Finagle's Law and the Finagle Point, a New Solution Concept for Two-Candidate Competition in Spatial Voting Games Without a Core

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Finagle's Law and the Finagle Point, a New Solution Concept for Two-Candidate Competition in Spatial Voting Games without a Core*

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We investigate the geometry underlying Finagle's Law, which states, "No matter what happens, you can come out ahead if you just know how to finagle," and we introduce a new solution concept for two-candidate sequential spatial voting games, the "finagle point." The finagle radius is the radius of a circle such that if a candidate locates at its center, some alternative in the circle can beat any alternative in the space. The finagle point is the point with minimum finagle radius—from it a candidate can, with only minuscule changes in his initial policy location, find a response to any challenger that will defeat that challenger. For each possible candidate location, we provide a geometric construction which gives an outer bound for its "finagle radius," a measure of the attractiveness of that location to a finagling politician. For three-voter games without a core, we provide an analytic solution for the point with minimal finagle radius that guarantees that the maximum finagle needed to defeat an opponent will, in general, be quite small relative to the Pareto set. We also show how the construction used to generate the finagle point in the three-voter case can be extended to the *n*-voter case. The basic idea underlying the finagle point is that it is unnecessary (and indeed usually impossible) to find a position that will defeat all challengers, but it is possible to find a position that is virtually invulnerable to challenge, since any position that beats it can be countered by shifting to a position very close to the original location that *will* defeat the challenger. Moreover, even if not at the point with minimal finagle radius, in searching for positions to respond optimally to the location of the other candidate, a candidate will in general not find it necessary to move far from an initial point located near the finagle point, since points located near the finagle point will also have a small finagle radius.

1. Introduction

In the world of social choice theory, deep pessimism has prevailed about the feasibility of stable majority decision processes; impossibility results and disequilibrium results abound (see, e.g., the review in Riker, 1982). Nonetheless, in the real world, Timothy Aloysius Finagle (to name but one prime example) has been elected time and time again, and the policies he has advocated throughout his electoral career have varied relatively little from election to election. Why the divorce between social choice theory and empirical reality? Why so much stability? (Tullock, 1981). A variety of answers have been given to this question (see review in Grofman and Uhlaner, 1985), including certain possibility results for probabilistic voting (Coughlin, 1984); emphasis on the importance of candidate uncertainty about voter preferences (Glazer, Grofman, and Owen, 1985); a focus on properties of the minmax set (Kramer, 1977); present constraints on a candidate's location caused by past publicly held positions (Samuelson, 1984); quasi-ideological constraints on the issue framing of alternatives (Niemi, 1983); and, of course, the ability of incumbents to send out free mail and speed Social Security checks (Mayhew, 1977; Fiorina, 1977).

We propose a different answer, to wit: Incumbents take advantage of Finagle's Law.

FINAGLE'S LAW: No matter what happens, you can always come out ahead if you just know how to finagle.

In other words, incumbent politicians are good finaglers. There is, of course, a process of natural selection: individuals who are *not* good finaglers are unlikely to remain politicians, much less remain incumbents.

What does it mean for a politician to be a good finagler? We shall provide an answer only for the important case of two-candidate sequential competition in an *n*-dimensional majority rule spatial voting game without a core. Other extensions will, no doubt, be obvious. In a nutshell, avoiding undue mathematical formalism, the answer is: To be a good finagler, you must find a *point with a small finagle radius* and locate yourself there.

Absent extremely restrictive symmetry conditions, it is well known (McKelvey, 1976, 1979; Schofield, 1978) that in majority rule voting games any point in the space can be made a potential outcome by appropriate choice of an agenda. Such voting games customarily lack a Condorcet winner, that is, a point that is majority preferred to each and every other point in the space. Nonetheless, we claim that we can find a point, which we shall call the *finagle point*, which has the property that every point in the space is defeated by some point very near the finagle point, and no point with a smaller finagle radius can be found. Suppose a candidate such as Finagle lo-

cates at such a finagle point. While there are, of course, points that a challenger could pick that would beat the point Finagle has chosen, for *every* response by a challenger that beats the finagle point, Finagle can find a *nearby* point that will return him to victory.[1]

Of course, to a counter by Finagle to the challenger, the challenger may in turn, have a counterresponse, but the process must end somewhere, and since Finagle never needs to go very far from his initial starting point, he can maintain credibility with the voters while still being assured of always having a winning riposte to any position the challenger might adopt.

Before we can specify how to find the point that will minimize the maximum finagle needed for victory, a number of definitions are useful.

> DEFINITION 1: A set of points is a *Von Neumann-Morgenstern externally stable solution set* (V-Mess) for a majority voting game if, for any point outside the set, there exists at least one point in the set that is majority preferred to the point outside the set (see, e.g., Owen, 1983).[2]

We shall be interested in V-Messes that are circles,[3] since the radius of such a circle tells us the length of the maximum possible move required from a starting point at the center of the V-Mess to find a point that can defeat any given point in the space.

> DEFINITION 2: In the spatial context the point in any V-Mess that minimizes the distance to the maximally distant point in the V-Mess we shall refer to as the *center of the V-Mess*.

> DEFINITION 3: In the spatial context the distance between the center of a V-Mess and the point in the V-Mess from which it is maximally distant we shall refer to as the *radius of the V-Mess*.

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[1]FINAGLE'S COROLLARY: Victory is always within a finagler's reach, if his reach is long enough and he knows which way to stretch. Cf. Ronald Reagan's celebrated ability to sidestep ever so slightly from his most extreme statements in such a way as to leave himself invulnerable to challenge on that issue.

[2]A V-Mess is said to be *minimal* if it contains no proper subset which is also a V-Mess. A V-Mess that has the property that no point *inside* the set is majority preferred to any other point in the set is said to be a *Von Neumann-Morgenstern internally stable solution set*. Such a set is what most of us think of simply as a *V-M solution*. We shall, however, not limit ourselves to V-Messes that are internally stable.

In graph theory a V-M internally stable set is the analogue to a one-basis, while a V-M externally stable set is the analogue to a one-cover. V-M solutions are always minimal V-Messes.

[3]For simplicity our illustrations will be in two dimensions. In higher dimensions we would look at V-Messes that are spheres.

DEFINITION 4: The *finagle radius* of a point is the radius of the minimal circular V-Mess centered around it.

DEFINITION 5: The point with minimal finagle radius is the *finagle point*.

Clearly, the candidates would like to find points in the space whose finagle radius is small, since this will permit them to respond successfully to any challenge with at most a small finagle. Clearly, too, there are costs to a candidate in attempting to shift his or her location in the policy space. The greater the shift the less likely it will be credible. The greater the shift the more effort it may take for candidates to sell it to the voters as reflecting a genuine change. Moreover, too great a shift may lead to accusations that a candidate lacks principles or is incompetent or wishy-washy.

2. Some Specific Finagle Games

We believe that there are many different types of situations in which candidates would find it useful to be located at the point that is defensible with the least movement, the finagle point. The previous section has attempted to create some intuition for the attractiveness of the finagle point. In this section we show that one might also wish to understand the finagle point (and its surrounding area) as the solution to particular election games. We provide three examples of situations that would lead to the finagle point (or its vicinity); these are intended as illustrations of the importance of the finagle point for a wide variety of situations.

