THE CORE AND THE STABILITY OF GROUP CHOICE IN SPATIAL VOTING GAMES

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The core of a voting game is the set of undominated outcomes, that is, those that once in place cannot be overturned. For spatial voting games, a core is structurally stable if it remains in existence even if there are small perturbations in the location of voter ideal points. While for simple majority rule a core will exist in games with more than one dimension only under extremely restrictive symmetry conditions, we show that, for certain supramajorities, a core must exist. We also provide conditions under which it is possible to construct a structurally stable core. If there are only a few dimensions, our results demonstrate the stability properties of such frequently used rules as two-thirds and three-fourths. We further explore the implications of our results for the nature of political stability by looking at outcomes in experimental spatial voting games and at Belgian cabinet formation in the late 1970s.

Two of the most important questions in politics are the extent to which the outcome of collective-decision-making procedures can be expected to be (1) predictable and (2) stable. The game-theoretic notion of a core in principle provides important insight into both these questions. The *core* of a voting game is the set of undominated outcomes, that is, those that once in place, cannot be overturned. If games have a core, we expect that outcomes will be in the core and thus predictable. We consider those group-decision-making situations where the set of possible outcomes are points in (Euclidean) W-dimensional space. We shall refer to the phenomena we examine as *spatial voting games*. In this context the conditions under which a core can exist are now well understood. Even when a core exists, however, it may be *structurally unstable,* in the case that arbitrary small changes in preference are sufficient to render the core empty. We shall argue that the existence of structurally stable core outcomes is of fundamental importance in understanding political decision making.

For a simple-majority-rule voting game with no ties, the core, if one exists, is simply the majority winner, that is, that alternative which can receive a majority in paired contest against each and every other alternative. The practical difficulty is that under majority rule we expect in general there will be no core except under highly restrictive symmetry conditions (Davis, DeGroot, and Hinich 1972; Feld and Grofman 1987; Matthews 1980; McKelvey and Schofield 1987; McKelvey and Wendell 1976; Plott 1967; Schofield 1983a; Sloss 1973) or in the case that there is only one dimension (Black 1958). Subsequent work has deepened this pessimistic conclusion by showing that in general not only will there be no majority winner, but *all* alternatives will cycle with each other (Bell 1978; Cohen 1979; Cohen and Matthews 1980; McKelvey 1976, 1979; Schofield 1978a, 1978b, 1980, 1983a, 1985). Thus, by manipulating the agenda through restricting choices to some finite set to be voted upon in a specified order, *any* alternative can be made the group choice. This set of instability results is sometimes referred to as the "chaos theorems." (See the review in Riker 1980, 1982 for further discussion and alternative interpretations.)

Recent work has, however, provided grounds for a more optimistic view. There have been seven important lines of research. (Cf. an alternative typology in Grofman and Uhlaner 1985.)

The dominant line of research has been one that looks for *structure-induced equilibria.* Such equilibria occur because of imposed limitations on the set of alternatives that can be considered, for instance, a germaneness restriction, a closed rule, a mandatory final vote against the status quo, a budget constraint, a yes-no vote on an alternative proposed by an agenda setter, and so on (McCubbins and Schwartz 1985; Riker 1980; Romer and Rosenthal 1978; Shepsle 1979; Shepsle and Weingast 1981, 1984).

In another line of attack (Ferejohn, Fiorina, and Packel 1980; Ferejohn, McKelvey, and Packel 1984; Packel 1981), which has not yet been absorbed into the mainstream of the social choice literature, a group of scholars has shown that in spatial voting games there may be *probabilistic* convergence of outcomes to a small and well-defined area of the space centered around what has been called the *yolk.* Closely related is the work on agenda construction that demonstrates, among other things, that agendas that move toward the center of the yolk are much easier to construct than agendas away from the center (Feld and Grofman 1986; Feld, Grofman, and Miller 1985).

A third promising line is work on the uncovered set and various subsets thereof (Banks 1985; Banks and Bordes 1987; Feld and Grofman n.d.; Feld et al. n.d.; McKelvey 1986; Miller 1977, 1980, 1983; Miller, Grofman, and Feld 1985; Moulin 1984; Shepsle and Weingast 1984). The uncovered set can be thought of as a weakening of the concept of the core. The *uncovered set* is the set of points that are majority-preferred to all other alternatives either directly *or* at one remove; that is, if x is uncovered, then for all y either $x P y$ or there exists z such that $x P z$ and $z P y$.

A fourth new approach has been to focus on the Copeland winner (also known as the strong point). The *Copeland winner* is the alternative that is defeated by the fewest other alternatives (Copeland 1951; Glazer, Grofman, and Owen 1985; Grofman 1972; Grofman et al. 1987; Henriet 1984; Owen and Shapley 1985; Straffin 1980). The strong point, too, can be thought of as a weakening of the concept of core.

A fifth line of research has been to look at *von Neumann-Morgenstern externally stable solution sets,* which have the property that for any alternative outside the set, there exists an alternative in the set that beats it. Wuffle et al. (n.d.) show that the V-M externally stable solution of minimal area can play a role analogous to that of the core and that it is identical to the core if one exists.

Riker's (1982) work on a "liberal" concept of democracy as a referendum on the status quo may be thought of as constituting a separate sixth line of attack. It involves "a conceptualization of the notion of democracy so that the chaos of the McKelvey-Schofield results is irrelevant" (Riker, personal communication, 1986).

The seventh line of research is one that looks at supramajoritarian decision making and the minmax set (for definition see next section). Early work in this tradition includes Craven 1971; Ferejohn and Grether 1974; Kramer 1977; Packel 1981; Simpson 1969; and Slutsky 1979. It is this seventh line of research with which we will deal.

Recent work in this research tradition (Greenberg 1979; McKelvey and Schofield 1986, 1987; Schofield 1983b, 1984a, 1984c, 1985, 1986; and Strnad 1985) has, for spatial voting games, found conditions (a) under which a core will be guaranteed to exist, and (b) such that it is possible to construct a core that will be structurally stable. The research above states results not just for supramajoritarian games but also for the widest possible class of noncollegial voting rules (where a *noncollegial voting rule* means one in which no one voter or set of voters has sufficient power to be in all winning coalitions). These papers are *quite* technical; they appear (with one exception) in economics journals and provide (with one partial exception) no discussion of any potential or actual empirical applications.

Our aim is threefold: *first*, to present the most empirically relevant of these results in a nontechnical fashion for a political science audience, since it will turn out that these results really are not hard to understand, even though they are quite hard to prove in their most general form; *second*, to extend the results by reformulating key theorems in a way which makes their practical significance clearer; in particular, we look at the stability implications of two-thirds and three-fourths rules in two and three dimensions and at the stability characteristics of weighted voting and veto rules; *third*, to demonstrate the importance of these results for the understanding of actual group-decision processes by reanalyzing some recent experiments on spatial voting games (Fiorina and Plott 1978; Wilson and Herzberg 1984) and by offering an illustrative example of how we can model the stability of coalitions in multiparty coalition governments—looking at the 1978 Belgian political-party system.

