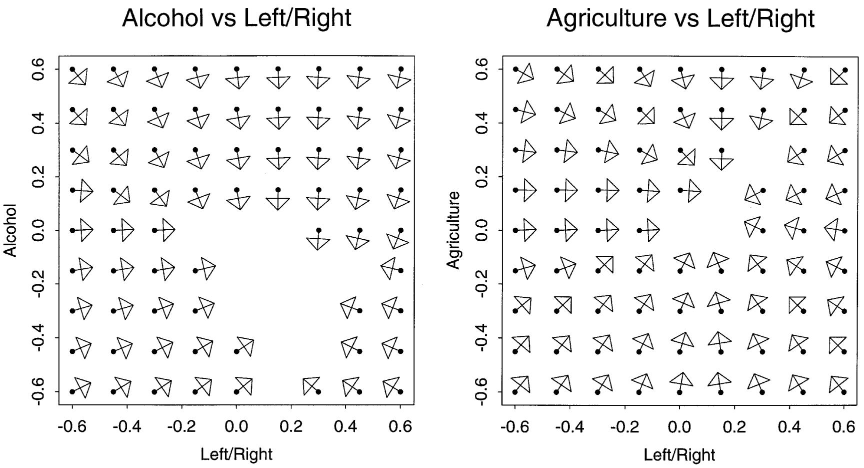

Fig. 7. Condorcet direction fields for Norway, 1989

function $g(\gamma) = 1/2 + 2\cos(\gamma)$ exceeds 1/2 on an interval of length π. Thus the Condorcet vacuum is empty (a small hole appears in the diagram due to statistical noise in the simulated sample). The third harmonic, however, yields a larger, pear-shaped region of Condorcet vacuum since the distribution is tripolar[11].

Direction fields for the 1989 Norwegian Election Study are plotted in Fig. 7 for alcohol and agricultural policy against left/right placement. The alcohol

[11] This conforms with the geometric shape of the Condorcet vacuum for a tripolar distribution obtained for finite electorates by Merrill and Grofman [5].

policy plot yields the pear-shaped Condorcet vacuum, characteristic of a tripolar voter distribution and indicative of the fact that three concentrations of voters can lead to directional as well as traditional spatial instability. The agricultural policy plot illustrates directional vectors pointing to a common center with only a small Condorcet vacuum, reflecting the strength of the second harmonic. In either case, the Condorcet vacuums depicted in these zoomed-in plots are small relative to that of the voter distribution.

5. Discussion

As a technique for decomposing the angular variations in a spatially-represented electorate, harmonic analysis lends itself well to directional modeling. We have associated a probability density and a support function with a two-dimensional spatial model and seen that the Fourier series of the latter is closely related to that of the former and depends only on its odd harmonics. This allows us to relate the existence and location of undominated (Condorcet) directional vectors with the pattern of humps (voter concentrations) in the electorate. In particular we have seen that candidates may be drawn to move toward a small, central region within which competition may be indeterminant. In approaching this region, small focused concentrations of electorate may overshadow larger, more diffuse conglomerations.

Appendix

Proof of Theorem 4. It will be convenient to use the sine form of the amplitude/phase-angle formulas. To prove Theorem 4, it suffices to show that

$$(1/2)[g(\gamma) - 1/2] = c_1 \sin\gamma + (-1)^k \frac{c_n}{n} \sin n(\gamma - \phi_n)$$

has exactly two zeros in the interval $(-\pi, \pi]$ where $g(\gamma)$ is the corresponding support function. Without loss of generality, set $c_1 = 1$. Write $p(\gamma) = \sin\gamma$ and

$$q(\gamma) = (-1)^k \frac{c_n}{n} \sin n(\gamma - \phi_n).$$

For convenience, if k is odd, we absorb the negative sign by replacing ϕ_n by $\phi_n - \pi$. Since $|q(\gamma)| \leq 1/n$ roots of the function

$$r(\gamma) = p(\gamma) + q(\gamma)$$

can only occur when $|p(\gamma)| \leq 1/n$, i.e., when γ is within $\delta = \arcsin(1/n)$ of 0 or π. By symmetry, it suffices to show there is precisely one zero in the δ-neighborhood of 0 and to assume that $\phi_n \geq 0$. If $\phi_n = 0$, then 0 is the only zero of $r(\gamma)$ in the neighborhood. Otherwise, if $\phi_n < \gamma < \delta$, then $q(\gamma) \geq 0$ and

$$q(\gamma) = \frac{c_n}{n} \sin n(\gamma - \phi_n) < \sin(\gamma - \phi_n) \leq \sin\gamma = p(\gamma),$$

hence $r(\gamma)$ has no root in this range. If $0 < \gamma \leq \phi_n$, then $p(\gamma) > 0$ but $q(\gamma) \leq 0$, so no root occurs.

Hence, any roots of $r(\gamma)$ near 0 must not exceed 0. One such root must occur by the intermediate value theorem. If there were two such roots, say, $\gamma_1 < \gamma_2 < 0$, by Rolle's theorem, $r'(\gamma_0) = 0$ for some $\gamma_0, \gamma_1 < \gamma_0 < \gamma_2$. But

$$q'(\gamma_0) = c_n \cos n(\gamma_0 - \phi_0) < c_n \cos n\gamma_0$$
$$\leq \cos n\gamma_0 \leq \cos \gamma_0 = p'(\gamma_0)$$

so that $r'(\gamma_0) > 0$. This contradiction shows that only one root exists near zero. □

References

1. Cohen L, Matthews SA (1980) Constrained Plott equilibria, directional equilibria and global cycling sets. Rev Econ Studies 47: 975–986
2. Davis O, DeGroot M, Hinich MJ (1972) Social preference orderings and majority rule. Econometrica 40: 147–157
3. Matthews SA (1979) A simple direction model of electoral competition. Public Choice 34: 141–156
4. McKelvey RD, Ordeshook P, Ungar P (1980) Conditions for voting equilibria in continuous voter distributions. S.I.A.M. J Appl Math 39: 161–168
5. Merrill S, Grofman, B (1996) Nash equilibrium strategies in directional models of two-candidate spatial competition. Paper presented at the annual meeting of the Public Choice Society, April 12–14, Houston, TX
6. Plott CR (1967) A notion of equilibrium and its possibility under majority rule. Am Econ Rev 57: 787–806
7. Rabinowitz G, Macdonald SE (1989) A directional theory of issue voting. Am Polit Science Rev 83: 93–121
8. Zygmund A (1959) Trigonometric series (2nd ed.). Cambridge University Press, Cambridge