EXAMPLE 1: Incumbents in election systems have different advantages and constraints from challengers. In particular, incumbents have known spatial locations at the onset of a political campaign, while relatively unknown challengers often have the freedom to locate anywhere in the space that they find strategically advantageous. While a campaign may involve successive readjustments of positions of each of the candidates in response to the other, incumbents often have the compensating advantage of being able to "interpret" (i.e., actually modify) their positions last. Thus, each round of the election game consists of the following: the incumbent's original position, followed by the challenger's chosen location, followed by successive readjustments of the candidates, concluded by the incumbent's last move, followed by the election. If the incumbent wins, he maintains his incumbent position to confront the next challenger; if the challenger wins, his position becomes his incumbent position from which to confront the next challenger.

If the incumbent had total freedom to readjust ("interpret") his position, his ability to go last before the election would always permit him to find some position that would be majority preferred to a particular position taken by a challenger. However, if his credibility with the voters does not

allow spatial relocations beyond some small adjustment, then he may not be able to find any position within those constraints that beat the challenger. The logic of the finagle point allows us to analyze this game and its "solution," if any.

Suppose that the electorate has a *finagle tolerance* (the maximum distance that an incumbent can credibly move). If the finagle tolerance is less than the finagle radius of the finagle point, then there will be no equilibrium; that is, there is no point that an incumbent can occupy that will always be defensible within the constraints of credibility. However, if the electorate's finagle tolerance is greater than the finagle radius of the finagle point, then the finagle point is a position that is always defensible; consequently, whenever a candidate arrives at the finagle point, he or she will remain there and win all subsequent elections (unless the voters' preferences change and there is a new finagle point). It should be noted that if the electorate's finagle tolerance is greater than the minimum necessary for stability, then there may be several points whose finagle radii are less than the voters' finagle tolerance and so are defensible, composing a *defensible zone*. In the case of three voters, we are able to show that the defensible zone for this game is always a convex region immediately surrounding the finagle point; we suspect that this may also be true in more complex situations.

We wish to show that, even in a game that allows the possibility that the challenger may be able to move last, the finagle point can still remain an attractive position at which to locate.

EXAMPLE 2: Suppose that once the incumbent and challenger have taken positions, then they alternate in readjusting their positions to counter that of their opponent, each within the finagle tolerance of the voters. If each always has an effective response to the other, then the one who gets to make the last readjustment before the election necessarily wins. With a fixed incumbent, a challenger can find many locations in the space that are completely defensible against the entire finagle tolerance of the incumbent, including some that may require very little finagling at all to beat any point around the incumbent. However, the *counter-finagle point* (the one with least radius that beats all points in the incumbent's finagle tolerance circle) may not be defensible against new challengers in subsequent elections. Since this challenger is concerned not only with winning this election but also in having a chance to win subsequent elections, he is best off locating within the defensible zone. In that way, if he goes last now, he wins and occupies a defensible position for the next elections. If he chose another position, he might have an equal chance of winning now, but he would necessarily lose in the following election. Thus, the anticipation of the subsequent elections leads each challenger to locate within the defensible zone rather than at the counter-finagle point.

It is interesting to consider that challengers who cannot or who will not run again are not subject to this motivation. They may be more likely to take account only of considerations in the present election contest; consequently, they are more likely than future-career oriented candidates to locate outside of the defensible zone, that is, farther from the finagle point.

EXAMPLE 3: It is possible that the voters will not have a specific finagle tolerance, but candidates will somehow be penalized according to how much they are shifting their positions. (E.g., former supporters of the candidate might be alienated in future elections.) Thus, ceteris paribus, to the extent that distance moved can be expected to be related to magnitude of shift in support coalition, candidates who are choosing a position will choose one that requires the shortest movements, the finagle point.

While this motivation encourages challengers to choose the counter-finagle point (because it requires the least adjustment from it to find the point that can beat those points that are around the finagle point), the anticipation of subsequent elections, as discussed in scenario 2, makes it likely that challengers will choose points that are closer in to the finagle point in order to minimize the amount of shifting that will be necessary in subsequent elections. The incumbent, who is fixed in position first, is always at a disadvantage with regard to the penalties for shifting because the challenger can always find a point that requires less shifting to counter the shifts of the incumbent. However, the incumbent may have many other advantages (e.g., being able to move last) that may more than compensate. In any case candidates will make the minimization of required shifts of positions an important consideration in choosing a location, and consequently, there will be a tendency for all candidates to locate at or near the finagle point.

These examples are intended to be illustrative; they represent sequential voting games where the maximum distance that can be moved is strictly constrained by the rules of the game, or where longer moves are merely more costly. In either case these games focus upon limits on spatial distance moved. To the extent that distance in an issue space is meaningful, we believe that voters will perceive and respond to the distance moved by candidates in the types of ways we have indicated.[4]

[4]Other costs and limits may be associated with changes in the coalitional structure, which we would expect to be associated with but not perfectly correlated with spatial distance.

Our general view is that in a multimove or multielection process it is sensible for candidates to concern themselves with how far they will have to move from their initial starting point in order to counter challengers and also to concern themselves with the total distance to and fro that they will have to move over a sequence of responses and counterresponses within or across campaigns. One reason for this is loss of credibility (and votes) by moves that take one too far (or too often) from where one started (since voters' choices are in part determined not just by ideological proximity but also by voter judgments about candidate competence and trustworthiness). A second reason is that candidates may suffer from antagonism from voters whom

3. Finding the Finagle Point

While a great deal of work has been done in the game theory literature on V-Messes and related ideas (e.g., the competitive solution, McKelvey, Ordeshook, and Winer, 1978), little is known about the area of V-Messes in the spatial context. Thus, we shall largely be exploring new ground (see, however, Ferejohn, McKelvey, and Packel, 1984). Borrowing useful geometric construction techniques from other recent papers on spatial modeling (e.g., Ferejohn, McKelvey, and Packel, 1984; Feld, Grofman, and Miller, 1985) and adapting them to our own purposes, in the next section we present a geometric construction that identifies the finagle point and allows us to specify the length of the maximum finagle required to find a position that will beat that of any challenger.

DEFINITION 6: The *yolk* is the circle of minimum radius which intersects all median lines[5] (Ferejohn, McKelvey, and Packel, 1984).

For the special case of three voters, we provide an analytic solution for the finagle point construction, which shows that the maximum finagle radius is never greater than roughly one-third the radius of the yolk and is usually smaller still. Because the yolk will in general be small relative to the Pareto set, our results ensure us that a candidate can find a location from which the maximum finagle required for victory against any possible challenger will be quite small. Interestingly, this location need not coincide with the center of the yolk.

First, we present a basic result about circular V-Messes.

LEMMA 1: Any circle which intersects all median lines is a V-Mess.

PROOF: To see this, draw any circle that intersects all median lines. For any point outside that circle we can drop a line through the center of the circle. The portion of that line within the circle is, of course, a diameter of the circle. The median line that is perpendicular to that diameter must pass through or be tangent to the circle (since the circle intersects all median lines). But then the intersection of the median line with the diameter to which it is perpendicular must lie on or in the circle. But this intersection

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they have "deserted," and these voters may be less willing (or even unwilling in the future) to vote for the candidate who has deserted, even if he shifts back to a position closer to their own. Third, candidates may be unable to convince voters that a large shift in position has in fact taken place. Thus, we expect candidates to be sensitive to the finagle radius of the point at which they locate, even if they do not (or cannot, because of some ideological or past position constraint) pick the point of minimal finagle radius.

[5]For dimensions higher than two, the notion of median lines must be replaced with median hyperplanes and circles replaced with spheres. A median hyperplane is a hyperplane such that at least half of all voter ideal points lie on it or to either side of it.

is majority preferred to the external point. Thus, we can always find some point on or in the circle which will be majority preferred to any given external point.   Q.E.D.

Thus, in particular, the yolk is a V-Mess. This fact was noted in Ferejohn, McKelvey, and Packel (1984).