Some of the most significant implications of the basic results we give are

1. When there are only two dimensions of choice, rules only marginally stronger (i.e., closer to unanimity) than a two-thirds rule will invariably give rise to a core, and rules that are only marginally stronger than simple majority may give rise to a structurally stable core if certain symmetry assumptions are met. Moreover, for two dimensions and few alternatives these conditions are not particularly stringent. In particular, in two dimensions, requiring only one voter above simple majority may make it possible to obtain a structurally stable core.
2. When there are only three dimensions of choice, rules only marginally stronger (i.e., closer to unanimity) than a three-fourths rule will invariably give rise to a core, and no rule that is less than two-thirds can give rise to a structurally stable core. Rules in between two-thirds and three-fourths may give rise to a structurally stable core if certain symmetry conditions are met.
3. Every voting situation in which there are one or more players with veto power has a core.
4. The core will always exist if the only feasible coalitions are those that are

connected (Axelrod 1970) with respect to some given single dimension.

Basic Results about the Minmax Set and Structurally Stable Voting Rules

Let n be the number of voters. Let q be the number of votes needed to replace the status quo with a new alternative. We shall refer to such decision rules as *quorum rules* or *q-rules*. For n odd, if $q = (n + 1)/2$ we have simply majority rule. If $q > (n + 1)/2$, that is, if decision making is supramajoritarian, then we may obtain a core. (Of course we must always obtain a core if $q = n$.)

To examine spatial voting games that are q-rules, we need to introduce a few concepts. We suppose that the set of outcomes is a subset of Euclidean space of dimension W. We generally assume the set is convex and compact (i.e., closed and bounded) unless we state otherwise. We also assume that individual preference is *convex* (i.e., for each alternative the set of preferred alternatives is a convex set).[1] To further simplify our exposition, we assume that the game is what is called a tournament; that is, there are no ties. Dealing with ties is, however, straightforward. For a given dimension W and society size n, we can find the *smallest* integer q such that any q-rule, with $q' \geqslant q$, must have a nonempty core. The core for the q-rule so defined is called the *minmax set* in dimension W (Simpson 1969; Kramer 1977).

Kramer (1977) showed that a sequential two-candidate spatial voting game, in which candidates sought to maximize their plurality, converged to outcomes in the minmax set. Let $q^* = q/n$; that is, since q is the minimum number of players in a winning coalition then q^* is the size of the minimal winning majority expressed as a proportion. Kramer (1977, 328, n. 4) conjectured that the smallest q^* that guarantees (i.e., is *sufficient* for) the existence of a core was given by $W/(W + 1)$, where W is the dimensionality of the space. We shall refer to the minimum q^* which guarantees the existence of a core as the *Kramer number,* and denote it q^*_k. Kramer's conjecture was proved by Greenberg (1979) and generalized further by Schofield (1983b, 1984a) and independently by Strnad (1985).

If a q-rule satisfies $q/n < q^*_k$, it is possible to show that there is a configuration of convex voter preferences such that the core is empty. However, it may also be possible to find a configuration of convex voter preferences for which a core *does* exist and is, moreover, structurally stable.

Results in Schofield (1986) can be used to prove a result closely related to Kramer's conjecture: for large electorates the minimum q^* for which there exists a configuration of voter preferences resulting in a structurally stable core is given by $(W - 1)/W$. We shall call this the *Schofield number* q^*_s. We shall also provide an exact expression that relates q and q^*_s to n and W.

In much of the literature, results are stated in terms of the lowest dimension in which instability occurs (the *instability dimension*). Here, we shall focus on finding what q-rules are sufficient for stability, or give rise to the possibility of a structurally stable core, when there are only a few dimensions. We believe such results are of considerable practical relevance, since much real-world decision making takes place with only a very limited set of issue dimensions under simultaneous consideration.

In this section we state in informal terms the key theorems and some implications derived from them.

THEOREM 1. *For any given n, the minimum value of q sufficient to guarantee the existence of a core in a q-rule game* *is given by the smallest integer q such that*

$$q > \frac{nW}{W + 1} \tag{1}$$

Greenberg (1979) shows that for supramajoritarian decision rules, the minimum dimensionality sufficient to guarantee a core is given by $W < q/(n - q)$. Theorem 1 follows after some elementary algebra.

Greenberg (1979) and Schofield (1984b) have also shown that if W is an integer and satisfies $W \geqslant q(n - q)$ then it is possible to find a set of preferences such that the core is empty. Thus, if $q \leqslant nW/(W + 1)$, the existence of a core cannot be guaranteed.

It is useful to express Theorem 1 in terms of proportions rather than absolute numbers. As before, let $q^* = q/n$.

COROLLARY 1 TO THEOREM 1. *For any given n, the Kramer number, q_k^*, i.e., the minimum special majority, q^*, sufficient to guarantee the existence of a core, is given by the smallest q_k^* which is a multiple of $1/n$ such that*

$$q_k{}^* > \frac{W}{W + 1} \tag{2}$$

Proof. Substitute $q^* = q/n$ in Relationship 1. QED

COROLLARY 2 TO THEOREM 1. *For $W = 1$, the one-dimensional case, for any value of n, there exists a majority-rule core.*

Proof. Substituting $W = 1$ in Relationship 1 we obtain $n/2$. The smallest integer greater than $n/2$ is $(n + 2)/2$ for n even, and $(n + 1)/2$ for n odd. QED

This corollary is familiar as Black's famous (1958) median-voter theorem, since, in one dimension, for simple-majority rule, the ideal point of the median voter is in the core.

COROLLARY 3 TO THEOREM 1. *For $W = 2$, the two-dimensional case, for any value of n, if $q^* > 2/3$ then a core is guaranteed.*

Proof. Substitute $W = 2$ in Relationship 2. QED

Analogously, for 3 dimensions, a core is guaranteed if $q^* > 3/4$, and so on.

Theorem 1 provides conditions sufficient to guarantee the existence of a core for q-rules.

Even though a core cannot be *guaranteed* if $q \leqslant nW/(W + 1)$, for a high enough q, a structurally stable core can be constructed by appropriately configuring voter preferences, as the following theorem proves:

THEOREM 2. *For any given n, any q-rule satisfying*

$$q \geqslant \frac{2 + n(W - 1)}{W} \tag{3}$$

a structurally stable core can be constructed.

Schofield (1984b, 1986) shows that for supramajoritarian decision rules, a structurally stable core can be constructed for some configuration of preferences if

$$W \leqslant \frac{n - 2}{n - q}. \tag{4}$$

Theorem 2 follows after some simple algebra.

It is useful to express Theorem 2 in terms of proportions rather than absolute numbers. Let $q^* = q/n$ as before.

COROLLARY 1 TO THEOREM 2. *For any given n, the Schofield number, q_s^*, i.e., the special majority, q^*, sufficiently large to permit the existence of a structurally stable core is given by the smallest q_s^* which is a multiple of $1/n$ such that*

$$q_s^* \geqslant \frac{(W - 1)n + 2}{Wn} = \frac{W - 1}{W} + \frac{2}{Wn}$$

Proof. This follows directly from Relationship 3 by substituting $q = q^*n$. However, we must be careful to confine ourselves to q^* which are multiples of $1/n$. QED

COROLLARY 2 TO THEOREM 2. *For $W \geqslant 2$, then as n gets large, the Schofield number, q_s^*, is given by*

$$\lim_{n\to\infty} q_s^* = \frac{W-1}{W}$$

Proof. Follows directly from Relationship 2. QED

COROLLARY 3 TO THEOREM 2. *For $W = 2$, for given n, the smallest integer q sufficiently large to permit the existence of a structurally stable core is given by $n/2 + 1$ for n even and $(n + 1)/2 + 1$ for n odd.*

Proof. If n is even, Relationship 4 gives $W \leqslant (n-2)/(n-q)$ or $q \geqslant n - n/2 + 1$ in the case $W = 2$. Hence $q \geqslant n/2 + 1$. If n is odd, the largest integer greater than $(n + 2)/2$ is simply $(n + 1)/2 + 1 = (n + 3)/2$. QED

Corollary 3 to Theorem 2 shows that it may be possible to generate a structurally stable core in two dimensions with q only one more than simple majority.