DEFINITION 7 (Shepsle and Weingast, 1984): *The win set of a point* is the set of points which defeat that point.

DEFINITION 8 (Feld, Grofman, and Miller, 1985): The *half-win set of a point* is the set of points which are obtained by uniformly reducing each ray in the win set of that point by a factor of 1/2.

It is easy to find an upper bound on the finagle radius of any point. Simply generate the win set of that point and then reduce that win set in half by taking the midpoints of the rays from the point to the extreme points of its win set:

LEMMA 2: The circle centered at a point which includes the furthest point on that point's half-win set is a V-Mess.

PROOF: A point on the boundary of the half-win set is majority preferred to all other points on its ray, since it is the median point (voter projection) on that ray (see Figure 1). Thus, the set of all such points (the boundaries of the half-win set) must constitute a V-Mess, since for any point in the space we can find a point on the boundary of that half-win set that will defeat it.   Q.E.D.

In Figure 1 we show such a half-win set of a point and the circle around it centered at the point.[6]

LEMMA 3: The center of the yolk is the point such that the minimum circle enclosing its half-win set has minimum radius.

PROOF: The midpoint of a ray from a point to an extreme point of its win set is the point at which the ray intersects a median line, but that is simply the boundary of the point's half-win set. Since the yolk is the smallest circle that intersects all median lines, it follows that the circle around the center of the yolk's half-win set is the minimal such circle.   Q.E.D.

For large electorates, under a reasonably broad class of symmetry conditions for the location of voter ideal points, the yolk will be very small relative to the Pareto set, that is, the space defined by the set of voter ideal points (McKelvey, 1983; Feld, Grofman, and Miller, 1985). Even with only

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[6]We illustrate this and subsequent examples with circular indifference curves. The general ideas can be extended to all quasi-concave utility functions, but the specific analytic results we give below are based on circular indifference.

FIGURE 1
The Win Set and Half-Win Set of a Point y
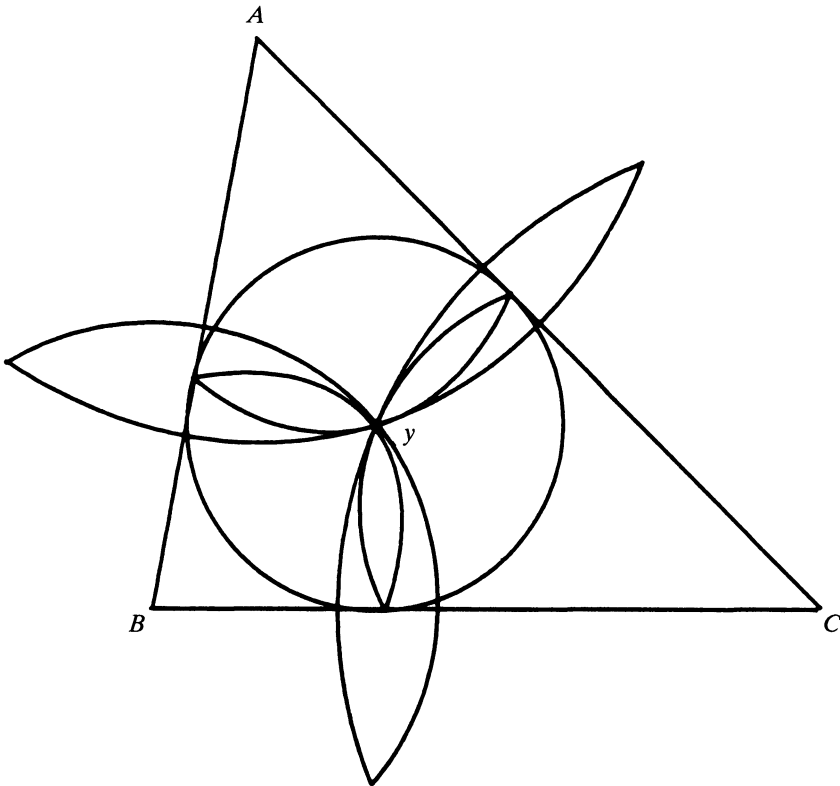
a few voters, the yolk may be small relative to the Pareto set. We show in Figure 2 an example adapted from one in Ferejohn, McKelvey, and Packel (1984).[7]

In this example the yolk, the minimum circle touching all median lines, is small and centrally located within the Pareto set—a situation which we believe to be the norm in spatial voting games. Thus, the center of the yolk

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[7]We have shown all of the voter ideal points to be on the hull of a convex polygon, but that is not necessary. In general, interior ideal points tend to shrink the size of the yolk.

FIGURE 2
A Seven-Voter Example Where the Yolk Is Small Relative to the Pareto Set
(Shape of Yolk Only Approximate)
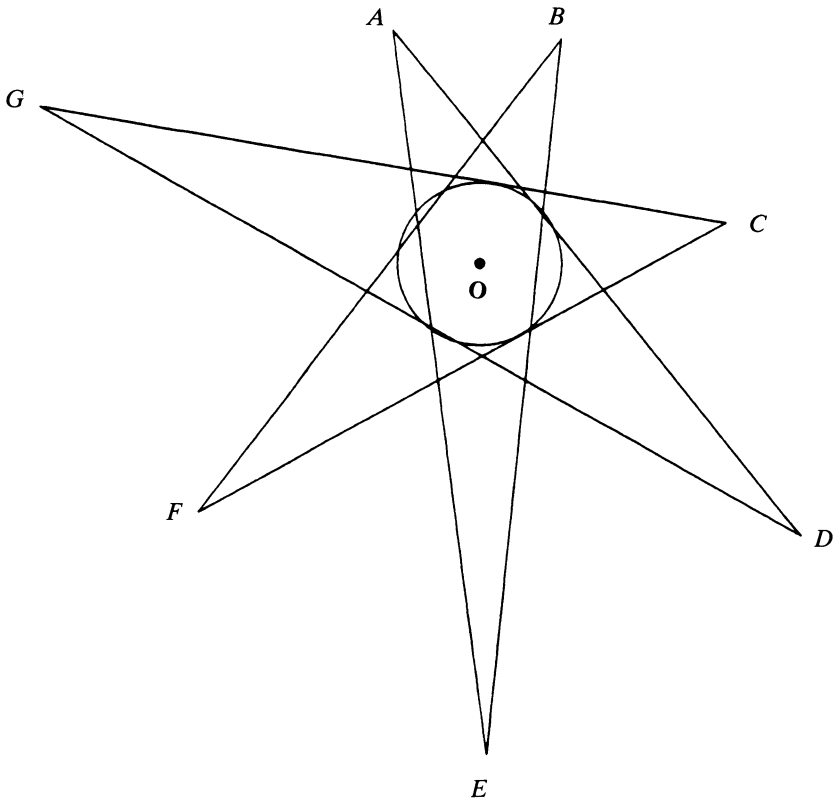
can be expected to be a point with relatively small finagle radius and would be attractive to candidates for that reason.[8]

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[8]We know (McKelvey, 1983) that the uncovered set (Miller, 1980) is located with four radii of the yolk. A point x is *covered* by a point y if yPx and yPz for all z: xPz, that is, one point covers another if it beats the first and can beat every point that the first point can beat (Miller, 1977, 1980; without real loss of generality we give the definition for the case of tournaments, where ties can be neglected). Points that are uncovered have a variety of desirable properties. For example, the set of sophisticated outcomes of standard amendment procedure is restricted to a subset of the uncovered set (Miller, 1980; Shepsle and Weingast, 1984; Banks, 1985).

THEOREM 1: The radius of the yolk, r, is an upper bound for the minimal finagle radius (i.e., the finagle radius of the finagle point).

PROOF: By Lemma 2 the finagle radius of the center of the yolk is at most r. By Lemma 3 this is the minimum bound for the radius of the finagle point that may be obtained by looking at half-win sets.   Q.E.D.

Looking at Theorem 1 might lead one to suspect that it would be impossible to find points whose finagle radius was much smaller than that of the yolk. This is, however, untrue. We might also expect that the point with minimal finagle radius will be the center of the yolk. This, too, is untrue.

THEOREM 2: For three voters the inner circle in Figure 3 is a V-Mess. This inner circle is tangent to each of the three larger circles. These outer circles are each centered at a vertex and pass through the nearest two tangency points of the yolk. Moreover, that inner circle is the smallest circular V-Mess that can be constructed.

PROOF: See Appendix.

There is another important implication of Theorem 2. Namely, the finagle radius of points will increase monotonically as we move along any ray from the finagle point. The implication of this is that the finagle radius of points is "well behaved" in the sense that its value increases continuously as we move out along any ray from the point of minimal finagle radius. Given a fixed finagle tolerance, the defensible zone is then a delimited area around the finagle point. Also, at least for three voters, the family of iso-finagle-radius lines will consist of portions of ellipses, and thus the defensible zone will be bounded by portions of ellipses.[9]

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[9] The finagle radius of a point is given by (see Figure A.2 in Appendix):

$$\frac{\max \{\overline{XA} + \overline{XB} - c, \overline{XA} + \overline{XC} - b, \overline{XB} + \overline{XC} - \alpha\}}{2}$$

We want to find the set of points that have the same finagle radius, say q. Let us take

$$\overline{XA} + \overline{XB} - c$$

to be the maximal element from among this triple for some set of alternatives. The set of points with finagle radius q is the set of points whose maximal element in the above expression is of value q. Now, we wish to find the locus of points such that
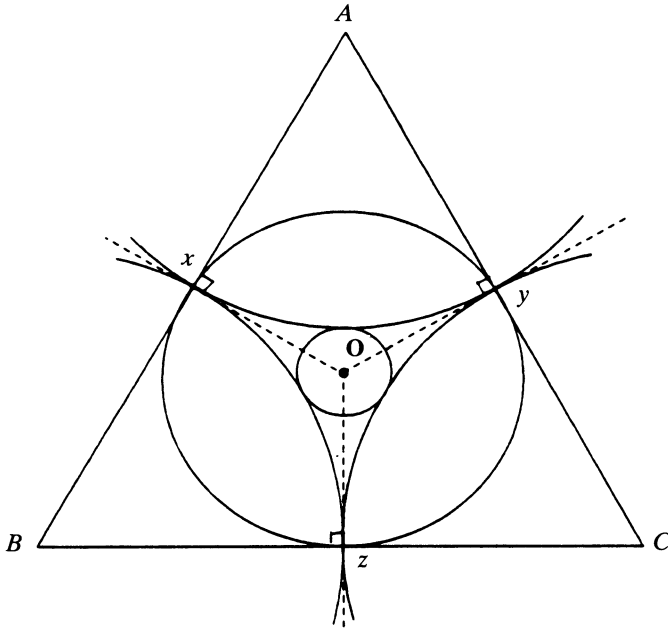
$$\frac{\overline{XA} + \overline{XB} - c}{2}$$

equals the constant q. This is the locus of points such that

$$\overline{XA} + \overline{XB} = 2q + c$$

The locus of points the sum of whose distances from two points is a constant is, of course, an ellipse.

FIGURE 3.A
Equilateral Triangle

THEOREM 3: The radius of the inner circle shown in Figure 3 is given by

$$\frac{1}{\alpha + \beta + \gamma + 2\sqrt{\alpha\beta + a\gamma + \beta\gamma}}$$

where p equals the semiperimeter of the triangle, i.e.,

$$p = \frac{\overline{AB} + \overline{BC} + \overline{AC}}{2}$$
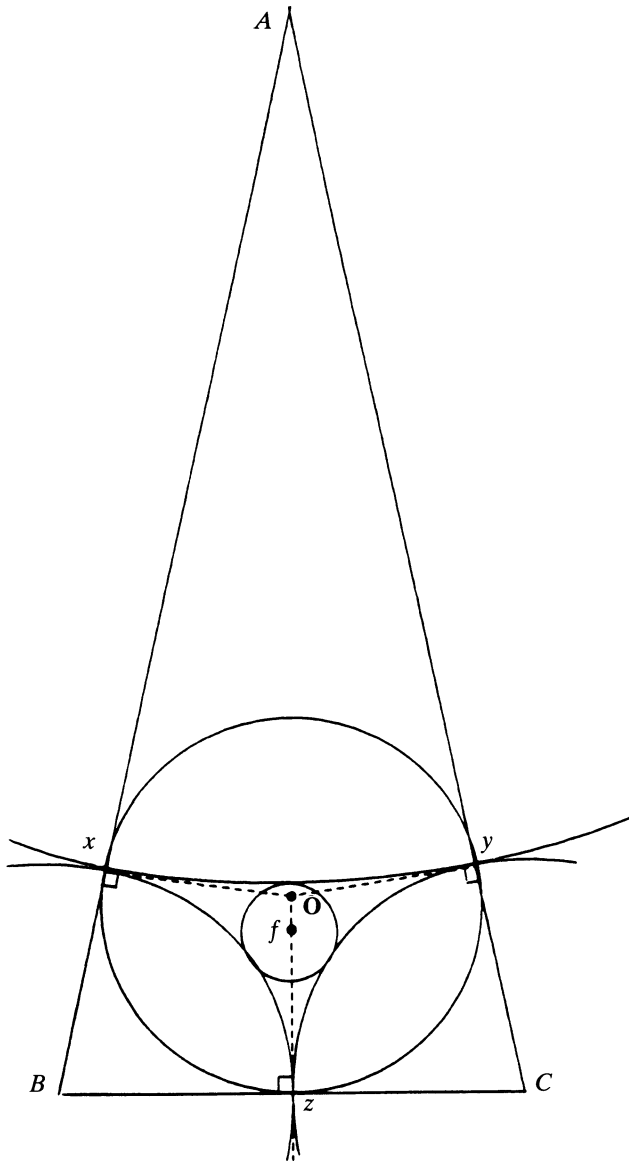
and

$$\alpha = \frac{1}{p - \overline{BC}}$$

$$\beta = \frac{1}{p - \overline{AC}} \tag{1}$$

$$\gamma = \frac{1}{p - \overline{AB}}$$

FIGURE 3.B
Acute Triangle

PROOF: The geometry in Figure 3 requires us to construct four circles that are mutually tangent. A standard result in analytic geometry is that four circles can be tangent only if their curvatures $\alpha + \beta + \gamma + \sigma$ satisfy

$$(\alpha + \beta + \gamma + \delta)^2 = 2(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) \tag{2}$$

The curvatures are, of course, the reciprocals of the radii. This gives us a quadratic equation in σ, in which we want the larger of the two roots. Solving, we obtain

$$\delta = \alpha + \beta + \gamma + 2\sqrt{\alpha\beta + \beta\gamma + \alpha\gamma} \tag{3}$$

Then, the radius of the desired circle is

$$\sigma = \frac{1}{\delta}$$

where δ is as given in equation (3).   Q.E.D.