More generally, Corollary 2 to Theorem 2 shows that for a very large electorate, it is always possible to construct a structurally stable core as long as a sufficiently high quorum rule $(W - 1)/W$ is chosen. For example, for $W = 3$ we can always find a way to construct a structurally stable core if $q^* > 2/3$ (i.e., we can always find a configuration of preferences in three-dimensions for which such a core will exist). It may be possible to find a structurally stable core for lower quorum rules, as the following necessary condition indicates.

THEOREM 3. *For any given n, then for a structurally stable core to exist it is necessary that*

$$q \geqslant \frac{W+n-1}{2} \tag{5}$$

McKelvey and Schofield (1986) have shown that if $W \geqslant 2q - n + 2$, then a structurally stable core can never exist on a compact set of alternatives. Thus a necessary condition for a structurally stable core is $W \leqslant 2q - n + 1$. Rearranging gives the desired result.

For $W = 2$, Relationship 5 is simply $q \geqslant (n + 1)/2$; that is, in two dimensions with n odd a structurally stable core can occur for simple-majority rule. However, the techniques used by McKelvey and Schofield (1986) show that it must then lie on the boundary of the set of alternatives (see also McCubbins and Schwartz 1985). Moreover, McKelvey and Schofield (1986) show that if the set of alternatives has an empty boundary, then a structurally stable majority-rule core in two dimensions with n odd is impossible. More generally, if the set of alternatives is open, then a necessary condition for a structurally stable core is that $q \geqslant (W + n)/2$.

Define $[(W + n)/2]$ to be the smallest integer greater than or equal to $(W+n)/2$ and denote $q_m^* = 1/n[(W + n)/2]$, the *McKelvey* number.

Writing the Kramer, Schofield, and McKelvey numbers as q_k^*, q_s^* and q_m^* note that $q_k^* \geqslant q_s^* > q_m^*$ for any dimension and electorate size.

As yet it is not known whether $q \geqslant (W + n)/2$ is sufficient in general to guarantee the possibility of a structurally stable core. In other words there are decision rules that lie between q_s^* and q_m^* for which we are not sure whether a structurally stable core configuration of voter preferences can be constructed.

Note that if a structurally stable core is possible, then the probability (P) of obtaining a core must be nonzero when preferences are randomly distributed. Thus we can rephrase the previous results as

Table 1. Supramajoritarian Decision Rules That Guarantee a Core and Those That Make Possible a Structurally Stable Core

	W (Dimension)										
	1		2		3		4			∞	
Number of Votes	q_k^*	q_s^*	q_k^*	q_s^*	q_k^*	q_s^*	q_k^*	q_s^*	. . .	q_k^*	q_s^*
3	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{3}{3}$	$\frac{3}{3}$	$\frac{3}{3}$			
4	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{4}{4}$	$\frac{4}{4}$	$\frac{4}{4}$			
5	$\frac{3}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{4}{5}$			
7	$\frac{4}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{6}{7}$			
9	$\frac{5}{9}$	$\frac{5}{9}$	$\frac{7}{9}$	$\frac{6}{9}$	$\frac{7}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{8}{9}$			
27	$\frac{14}{27}$	$\frac{14}{27}$	$\frac{19}{27}$	$\frac{15}{27}$	$\frac{21}{27}$	$\frac{19}{27}$	$\frac{22}{27}$	$\frac{21}{27}$			
43	$\frac{22}{43}$	$\frac{22}{43}$	$\frac{29}{43}$	$\frac{23}{43}$	$\frac{33}{43}$	$\frac{30}{43}$	$\frac{35}{43}$	$\frac{33}{43}$			
101	$\frac{51}{101}$	$\frac{51}{101}$	$\frac{68}{101}$	$\frac{52}{101}$	$\frac{76}{101}$	$\frac{68}{101}$	$\frac{81}{101}$	$\frac{77}{101}$			
501	$\frac{251}{501}$	$\frac{251}{501}$	$\frac{335}{501}$	$\frac{252}{501}$	$\frac{376}{501}$	$\frac{335}{501}$	$\frac{401}{501}$	$\frac{377}{501}$			
∞	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{4}{5}$	$\frac{3}{4}$	. . .	1	1

Note: q_k^* is the Kramer number, the decision rule that guarantees that a core will exist; q_s^* is the Schofield number, the decision rule such that a structurally stable core is possible for some configuration of voter preferences.

follows: P(core / $q^* \geqslant q_k^*$) = 1; P(core / $q^* \geqslant q_s^*$) > 0; and P(core / $q^* < q_m^*$) = 0.

In Table 1, for given values of n and W, we show the Kramer number (i.e., the values of q^* sufficient for a core) and the Schofield number (i.e., the values of q^* for which it is always possible to find a set of preferences with a structurally stable core). For example, if there are nine voters, then in two dimensions a core is guaranteed if $q = 7$ (i.e., using a seven-ninths rule), while a structurally stable core is possible if $q = 6$ (i.e., a two-thirds rule). For the case of four voters in two dimensions, a three-fourths rule guarantees a core. In like manner we may read out results from Table 1 for any given values of W, n, and q. Note that, as indicated above, $q_k^* \geqslant q_s^*$ for all W and n.

We shall provide a graphical illustration of a game without a core, a game with a structurally unstable core, and a game with a core that is structurally stable. First, however, we look at some additional results for collective-decision rules in which not all players are weighted equally.

Weighted Voting Games

Most political systems do not use majority rule for all decisions; not only do they require extraordinary majorities (q^* rules) for certain kinds of decisions (e.g., in situations ranging from constitutional changes to confirmation of ambassadorial appointments), but they also make use of hierarchies of committees or constituencies and veto and override procedures. Results similar to those in the theorems given above can be obtained for these more general kinds of voting rules (for work in this direction, see Schofield n.d.[a]). We provide below some important results for the class of weighted voting games.

In a *weighted voting game* each player, i, has a weight, p_i, and there is a quorum rule, q^*, such that a motion passes only if the sum of the weights of the players who support it is at least q^*. For convenience, we let $\Sigma p_i = 1$. For example, we may have a five-voter game where weights are 4/11, 3/11, 2/11, 1/11, and 1/11, respectively, with a simple majority, six-elevenths, needed for passage, or we might increase the number of votes needed for passage, say, to eight-elevenths. The first game is commonly denoted (6;4,3,2,1,1), and the second (8;4,3,2,1,1). In the first game, the extension of Greenberg's Theorem (Greenberg 1979) by Schofield (1985) shows that a core can only be guaranteed in one dimension but that a structurally stable core exists in two dimensions. (See Theorem 5 below.) In the second game, the player with a 4/11 weight belongs to every winning coalition, and so that player's most preferred point is the core (see Theorem 4 below).