THEOREM 4: For a three-voter majority rule game, the ratio of the minimum finagle radius to the radius of the yolk is given by

$$\frac{\sqrt{p\alpha\beta\gamma}}{(\alpha + \beta + \gamma + 2\sqrt{\alpha\beta + \beta\gamma + \alpha\gamma})} \tag{4}$$

PROOF: In general, the equation for the inscribed circle in a triangle is given by

$$r = \frac{1}{\sqrt{p\alpha\beta\gamma}} = \frac{\sqrt{(p - \overline{AB})(p - \overline{AC})(p - \overline{BC})}}{p}$$

Thus, by straightforward algebra, we obtain the desired result.   Q.E.D.

THEOREM 5: The ratio of the radius of the inner circle to that of the yolk is maximized for an equilateral triangle.

PROOF: See Appendix.
For an equilateral triangle, with

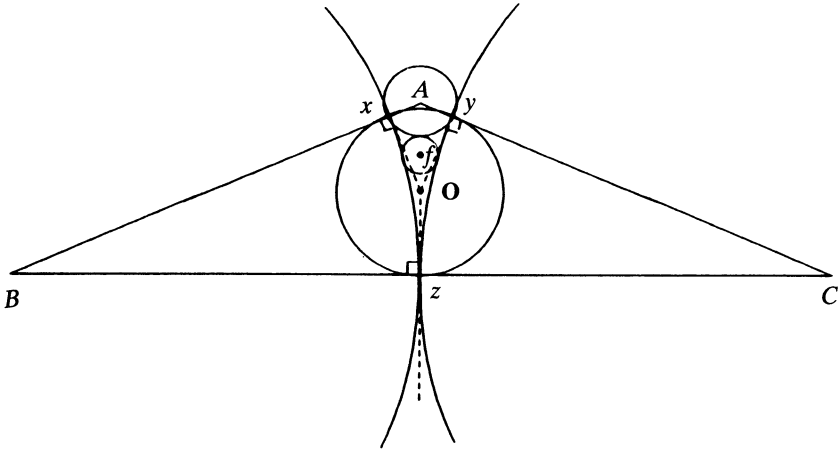$$\overline{AB} = \overline{BC} = \overline{AC} = 1,$$

some simple geometry establishes that the radius of the yolk (the inscribed circle) is given by

$$\frac{\sqrt{3}}{6} \simeq .288$$

while

$$\sigma = \frac{1}{6 + 4\sqrt{3}} \simeq .08 \tag{5}$$

FIGURE 3.C
Obtuse Triangle

Taking ratios and performing some simple algebra, we obtain

$$\frac{\sigma}{r} = \frac{1}{2+\sqrt{3}} = .267 \qquad (6)$$

Thus, at most the finagle radius is not even one-third the radius of the yolk. Moreover, for obtuse triangles we can make the ratio of the finagle radius to the radius of the yolk as small as we like by making the angle more and more obtuse (see Figures 3.C, 3.D, and 3.E).

Note that when the three-voter ideal points do not form an equilateral triangle, the finagle point (the center of the inner circle) and the center of the yolk will *not* coincide (see Figure 3).

The construction technique given for the three-voter case specified in Theorems 2, 3, and 4 shown in Figure 3 can be extended to the *n*-voter cases shown in Figure 4. For the five-voter case, we can find a point whose finagle radius is *much* smaller than the radius of the yolk (see Figure 4). Indeed, for symmetric polygons, as *n* increases, the finagle radius gets smaller and smaller relative to the yolk.[10] For triangles and symmetric polygons, this construction guarantees that the finagle point and its entire circle will be within the yolk. We conjecture that the finagle point is always within the

[10]A similar construction can be done for asymmetric polygons and ones with interior voter ideal points.
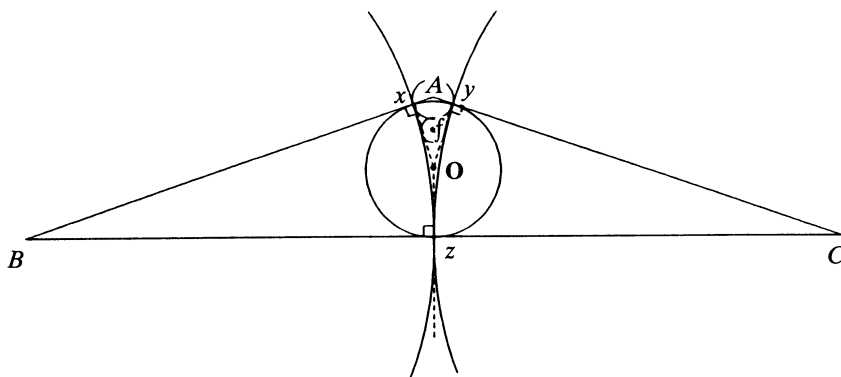
FIGURE 3.D
Obtuse Triangle

FIGURE 3.E
Obtuse Triangle
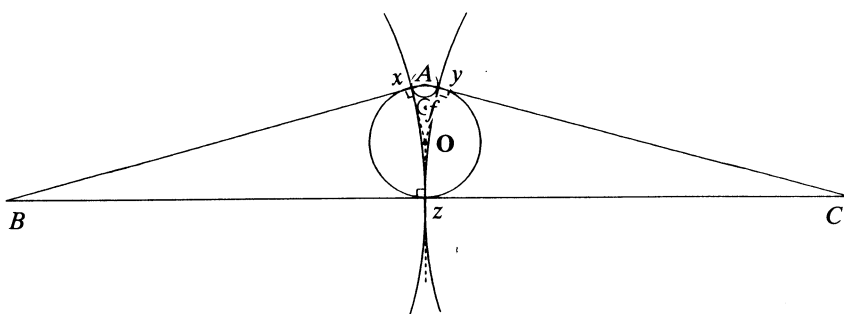
yolk, but we have been unable to prove it. The best we have been able to do so far is to prove that the finagle point is always within 2.5 radii of the center of the yolk.

We believe that the finagle point, although uniquely motivated, can be related to other solution concepts such as the nucleolus (Owen, 1982) and the competitive solution (McKelvey, Ordeshook, and Winer, 1978). If we look at the construction shown in Figure 4, we observe that the arcs that de-

FIGURE 4.A
Finagle Point for an Equilateral Pentagon
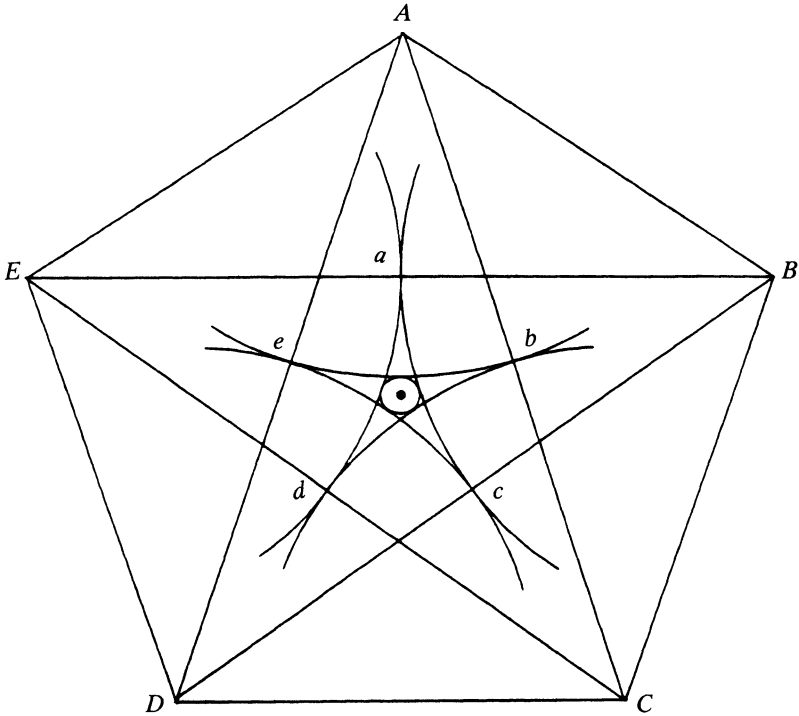
fine the tangency points of the finagle circle are the same as those whose intersections specify the points in the competitive solution (McKelvey, Ordeshook, and Winer, 1978). This appears always to be true in the case of symmetric polygons whose voter ideal points are located on the convex hull. We believe it may also be true in general, but proving this equivalence is not straightforward. In any case the competitive solution and the finagle radius have quite different analytic motivations: the one is a discrete collection of points related to coalitional bargaining offers, while the other is a unique point which is the center of a circle of minimal finagle radius and motivated in terms of a two-party political contest.

FIGURE 4.B
Finagle Circle for an Equilateral Nonagon
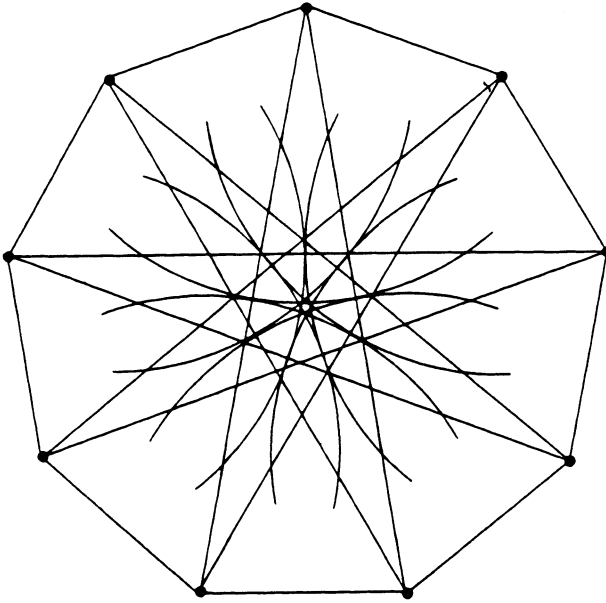
4. Discussion

We believe that too little attention has been paid to constraints on the movement of candidates among positions in an issue space. In general, we believe that there are multiple pressures on candidates to locate impositions that they can defend against challengers with as little finagling (movement from their initial position) as possible. We show that every point in the space can be characterized by a "finagle radius," the maximum distance from that point that a candidate at that position may be required to move in order to beat a potential challenger. A major contribution of this paper is the identification of a new and potentially quite powerful solution concept, the finagle point, the point with minimum finagle radius. We have identified various two-candidate sequential election games in which strategically motivated individuals will be led to locate at or near the finagle point.

In three-voter situations, we have shown that the finagle point is centrally located and has a very small finagle radius; that is, a candidate located at the finagle point can defeat any challenger with only slight shifts of his or her position. In addition, we have provided a relatively general geometric

construction to find the finagle point and its finagle radius.[11] Also, we have provided an analytic solution to find the finagle point and its finagle radius in the case of three voters and to show that the finagle radius of points increases as they are further away from the finagle point along any ray. For this case we have shown that the finagle circle is contained within the yolk, but that the finagle point will not in general coincide with the center of the yolk.

Location at a point with a small finagle radius makes it possible for a candidate to have an "easy-to-get-to" winning *counter* to any move of his opponent (i.e., a counter that is only incrementally different from his initial location in the space and is thus "credible" with the voters).[12,13] Thus, if an incumbent is able to locate at a point with a small finagle radius and makes (what are, in the perception of the voters) the right moves thereafter; that is, if he is a successful finagler, he is likely to retain office.[14] The positions with the smallest finagle radius are in or near the yolk. Thus, the expected outcomes of two-party electoral competition in a world where at least one candidate locates at or near the finagle point will be confined to a small and

[11]Further investigation of the location of the finagle point in the asymmetric *n*-voter case is, however, still needed.

[12]Since the platform of one's challenger is in general not known with certainty in advance, it is always desirable to be able to defeat *any* possible opponent, that is, to be the center of a V-Mess. It is also reasonable to believe that, ceteris paribus, small radius V-Messes are better than big ones. Thus, ceteris paribus, we might expect candidates to seek the smallest V-Mess they can find, that is, that which minimizes the maximum finagle. The reason for that is quite simple: there are likely to be costs to changing positions, and these costs are likely to rise the further away from his or her starting point a candidate tries to move. Such costs can arise because candidates are punished by voters for being untrustworthy in too radically shifting policies for the sake of expediency, or simply because credibly conveying to voters a slight switch in views is a lot easier than, say, convincing voters that yesterday's free spender is today's fiscal conservative (look at the fate of Mondale). Page (1978) suggests that McGovern lost support when his attempts to change positions as the campaign progressed (e.g., on his tax plan) cost him his reputation of being a man of principle and branded him as indecisive.

[13]Of course, to the extent that candidates (incumbents in particular) are constrained by their *previous* positions, candidates may be unable to locate at the finagle point itself. Also, if a V-Mess is small in radius, it may be difficult for any challenger to credibly stake out a claim to a position in the same V-Mess in which another candidate has located (especially if the latter is an incumbent). Indeed, the incumbent may, in effect, deny this option by creating a kind of uncertainty around his own true position within which no one challenger can enter without being accused of playing tweedledum-tweedledee politics in a way that will cost him or her votes.

[14]If the challenger also picks a point with a small finagle radius and thus also has credible countermoves, the last player to "move" will win, or chance will determine the outcome because voters will be unable to make sufficiently fine-tuned discriminations between the two candidates' positions, or the election will be determined by nonspatial factors (e.g., partisan bias in ascribing to the candidate of the voter's own party a position closer to the voter's own position than is in fact the case).

rather precisely delimited domain of the Pareto set. Thus, having a good fi-
nagler around redounds to the public benefit because it makes politics more
stable. Note, too, that one clear implication of our results is that there is no
need for an incumbent to "jump around" the space in order to defeat any
challenger—as might have been suggested by a too careless reading of the
McKelvey (1976, 1979) results showing that the entire space can be in the
top cycle set.

We should also note an important alternative interpretation of our re-
sults. If candidate positions are largely fixed, then we can think of a point
with a small finagle radius as what *political parties* (rather than candidates)
ought to seek; that is, a party will wish to be able to field candidates from any-
where in some small finagle radius and thus assure itself of being able to find
a candidate able to defeat any *given* opponent if that opponent has a known
and *fixed* location. Such an interpretation suggests that parties that are not
located at a point with a small finagle radius cannot expect to compete ef-
fectively against a full range of ideological challenge (cf. Page, 1978). We
hope to explore these ideas in future research.

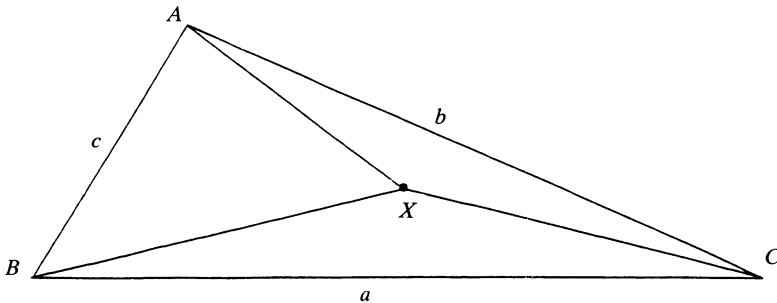*Manuscript submitted 19 July 1985*
Final manuscript received 25 July 1988

APPENDIX

Proof of Theorems on the Finagle Point in the Three-Voter Case

In triangle ABC, let the sides have length a, b, c, respectively. See Figure A.1.