Weighted voting games are a much more general concept than they might first appear. Almost all multitiered representation systems with veto power or veto power with override that are in common use can be reexpressed as weighted voting games (Brams 1975; Straffin 1980). For example, in the UN Security Council, the Big Five (U.S., USSR, Great Britain, France, China) have only one vote apiece but they also have veto power (a negative vote cast by any of them defeats any motion); the 10 other members of the Security Council have 1 vote, and, absent a veto, 9 of 15 votes are needed for passage. This voting situation can be reexpressed in weighted-voting terms as (39;7,7,7,7,7,1,1,1,1,1,1,1,1,1,1). Note that we now have a thirty-nine forty-fifths q^* rule (Lucas 1976). To see that this weighted voting game is equivalent to the five-power veto, merely note that a bill can pass if and only if at least 9 countries vote for it, including all 5 of the big powers. Call a game with a veto player a *collegial game.*

In illustrating the existence of a core, it is often convenient to assume that each voter has preferences that can be represented in terms of the distance from the voter's ideal point. Call such a situation a *Euclidean voting game.* In general, when the core exists, it will be the *convex hull* of some set of voters' ideal points (i.e., the smallest convex set containing the particular points).

There is a simple result for weighted voting games that are collegial.

THEOREM 4. *All collegial weighted voting games have a core. If voter preferences are Euclidean, then the core contains the convex hull of the vetoers' ideal points. (In particular, games with a single veto player have the core located at that player's ideal point.)*

This result follows straightforwardly from Schofield (1985, Corollary 4.3.8).

For noncollegial weighted voting rules, a quite general result can be stated. First, a definition is essential.

DEFINITION. *The Nakamura number, N is the cardinality of the smallest set of winning coalitions with the property* *that the intersection of its members is empty* (Nakamura 1979; Schofield 1984a).

Since this definition is not at all obvious, an example will help. Consider the five-voter game (6;4,3,2,1,1). Label the players A, B, C, D, and E in that order. The minimal winning coalitions are AB, AC, ADE, BCD, and BCE. Any two minimal winning coalitions have at least one member in common, that is, their intersection is nonempty. Such games are said to be *proper*. If, however, we consider the coalitions {A,B}, {A,C} and {B,C,D}, these three have an empty intersection. With three minimal winning coalitions we can get a nonempty intersection, but with two coalitions the intersection is nonempty. Hence the Nakamura number for this example is 3.

THEOREM 5. *Let N be the Nakamura number. If $N \geqslant W + 2$ then the core is nonempty.*

For a proof see Schofield 1984a.

To see how Theorem 5 generalizes Theorem 1, consider a q-rule with $n = 5$ and $q = 4$. The winning coalitions {ABCD}, {ABCE}, {ABDE}, {ACDE}, and {BCDE} do not intersect, and so $N = 5$. By Theorem 5 a core is guaranteed only if $W \leqslant 3$. Greenberg's result (Theorem 1) requires that $W < 4/(5 - 4)$; that is $W \leqslant 3$.

For weighted voting games (and other more complex games), the Nakamura number tells us in what dimensions it will be possible to construct a core. The higher the Nakamura number, the higher the dimensionality in which outcomes of the game are predictable. For example, if $N = 3$, then a core can exist in only one dimension. By convention, if a game has a veto player, then we shall define the Nakamura number to be infinity. Hence, if a game has a veto player, by Theorem 5 it always has a core regardless of the dimensionality of the space; that is, Theorem 4 can be taken to be a direct corollary of Theorem 5.

One important way in which a core can be created in weighted voting games is if certain coalitions are prohibited. This can have the effect at making the game collegial. For example, take the game (6;4,3,2,1,1) that we previously mentioned. If we now require that the only coalitions that can form are those that are connected along the *ABCDE* dimension (i.e., if, say, A and C are in a coalition, then B, who is between them, must also be in that coalition) then the only minimal winning (connected) coalitions are {A,B} and {B,C,D} (see Axelrod 1970 and Grofman 1982). Hence the game is now collegial. Let us take a further example (9;4,3,2,1,1,1,1): if we imposed a connectedness requirement, then the only feasible coalitions would be {A,B,C} and {B,C,D,E,F,G} since most of the (minimal) winning coalitions are not connected. Again, we would have a collegial voting game and thus a core.

COROLLARY 1 TO THEOREM 5. *If the only feasible winning coalitions are those that are connected in some given dimension and the game is proper (i.e., the complement of any winning coalition is losing), then there always exists a core.*

Outline of Proof. Since the game is proper, the Nakamura number is at least 3, so by Theorem 5 there always exists a core in one dimension (as in Black 1958). But then there exists at least one player who belongs to every connected feasible winning coalition. Hence the game is collegial, and by Theorem 4 the core is nonempty. QED

Many examples where certain coalitions are "ruled out" occur in real life; for instance, the juxtaposition of ideological extremes in the same coalition is quite unlikely and certain parties (e.g., the

Communists and the far right in many European countries) are not regarded as possible coalition partners. Also, some parties choose not to be part of the governing coalition. Reducing the number of relevant players in the set of feasible winning coalitions increases the likelihood that winning coalitions will intersect. That is to say, the existence of parties who choose to remain outside the government may increase the size of the Nakamura number and thus promote stability. More generally, the fewer the parties who enter the coalition game, the more likely the game is to be collegial. This line of argument indicates that a multiplicity of parties intent on entering government can make it less likely that there will be a core —and the absence of a core makes for political instability in the form of an absence of enforceable political consensus.

Stability may also occur if groups in a weighted-voting-rule setting (of which simple-majority rule is a special case) operate under a de facto rule that is more than the de jure rule. Then, such a consensus-oriented norm (even though slight) may allow a core to exist. For example, while (8;4,3,2,1,1,1,1) need not have a core in two dimensions, (9;4,3,2,1,1,1,1) does, as we shall show below. Such pressure toward larger than minimal majority coalitions may occur if certain decisions, such as veto override, require more than simple majority or if coalitional allegiances are uncertain or simply if social norms exist that lead voters to try to seek consensus in order to avoid conflict. More generally, Schofield (1986) has conjectured that the imposition of multiple tiers or other forms of complex (but noncollegial) weighted voting rules will make likely the existence of a structurally stable core even in issue spaces of relatively high dimensions.

When there are several key actors who are found in almost all minimal winning coalitions, the Nakamura number will be relatively high, so we will be able to construct a core in two or perhaps even three or more dimensions. We provide an example of such a game: (9;4,3,2,1,1,1,1) with players A, B, C, D, E, F, G. Here the minimal winning coalitions are ABC, ABDE, ABDF, ABDG, ABEF, ABEG, ABFG, ACDEF, ACDEG, ACDFG, ACEFG, and BCDEFG. Any two or any three minimal winning coalitions have a nonempty intersection. But, for example, {A,B,C}, {A,B,D,E}, {A,C,D,E,F} and {B,C,D,E,E,G} have an empty intersection. Hence the Nakamura number is 4. Since $4 - 2 = 2$, there must exist a nonempty core in two dimensions.