FIGURE A.1

For an arbitrary point X, define

$$g_1(X) = \overline{XA} + \overline{XB} - c$$

$$g_2(X) = \overline{XA} + \overline{XC} - b$$

$$g_3(X) = \overline{XB} + \overline{XC} - a$$

where $\overline{XA}$ is the Euclidean distance from X to A, etc. As is well known, Euclidean distance is a convex function; since the sum of two convex functions is convex, then for fixed A, B, and C, the three functions $g_j(X)$ are convex. Moreover, $h(X) = \max\{g_1(X), g_2(X), g_3(X)\}$, as the maximum of convex functions, is also convex.

Let p be the semiperimeter

$$p = \frac{a+b+c}{2}$$

then circles of radius $p-a$, $p-b$, $p-c$ centered at A, B, C, respectively, will all be mutually tangent. A small circle, centered at S, will be tangent to all three of these circles. Its radius σ can be obtained from the formula

$$\left(\frac{1}{\sigma} + \frac{1}{p-a} + \frac{1}{p-b} + \frac{1}{p-c}\right)^2 = 2\left(\frac{1}{\sigma^2} + \frac{1}{(p-a)^2} + \frac{1}{(p-b)^2} + \frac{1}{(p-c)^2}\right)$$

where the larger of the two roots, $1/\sigma$, is to be taken.[1]

Now the distance between the centers of two tangent circles is the sum of their radii; hence,

$$SA = \delta + p - a$$

$$SB = \delta + p - b$$

$$SC = \delta + p - c$$

which gives us

$$\overline{SA} + \overline{SB} = 2\delta + 2p - a - b = 2\delta + c$$

[1] As a mnemonic, the reader may wish to memorize Soddy's well-known poem, "The Kiss Precise":

Four circles to the kissing come;
The smaller are the benter.
The bend is just the inverse of
The distance from the center.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.
Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

(Frederick Soddy, "The Kiss Precise," *Nature* [1936], p. 1021. Cf. Harold Scott Coxeter, *Introduction to Geometry* [New York: Wiley, 1961].)

FIGURE A.2
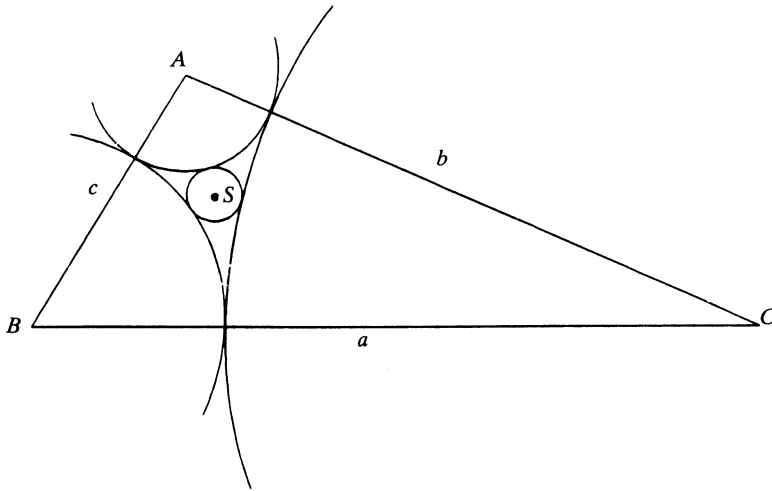
and similarly

$$\overline{SA} + \overline{SC} = 2\delta + b$$

$$\overline{SA} + \overline{SC} = 2\delta + a$$

and so

$$g_1(S) = g_2(S) = g_3(S) = 2\delta$$
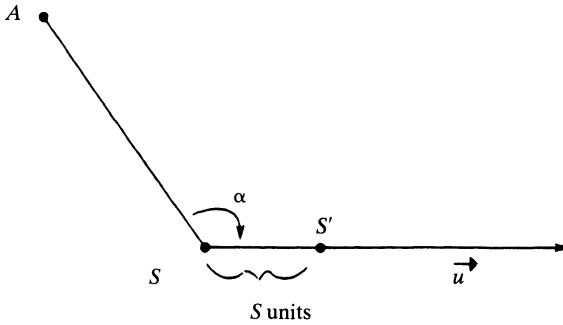
and

$$h(S) = 2\delta$$

We claim, now:

LEMMA: S is the (unique) minimum of the function h.

PROOF: Since h is convex, it suffices to prove that S is a local minimum of h.
Let $\vec{u}$ be a direction (i.e., a vector of magnitude 1 and consider a "nearby point," $S' = S + t\vec{u}$ where t is a small positive scalar (i.e., S' is t units from S in the direction $\vec{u}$).

FIGURE A.3

Let α be the angle $\angle$ ASS'. Then

$$\overline{AS'}^2 = \overline{AS}^2 + \overline{SS'}^2 - 2 \cos \alpha \overline{AS} \times \overline{SS'}$$

or, since $\overline{SS'} = t$,

$$\overline{AS'}^2 = \overline{AS}^2 - 2t \cos \alpha \overline{AS} + t^2$$

or

$$\overline{AS'} = \sqrt{\overline{AS}^2 - 2t\overline{AS} \cos \alpha + t^2}$$

so

$$\frac{d(\overline{AS'})}{dt} = \frac{-\overline{AS} \cos \alpha + t}{\sqrt{\overline{AS}^2 - 2t\overline{AS} \cos \alpha + t^2}}$$

and, setting $t = 0$,

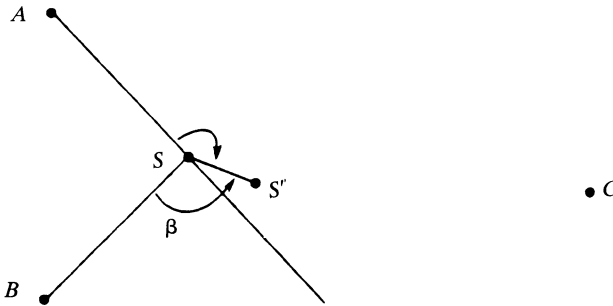$$\frac{d(\overline{AS'})}{dt} = -\cos \alpha$$

In a similar way,

$$\frac{d(\overline{BS'})}{dt} = -\cos \beta$$

$$\frac{d(\overline{CS'})}{dt} = -\cos \gamma$$

where β and γ are the angles BSS' and CSS', respectively. Then, at point S the directional derivatives in the direction $\vec{u}$ are

$$\frac{\partial g_1}{\partial \vec{u}} = -\cos \alpha - \cos \beta$$

FIGURE A.4

$$\frac{\partial g_2}{\partial \vec{u}} = -\cos \alpha - \cos \gamma$$

$$\frac{\partial g_3}{\partial \vec{u}} = -\cos \beta - \cos \gamma$$

We claim at least one of these three is positive.

To see this we first point out that at least one of the three angles α, β, γ must be obtuse. For if all were acute (or right) then the line ℓ through S, orthogonal to $\vec{u}$, would be a support for ABC (i.e., all three of A, B, C would be on the same side of ℓ). But this is not possible, since S is in the interior of ABC.

At least one of α, β, γ is obtuse. Without loss of generality, assume α is obtuse. Again, we note that B and C cannot both be on the same side of line AS, since this would mean AS was a support for ABC. Now, either $\vec{u}$ is parallel to AS or it lies to one side of AS. Without loss of generality, once again, we can assume that $\vec{u}$ lies to the same side of AS as C (or else parallel to AS). In this case, we see that $\alpha + \beta = 2\pi - \angle ASB$, or $\alpha + \beta > \pi$. Thus α, β are proper angles (not larger than π radians) whose sum is larger than π. We have $\beta > \pi - \alpha$, and so $\cos \beta < \cos (\pi - \alpha) = -\cos \alpha$, and therefore $-\cos \beta - \cos \alpha > 0$. Thus

$$\frac{\partial g_1}{\partial \vec{u}} > 0, \text{ i.e., } g_1(S') > g_1(S)$$

for points S' close to S in this direction. It follows that $h(S') \geqslant g_1(S') > 2\alpha$, and since $\vec{u}$ is an arbitrary direction, we find that S is a strict local minimum of h. By convexity, S is the unique global minimum.

We can now prove the following:

THEOREM A.1: The Finagle radius of S is σ; i.e., for any X and any $\epsilon > 0$, there is some point T dominating X such that $\overline{ST} \leqq \sigma + \epsilon$.

PROOF: Let $X \neq S$. We know $h(X) > h(S) = 2\sigma$. Thus, let us write $h(X) = 2\sigma + \delta$ where $\delta > 0$.

FIGURE A.5
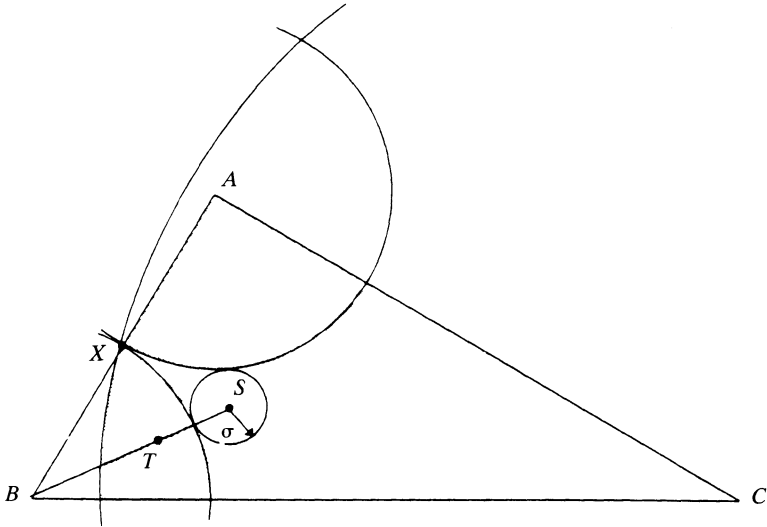
Consider the differences $\overline{XA} - \overline{SA}$, $\overline{XB} - \overline{SB}$, $\overline{XC} - \overline{SC}$. There is no loss of generality in assuming

$$\overline{XA} - \overline{SA} \leqslant \overline{XB} - \overline{SB} \leqslant \overline{XC} - \overline{SC} \tag{4}$$

Let q be the second of these:

$$q = \overline{XB} - \overline{SB}$$

If $q > 0$, then both B and C prefer S to X, so S dominates X. Suppose, then, $q \leqq 0$.

We know $\overline{XA} + \overline{XB} \geqslant \overline{AB} = c$, and so $2q \geqslant \overline{XA} - \overline{SA} + \overline{XB} - \overline{SB} \geqslant c - \overline{SA} - \overline{SB} = g_1(S) = -2\sigma$, and so $q \geqslant -\sigma$.

Let now $0 < \epsilon < \delta$. Let T be the point obtained by moving $-q + \epsilon$ units from S toward B. Then $\overline{ST} = -q + \epsilon \leqslant \sigma + \epsilon$. We claim T dominates X through $\{B, C\}$. In fact, $\overline{TB} = \overline{SB} + q - \epsilon = \overline{XB} - \epsilon$, and so B prefers T to X.

Now,

$$g_1(X) = \overline{XA} + \overline{XB} - c = \overline{XA} - \overline{SA} + \overline{XB} - \overline{SB} + 2\sigma$$

$$g_2(X) = \overline{XA} + \overline{XB} - b = \overline{XA} - \overline{SA} + \overline{XC} - \overline{SC} + 2\sigma$$

$$g_3(X) = \overline{XB} + \overline{XC} - a = \overline{XB} - \overline{SB} + \overline{XC} - \overline{SC} + 2\sigma$$

By equation (4), $g_3(X)$ is the largest of these, and so $g_3(X) = h(X) = 2\sigma + \delta$, and by equation (5), $XC - SC = \delta - q$, or $\overline{XC} = \overline{SC} + \delta - q$.

Now, $\overline{TC} \leq \overline{SC} + \overline{ST} = \overline{SC} + \epsilon - q$. Since $\epsilon < \delta$, we see $\overline{TC} < \overline{XC}$, and C also prefers T to X. Hence, T dominates X and lies the desired $\sigma + \epsilon$ units from S.

We show, finally, that point S is best, in this sense.

THEOREM A.2: Let X be any point other than S. Then X has a finagle radius greater than σ.

PROOF: Since $X \neq S$, we must have $h(X) > h(S) = 2\sigma$. There is no loss of generality in assuming $h(X) = g_1(X)$.

Thus, $\overline{XA} + \overline{XB} - c = h(X)$.

On segment AB, now choose Y so that

$$\overline{YA} = \overline{XA} - \frac{h(X)}{2}$$

then

$$\overline{YB} = \overline{AB} - \overline{YA} = c - \overline{YA}$$

but

$$c = \overline{XA} + \overline{XB} - \frac{h(X)}{2}$$

To dominate Y, a point Z must be preferred (to Y) by at least one of voters A and B. Thus, either $\overline{ZB} < \overline{YB}$ or $\overline{ZA} < \overline{YA}$. In either case we must have $\overline{XZ} > h(X)/2$, and so the finagle radius of X is at least $h(X)/2$. Since $h(X) > 2\sigma$, this is greater than σ. Thus, S has the minimal finagle radius.

Combining Theorems A.1 and A.2, we have Theorem 2 in the text.

PROOF OF THEOREM 5 IN THE TEXT: We wish to show the ratio α/r is maximized for an equilateral triangle. For such, we have seen that it equals

$$\frac{1}{2 + \sqrt{3}} \simeq .268$$

To simplify the notation, we set

$$f = p - \overline{AB}$$

$$g = p - \overline{AC}$$

$$h = p - \overline{BC}$$

then

$$\sigma = \frac{1}{f}, \ \beta = \frac{1}{g}, \ \gamma = \frac{1}{h}, \ and \ p = f + g + h$$

We have now

$$\frac{1}{\alpha} = \frac{1}{f} + \frac{1}{g} + \frac{1}{h} + 2\sqrt{\frac{1}{fg} + \frac{1}{fh} + \frac{1}{gh}}$$

which reduces to

$$\frac{1}{\sigma} = \frac{fh + fh + gh}{fgh} + 2\frac{\sqrt{f + g + h}}{fgh}$$

On the other hand, the radius of the inscribed circle is

$$r = \frac{\sqrt{fgh}}{f+g+h}$$

and so

$$\frac{r}{\alpha} = 2 + \frac{fg+fh+gh}{\sqrt{fgh(f+g+h)}}$$

For an equilateral triangle, $f = g = h$, and so

$$\frac{r}{\alpha} = 2 + \sqrt{3}$$

For other cases, we use the identity

$$(fg+fh+gh)^2 - 6fgh(f+g+h) = f^2(g-h))^2 + g^2(f-h)^2 + h^2(f-g)^2$$

The right side of this, being the sum of squares, is nonnegative. So, then, is the left side, i.e., $(fg + fh + gh)^2 \geqslant \sqrt{3}$; thus r/α is always at least $2 + \sqrt{3}$, and is minimized for equilateral triangles. This means, of course, that α/r is maximized for such triangles.

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