One further observation worth emphasizing is that, for Euclidean preferences, in the highest dimension that a structurally stable core for a particular game is possible, the core may be at the preferred point of one of the players, that is, the minmax set will be a point. In general that player will be the most powerful player in the game. We now discuss an example to illustrate this possibility and also provide an example of a weighted voting game that occurred in Belgium in 1978 for which no majority-rule core exists. We then show how a structurally stable core for this game can be created in two dimensions.

Applications of Basic Results to Experimental Games and Cabinet Formation

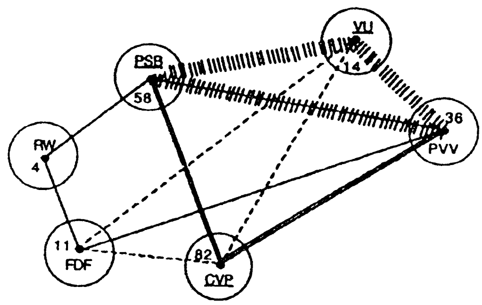
Our results suggest that at least when there are only a few dimensions (e.g., $W = 2$ or 3), social-decision processes are apt to result in stable choice within a delimited area of the space if groups use a supramajoritarian procedure or a weighted voting rule. For example, in each of three games with equally weighted players where there is no majority-rule core—a game in two dimensions with five players used in experiments by Wilson and Herzberg (1984, Fig. 3), a game in two dimensions with seven players considered in Kramer (1977, 321), and another two-dimensional five-voter game, Game 3 of the experiments by Fiorina and Plott (1978, 579, Fig. 3)—the minmax set appears to have useful predictive powers.[2] Also, in all three games the minmax set is relatively small.[3] In each game we find that a rule of one more than simple majority gives rise to a core, since 4/5 > 2/3 and 5/7 > 2/3 (see Corollary 2 to Theorem 1).

We now wish briefly to consider the results of the 1978 Belgian general election. We show in Figure 1 a two-dimensional spatial map of the Belgian party system in 1978 based on data prepared by Hearl under the auspices of the European Party Manifestos Project directed by Ian Budge (Budge, Robertson and Hearl 1987).

The core cannot be guaranteed in two-space in that election since three of the minimal coalitions, namely {CVP,PSB}, {CVP,PVV}, and {VU,PVV,PSB} have an empty intersection. Hence the Nakamura number is 3, and by Theorem 5, the core is guaranteed only in one dimension. For purposes of exposition we shall assume that each party has Euclidean preferences based on its ideal point. In Figure 1 we associate to each winning coalition the convex set generated from its members' ideal points. For the core to be nonempty, the five relevant sets must intersect, which they clearly do not.

Another way to see this is to use the notion of "median lines." Consider the line through the VU and CVP positions. The parties on and to the "left" of this line comprise a winning coalition with 169 seats, while the parties on and to the right of the line also comprise a winning coalition with 132 seats. Such a line is a median line, and for the core to be nonempty it must lie on the intersection of all median lines. However, the median lines VU-CVP, PVV-FDF and CVP-PSB do not intersect. (For a nontechnical exposition of the geometry, see Feld and Grofman 1987. For an extension of this analysis to a situation with non-Euclidean preferences, see McKelvey and Schofield 1987).

Figure 1. Belgian Political-Party Space and Seats Held in 1978



PSB	Socialist party
VU	Volksunie
FDF	Francophone Front
RW	Rassemblement Wallon
PVV	Liberals
CVP	Christelijke Volksparti

Minimal Winning Coalitions

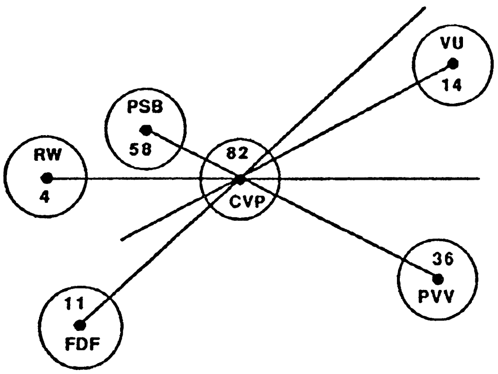
CVP, PSB	140 seats
CVP, PVV	118 seats
RW, FDF, PVV, PSB	110 seats
PSB, VU, PVV	108 seats
FDF, CVP, VU	107 seats

Source: Adapted from Derek Hearl, Belgium 1946–1981, in *Ideology, Strategy, and Party Change,* ed. Ian Budge, David Robertson, and Hearl (Cambridge: Cambridge University Press, 1987).

Note: Underlined parties in the figure are members of the cabinet (PSB and CVP). Independents (I, 3 seats) and Communists (KPB, 4 seats) are not shown. $N = 212$; 107 seats are needed for majority.

However, if we move the CVP to be colinear with the PVV and the PSB, as shown in Figure 2, then there is a core at the location of the CVP. Note, however, that if the CVP position is moved slightly, then the core becomes empty since once again the set of median lines does not intersect at a single point. To see this, note that the line connecting the PSB and PVV remains a median line even though it

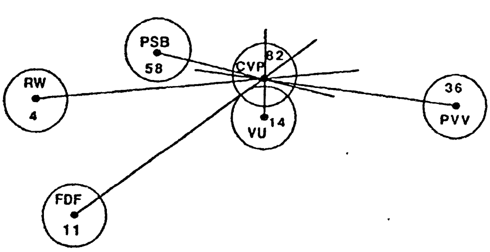
Figure 2. Hypothetical Transformation of Figure 1 to Create a Structurally Unstable Majority-Rule Core



no longer passes through CVP. Thus the core in Figure 2 is structurally unstable. However, we can move party locations so as to create a core. For example, if both the CVP and VU positions are changed, as in Figure 3, then there exists a structurally stable core at the preferred point of the CVP, because now the line between PSB and PVV is not a median line and all median lines pass through CVP.

There is some evidence that the parties tend to change position in this policy space, as though "hunting the core." We can see a possible reason for this phenomenon. In a situation such as represented in Figure 1, no matter which coalition were to form, there always exists another majority group that stands to

Figure 3. Hypothetical Transformation of Figure 1 to Create a Structurally Stable Core



gain by bringing down the government. One would therefore expect governments to be quite short-lived. Indeed, in Belgium in the period since 1971, average coalition-government duration is 16 months (Schofield n.d.[b]). However, as Figures 2 and 3 illustrate, if the party positions were different, then a core could exist. Suppose the core were nonempty but structurally unstable, as in Figure 2. No winning coalition has in fact an incentive to bring down the government, but the smallest error in judgment or perception would lead parties to the belief that there were such an incentive. Only for the structurally stable core situation would one expect relatively long-lived governments.

Short-lived government does impose a cost on parties. If parties are willing to make a trade-off between their preferred policy positions and the advantages accruing from membership of government, then they may be aware of the fact that changing policy position somewhat can in fact lead to a core situation. Moreover, if one party makes the correct type of policy movement, then its preferred point would indeed be at the structurally stable core. In that case the party would know that it would belong to the government coalition and would also expect that the government coalition would be relatively long-lived. Thus the game has a high strategic context as parties become aware of the advantages of finding the structurally stable core.

Instead of moving party locations, another way to obtain a stable core is to impose a supramajoritarian rule on a coalition game. In particular, for Belgium in 1978, if we require 125 or so votes to have a "comfortable" (but not unnecessarily large) majority, then the only minimal winning coalitions (with Communists and independents excluded) are those that include the CVP.

A third way we can create a core is by treating one of the dimensions as more important, and positing that all coalitions will be connected with respect to that dimension. In the example of Figure 1, if we collapse to the x-axis (i.e., look at party projections on that axis), then CVP is in the core.

A point worth noting from the Belgian case that we just examined is that the CVP is the only party whose preferred point can be a structurally stable core *point.* The next largest party, the PSB, with just under 30% of the seats, is too small to be in a structurally stable core position. (See Schofield 1986 for a fuller discussion of this example.) It is evident that this feature endows the CVP with far greater bargaining power than any other party.

To adapt Daalder's (1971) terminology, it is reasonable to identify the Belgian political system as *unipolar.* The same may be said of the Netherlands and Italy, as well as certain other countries. In some countries, on the other hand, the dimensionality of the policy space is such that *two* quite different structurally stable cores may occur, each one at the preferred position of one of two parties. Such a system could be called *bipolar.* This notion could be carried further to identify *multipolar* systems as those where three or more parties may occupy the structurally stable core position. All such parties could be regarded as being of equal power, irrespective of the differences in the actual number of seats they control. Note that this suggestion, if valid, gives a very precise method of formalizing the qualitative analysis of political party systems carried out by Daalder (1971), Sartori (1966) and others. In other words, by specifying how many of the parties in a given system could be core points, we can provide a nonarbitrary classification for party systems as unipolar, bipolar, tripolar, and so on (see Schofield n.d.[a]).

The 1978 Belgium example is intended to be purely illustrative. We hope in further work to pursue the question of adequately modeling the dynamics of party evolution over time (in Belgium and elsewhere) and to trace the link between the existence of core points in party spatial arrays and features of political life such as cabinet stability, voter volatility, and sharpness of ideological conflict.

Discussion

We believe that understanding the conditions under which group choice can be expected to be stable is important. Policy cannot be intelligibly formulated or intelligibly implemented in an environment where decisions can always be revoked and no outcome is final. The results we have presented here suggest one important reason why, in the real world, stable outcomes occur. Outcomes in the minmax set offer the minimum consensus needed to achieve stability. Once an outcome in the minmax set has been picked, it is unlikely to be altered, especially if the core is structurally stable.[4]

Moreover, we believe that outcomes in the minmax set occur even in situations where the *formal* voting rule is one that is insufficient to give rise to a core. The reason for that view is quite simple. In real groups there is a reluctance to permit "bare"-majority decision making. Rather, consensus is sought (Zablocki, 1971, 155–58). Also in the political-party context, bare-minimum majorities are unsafe, because they are too vulnerable to blackmail through threats of defection. Thus, the de facto rule in group decision making is apt to be larger than the de jure rule.[5] This may also help explain why, in experiments on spatial voting games, outcomes may tend to be "tightly clustered" (Fiorina and Plott 1978, 590).

For decision processes where all voters have the same weight, we have shown that stability can occur with relative ease when the dimensionality of the space is low. In this case decision rules very near to bare majority may give rise to a structurally stable core, and a two-thirds-plus rule (in the case of $W = 2$), or a three-fourths-plus rule (in the case of $W = 3$) will *guarantee* a core. We believe this finding can account in part for the popularity of such supramajoritarian rules. We have also shown that, even if the dimensionality of the space is relatively high, then (as with the UN Security Council example) veto rules or other special features may create either collegial voting rules or weighted voting games whose spatial array is such as to give rise to a core. These results significantly ameliorate the pessimism that seemed warranted by the generic instability results for simple-majority rule.

Finally, we have suggested ways in which the structural features of voting games can be used to provide a nonarbitrary classification of party systems as unipolar, bipolar, and so on. Also, we have identified hypotheses about electoral dynamics and cabinet longevity that are based on the spatial array and relative strength of political parties and the conditions of a structurally stable core, which suggests a new and promising line of research on a topic that has been much studied but is still not well understood (Grofman n.d.; Schofield n.d.[a]).

Notes

A portion of this research was conducted while Schofield was the Sherman Fairchild Scholar in the Division of Humanities and Social Sciences, California Institute of Technology. The work has benefitted from conversations with Charles Plott and earlier collaborative research with Richard McKelvey. We are indebted to the staff at the Word Processing Center, School of Social Sciences, University of California, Irvine for typing earlier drafts of this manuscript; to Cheryl Larsson for preparation of the figures; and to Dorothy Gormick for bibliographic assistance. This research was partially supported by NSF Grant no. SES 84-18295 to Schofield, and no. SES 85-06376, Program in Decision and Management Science, to Grofman. It was prepared for presentation at the International Conference on European Cabinet Coalitions, European University Institute, Fiesole, Italy, 1987. Participation at the conference by the first two authors was funded by NSF Grant no. SES 85-21151, Political Science Program.

1. If the convexity and compactness assumptions are dropped, then a "local core" is obtained. See Kramer and Klevorick 1974 and Schofield 1985 for the technical details. That readers grasp the exact nature of the compactness and convexity assumptions is not at all critical for understanding what follows. Readers may act as if all results refer to the familiar Cartesian coordinate space, thereby treating distance between alternatives as if that distance satisfied those properties it would satisfy in ordinary life.

2. Wilson and Herzberg (1984) in their experiments on five-voter spatial games used both simple-majority rule and a four-fifths rule. In both cases the majority of the outcomes was either within the minmax set (10 of 18 cases for the four-fifths rule, 8 of 18 for the three-fifths rule) or very near to it (an additional 6 of 18 cases for the four-fifths rule and an additional 7 of 18 cases for the three-fifths rule). Moreover, in the Wilson and Herzberg (1984) experiments, of the few outcomes that were not in or near the minmax set, half occurred as outcomes that were imposed when time ran out.

In the Fiorina and Plott (1978) experiment of Game 3, while only 3 of 15 observations fall directly within the minmax set, if we enlarge the bounds of the minmax set slightly, we can form a polygon in which 13 of the 15 observations fall. Even the remaining two observations are not so far away from the minmax set. Indeed, the mean outcome (45,62) in this voting game is within the minmax set (albeit barely). Because Fiorina and Plott (1978) counted success only when an outcome was inside the minmax set, they were not impressed with the fit of the minmax set to their data (cf. Ferejohn and Fiorina 1975). Although they concluded that "the minmax set does best among the models considered" (Fiorina and Plott 1978, 592), they then went on to say that was "only because of very weak competitors." However, there were 14 of these so-called "weak" competitors and we believe that *visually*—the most powerful statistical test known—the Fiorina and Plott experiments show the minmax set to fit the data quite well (see Fiorina and Plott 1978, 589, Fig. 8).

Also, we should note that other explanations of the location of outcomes in the Fiorina and Plott (1978) experiments on games without a core have been offered by Margolis (1982) and Grofman et al. (1987). A variety of models to predict expected outcomes in such games have been proposed. See Fiorina and Plott (1978) and Straffin and Grofman (1984) for relatively nontechnical reviews.

Wilson and Herzberg (1984) also ran experiments on the same five-voter two-dimensional game that we discussed, but now with one player (Player 2) given a veto. Though Wilson and Herzberg (1984) do not treat the game this way, it can be reexpressed as the weighted voting game (5;3,1,1,1,1). This game is collegial (i.e., there is a veto player who is in all minimal winning coalitions). Giving Player 2 a veto power shifts the location of the core away from the center of the space and to the ideal point of that player. In the Wilson and Herzberg (1984) data, 5 of 11 outcomes fall near the core point and another 3 outcomes lie in the inner polygon just below it. While the fit of the core is far from ideal, even here it appears to have considerable predictive power. We conjecture that the reason that the veto player does not obtain his ideal point as outcome is that motion toward the veto player requires continued shifts in the composition of the winning coalition.

3. For games where players are equally weighted there is reason to believe that the area of the minmax set will shrink as the number of voters increases (Demange 1982; cf. Feld, Grofman, and Miller 1985).

4. Of course, there are decision costs to searching out new alternatives; thus we may never get to the minmax set. The work of Kramer (1977) provides one plausible agenda process that will lead us toward the minmax set. Kramer (1977) proposed that at each iteration of a sequential decision process actors would seek out the alternative that defeats the status quo by the greatest margin. Kramer's discussion is couched in terms of two-candidate competition, but it can just as easily be restated as a process directed toward finding consensus (i.e., one seeking outcomes that beat the status quo by the largest margin).

5. An exception would be where the group is operating under a de jure unanimity rule. Here the group pressure would probably act so as to exert pressure on "deviants" from the group consensus to reduce the effective rule to prevent minority veto power. This appears to be true in juries (Grofman 1981).

References

Axelrod, Robert. 1970. *Conflict of Interest: A Theory of Divergent Goals with Applications to Politics.* Chicago: Markham.

Banks, Jeffrey S. 1985. Sophisticated Voting Outcomes and Agenda Control. *Social Choice and Welfare* 1:295–306.

Banks, Jeffrey S., and Georges Bordes. 1987. Voting Games, Indifference and Consistent Sequential Choice Rules. University of Rochester. Typescript.

Bell, Colin E. 1978. What Happens When Majority Rule Breaks Down: Some Probability Calculations. *Public Choice* 33:121–26.

Black, Duncan. 1958. *The Theory of Committees and Elections.* Cambridge: Cambridge University Press.

Brams, Steven J. 1975. *Game Theory and Politics.* New York: Free Press.

Budge, Ian, David Robertson, and Derek Hearl, eds. 1987. *Ideology, Strategy and Party Change: A Spatial Analysis of Post-War Electoral Programmes in Nineteen Democracies.* Cambridge: Cambridge University Press.

Cohen, Linda R. 1979. Cyclic Sets in Multidimensional Voting Models. *Journal of Economic Theory* 20:1–12.

Cohen, Linda R., and Steven Matthews. 1980. Constrained Plott Equilibria, Directional Equilibria, and Global Cycling Sets. *Review of Economic Studies* 47:975–86.

Copeland, Arthur H. 1951. A Reasonable Social Welfare Function. University of Michigan. Typescript.

Craven, John. 1971. Majority Voting and Social Choice. *Review of Economic Studies* 38:265–67.

Daalder, Hans. 1971. Cabinets and Party Systems in the Smaller European Democracies. *Acta Politica* 6:282–303.

Davis, Otto, Morris H. DeGroot, and Melvin J. Hinich. 1972. Social Preference Orderings and Majority Rule. *Econometrica* 40:147–57.

Demange, Gabrielle. 1982. A Limit Theorem on the Minmax Set. *Journal of Mathematical Economics* 9:145–62.

Feld, Scott L., and Bernard Grofman. 1986. Agenda Constraints. University of California, Irvine. Typescript.

Feld, Scott L., and Bernard Grofman. 1987. Necessary and Sufficient Conditions for a Majority Winner in *n*-Dimensional Spatial Voting Games: An Intuitive Geometric Approach. *American Journal of Political Science* 31:709–28.

Feld, Scott L., and Bernard Grofman. N.d. Majority Rule Outcomes and the Structure of Debate in One-Issue-at-a-Time Decision Making. *Public Choice.* Forthcoming.

Feld, Scott L., Bernard Grofman, Richard Hartley, Mark Kilgour, Nicholas Miller, and Nicholas Noviello. N.d. The Uncovered Set in the Spatial Context. *Theory and Decision.* Forthcoming.

Feld, Scott L., Bernard Grofman, and Nicholas Miller. 1985. Cycle Lengths and Other Features of Majority Preference in the Spatial Context. Paper presented at the Weingart Conference on Models of Voting, California Institute of Technology, Pasadena.

Ferejohn, John A., and Morris Fiorina. 1975. Closeness Counts Only in Horseshoes and Dancing. *American Political Science Review* 69:920–25.

Ferejohn, John A., Morris Fiorina, and Edward Packel. 1980. Nonequilibrium Solutions for Legislative Systems. *Behavioral Science* 25: 140–48.

Ferejohn, John A., and David Grether. 1974. On a

Class of Rational Social Decision Procedures. *Journal of Economic Theory* 8:471–81.
Ferejohn, John A., Richard D. McKelvey, and Edward Packel. 1984. Limiting Distributions for Continuous State Markov Models. *Social Choice and Welfare* 1:45–67.
Fiorina, Morris, and Charles Plott. 1978. Committee Decisions under Majority Rule: An Experimental Study. *American Political Science Review* 72: 575–98.
Glazer, Amihai, Bernard Grofman, and Guillermo Owen. 1985. Imperfect Information Models of Spatial Competition in Electoral Politics. Paper presented at the annual meeting of the Public Choice Society, New Orleans.
Greenberg, Joseph. 1979. Consistent Majority Rules over Compact Sets of Alternatives. *Econometrica* 47:627–36.
Grofman, Bernard. 1972. A Note on Some Generalizations of the Paradox of Cyclical Majorities. *Public Choice* 12:113–14.
Grofman, Bernard. 1981. Mathematical Models of Jury/Juror Decision Making. In *Perspectives in Law and Psychology*, Vol. 2, *The Jury, Judicial, and Trial Processes*, ed. Bruce D. Sales. New York: Plenum.
Grofman, Bernard. 1982. A Dynamic Model of Protocoalition Formation. *Behavioral Science* 27:77–90.
Grofman, Bernard. N.d. Applications of a Dynamic Model of Protocoalition Formation. In *Coalitions in Western Europe*, ed. Norman Schofield. Norwell, MA: Kluwer-Nijhoff. Forthcoming.
Grofman, Bernard, Guillermo Owen, Nicholas Noviello and Amihai Glazer. 1987. Stability and Centrality of Legislative Choice in the Spatial Context. *American Political Science Review* 81: 538–53.
Grofman, Bernard, and Carole Uhlaner. 1985. Metapreferences and the Reasons for Stability in Social Choice. Thoughts on Broadening and Clarifying the Debate. *Theory and Decision* 19: 31–50.
Henriet, Dominique. 1984. The Copeland Choice Function: An Axiomatic Characterization. Laboratoire D'Econometrie de l'Ecole Polytechnic, Paris. Typescript.
Kramer, Gerald H. 1977. Dynamical Model of Political Equilibrium. *Journal of Economic Theory* 16:310–34.
Kramer, Gerald H., and Alvin K. Klevorick. 1974. Existence of a "Local" Cooperative Equilibrium in a Class of Voting Games. *Review of Economic Studies* 41:539–48.
Lucas, William F. 1976. *Measuring Power in Weighted Voting Systems, Case Studies in Applied Mathematics.* Washington: Mathematics Association of America.
McCubbins, Mathew D., and Thomas Schwartz. 1985. The Politics of Flatland. *Public Choice* 46: 45–60.
McKelvey, Richard D. 1976. Intransitivities in Multidimensional Voting Models and Some Implications for Agenda Control. *Journal of Economic Theory* 12:472–82.
McKelvey, Richard D. 1979. General Conditions for Global Intransitivities in Formal Voting Models. *Econometrica* 47:1085–1112.
McKelvey, Richard D. 1986. Covering, Dominance, and Institution Free Properties of Social Choice. *American Journal of Political Science* 30: 282–314.
McKelvey, Richard, and Norman Schofield. 1986. Structural Instability of the Core. *Journal of Mathematical Economics* 15:179–98.
McKelvey, Richard, and Norman Schofield. 1987. Generalized Symmetry Conditions at a Core Point. *Econometrica* 55:923–33.
McKelvey, Richard D., and Richard E. Wendell. 1976. Voting Equilibria in Multidimensional Choice Spaces. *Mathematics of Operations Research* 1:144–58.
Margolis, Howard. 1982. *Selfishness, Altruism, and Rationality: A Theory of Social Choice.* New York: Cambridge University Press.
Matthews, S. A. 1980. Pairwise Symmetry Conditions for Voting Equilibria. *International Journal of Game Theory* 9:141–56.
Miller, Nicholas. 1977. Graph-Theoretical Approaches to Theory of Voting. *American Journal of Political Science* 21:769–803.
Miller, Nicholas. 1980. A New Solution Set for Tournaments and Majority Voting. *American Journal of Political Science* 24:68–96.
Miller, Nicholas. 1983. Pluralism and Social Choice. *American Political Science Review* 77:734–47.
Miller, Nicholas, Bernard Grofman, and Scott L. Feld. 1985. Cycle Avoiding Trajectories and the Uncovered Set. Paper presented at the Weingart Conference on Models of Voting, California Institute of Technology, Pasadena.
Moulin, Herve. 1984. Choosing from a Tournament. Virginia Polytechnic Institute and State University. Typescript.
Nakamura, Kenjiro. 1979. The Vetoers in a Simple Game with Ordinal Preferences. *International Journal of Game Theory* 8:55–61.
Owen, Guillermo, and Lloyd Shapley. 1985. Results on the Strong Point and the Modified Shapley Value. Naval Postgraduate School, Monterey, CA. Typescript.
Packel, Edward W. 1981. A Stochastic Solution Concept for *n*-Person Games. *Mathematics of Operations Research* 6:349–62.
Plott, Charles R. 1967. A Notion of Equilibrium and Its Possibility under Majority Rule. *American Economic Review* 57:787–806.
Riker, William H. 1980. Implications from the Disequilibrium of Majority Rule for the Study of Institutions. *American Political Science Review*

74:432–46.
Riker, William H. 1982. *Liberalism versus Populism.* San Francisco: W. H. Freeman.
Romer, Thomas, and Howard Rosenthal. 1978. Political Resource Allocation, Controlled Agendas and the Status Quo. *Public Choice* 33:27–43.
Sartori, Giovanni. 1966. European Political Parties: The Case of Polarized Pluralism. In *Political Parties and Political Development,* ed. Joseph La Palombara and Myron Weiner. Princeton: Princeton University Press.
Schofield, Norman. 1978a. Instability of Simple Dynamic Games. *Review of Economic Studies* 65:575–94.
Schofield, Norman. 1978b. The Theory of Dynamic Games. In *Game Theory and Political Science,* ed. Peter C. Ordeshook. New York: New York University Press.
Schofield, Norman. 1980. Generic Properties of Simple Bergson-Samuelson Welfare Functions. *Journal of Mathematical Economics* 7:175–92.
Schofield, Norman. 1983a. Generic Instability of Majority Rule. *Review of Economic Studies* 50: 696–705.
Schofield, Norman. 1983b. Equilibria in Simple Dynamic Games. In *Social Choice and Welfare,* ed. P. K. Pattanaik and M. Salles. Amsterdam: North Holland.
Schofield, Norman. 1984a. Social Equilibrium and Cycles on Compact Sets. *Journal of Economic Theory* 33:59–71.
Schofield, Norman. 1984b. Classification Theorem for Smooth Social Choice on a Manifold. *Social Choice and Welfare* 1:187–210.
Schofield, Norman. 1984c. Existence of Equilibrium on a Manifold. *Mathematics of Operations Research* 9:545–57.
Schofield, Norman. 1985. *Social Choice and Democracy.* Heidelberg: Springer Verlag.
Schofield, Norman. 1986. Existence of a "Structurally Stable" Equilibrium for a Noncollegial Voting Rule. *Public Choice* 51:267–84.
Schofield, Norman. N.d.(a). The Spatial Theory of Democracy and Coalition Governments in Europe. In *Coalitions in Western Europe,* ed. author. Norwell, MA: Kluwer-Nijhoff. Forthcoming.
Schofield, Norman. N.d.(b). Stability of Coalition Governments in Western Europe: 1945–1986. *European Journal of Political Economy.* Forthcoming.
Shepsle, Kenneth A. 1979. Institutional Arrangements and Equilibrium in Multidimensional Voting Models. *American Journal of Political Science* 23:27–59.
Shepsle, Kenneth A., and Barry R. Weingast. 1981. Structure-Induced Equilibrium and Legislative Choice. *Public Choice* 37:503–19.
Shepsle, Kenneth A., and Barry R. Weingast. 1984. Uncovered Sets and Sophisticated Voting Outcomes with Implications for Agenda Institutions. *American Journal of Political Science* 28:49–74.
Simpson, Phillip. 1969. On Defining Areas of Voter Bias: Professor Tullock on Stable Voting. *Quarterly Journal of Economics* 83:478–90.
Sloss, Judith. 1973. Stable Outcomes in Majority Rule Voting Games. *Public Choice* 15:19–48.
Slutsky, Stephen. 1979. Equilibrium under α-Majority Voting. *Econometrica* 47:1113–25.
Straffin, Philip. 1980. *Topics in the Theory of Voting.* Cambridge, MA: Birkhauser.
Straffin, Philip, and Bernard Grofman. 1984. Parliamentary Coalitions: A Tour of Models. *Mathematics Magazine* 57:259–74.
Strnad, James F. 1985. The Structure of Continuous-Valued Neutral Monotonic Social Functions. *Social Choice and Welfare* 2:181–95.
Wilson, Rick, and Roberta Herzberg. 1984. Voting Is Only a Block Away: Theory and Experiments on Blocking Coalitions. Paper presented at the annual meeting of the Public Choice Society, Phoenix.
Wuffle, A, Scott L. Feld, Guillermo Owen, and Bernard Grofman. N.d. Finagle's Law and the Finagle Point. *American Journal of Political Science.* Forthcoming.
Zablocki, Benjamin. 1971. *The Joyful Community.* New York: Pelican.

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