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ON THE POSSIBILITY OF FAITHFULLY REPRESENTATIVE COMMITTEES

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By faithful representation *we mean the delegation of decision making to a relatively small committee that, using a weighted voting rule, will for each pair of alternatives make sincere choices identical to those that would be made by the society as a whole, and with the same vote margins. We show that for any society, no matter how large, faithful representation is possible by a committee with no more than* $m(m-1)/2$ *members, where* m *is the number of alternatives. We also show that for any society, no matter how nonideological the bulk of its electorate, social preferences can be faithfully represented by a committee whose members all have single-peaked or single-troughed preferences. Thus, all societies can be faithfully represented by a committee whose members see the world in unidimensional terms—that is, representatives can share a coherent ideological perspective even though the electorates they represent lack such a perspective. We further show that the usual mechanisms of proportional representation and the modified form of proportional representation recently proposed by Chamberlin and Courant (1983) do not guarantee faithful representation, and we discuss mechanisms that may provide faithful representation, even in a context in which new alternatives can arise.*

Faithful representation has long been advocated as the ideal form of representation. As Chamberlin and Courant (1983, p. 718) characterize this view, "Representative democracy is at best a working model of direct democracy and is most successful when it generates . . . decisions as close as possible to those that would be generated in a direct democracy."[1] We shall show that, contrary to what common sense might expect, even societies having millions of voters (and whose voters have millions of distinct preference orderings over the various alternatives) can often be *faithfully represented* by a relatively small group of representatives: i.e., the representatives' weighted sincere votes will perfectly mirror the directions of all pairwise preferences between alternatives, and the margins by which the voters prefer one alternative over another. Moreover, we shall show that the means of implementing such faithful representation is not any of the usual forms of proportional representation.

We believe that the idea of faithful representation is of theoretical and practical importance. Direct democracy is rare; usually voters must choose a committee (e.g., a legislature) to decide on their behalf (Black, 1958). The size of electoral margins may influence policies over and above the simple directionality of vote outcomes. Vote margins may affect future policy proposals as well as perceptions of the likelihood of reversal of present decisions.

There are many instances of the importance of vote margins. For example, New York State school budgets are voted on by residents of school districts, but the subsequently proposed budgets are determined by the school boards. Feld and Grossman (1984) show that the size of the budget referendum majorities affects the subsequently proposed budgets over and above the decision to either pass or defeat a particular proposed budget. Journalists, political scientists, and politicians themselves commonly use an elected official's margin of victory as an indication of the size of his or her political mandate.[2] Observers and participants in legislative decisions use legislative margins as an indication of the stability or changeability of a particular policy direction (Grofman and Uhlaner, 1985). A strong notion of representation requires that a committee not only make the same decision as would the entire set of voters (we shall refer to this as *order representation*), but also that it would make decisions by the same margins, so as to indicate the strength as well as the direction of preference of the voters. We shall refer to the latter form as *faithful representation.*[3]

It might appear that faithful representation is impossible. Certainly the social choice literature is filled with impossibility results, and faithful representation seems to be such a strong requirement that an impossibility result would seem almost inevitable. As we shall see, however, although the usual forms of representation (e.g., plurality or list proportional representation) cannot guarantee faithful representation, it is possible to devise a weighted voting scheme that will in general provide faithful representation. A *weighted voting scheme* is one in which each committee member has multiple votes to cast. Such schemes are used in proxy-voting for corporate boards of directors, in voting for labor union officers in labor federations such as the AFL-CIO, in most county governments in the State of New York, in the European Economic Community, and in various other organizations (Brams, 1975; Grofman and Scarrow, 1981, 1982; Straffin, 1980). Moreover, the committee size needed to implement a weighted voting scheme to carry out faithful representation is remarkably small; and the committee can be constructed so that all of its members will have single-peaked or single-troughed preferences with respect to any given linear ordering, even though in the society as a whole only a handful of voters have orderings that are single-peaked or single-troughed with respect to that range of alternatives.[4] Thus, we have the quite counterintuitive result that we can have faithful representation from a legislature whose members have well-defined ideologies underlying their preferences (i.e., they have single-peaked or single-troughed preferences with respect to a given linear ordering) even though the society they faithfully represent is not at all ideological in character.[5] Furthermore, a committee that is selected to be faithfully representative for a relatively small set of alternatives can, under reasonable circumstances, continue to be faithfully representative of the electorate over a much wider set of alternatives that may subsequently arise.

Proportional Representation is Not Faithful Representation

Proportional representation in multimember districts is generally considered an improvement over single-member district-based plurality (or majority runoff) representation, because the legislature or committee chosen is expected to mirror more closely the composition (and thus the preferences) of the electorate. However, list proportional representation (list PR), the most common form of PR, is generally inadequate for faithfully representing pairwise vote margins for more than two alternatives, especially when multiple alternatives cannot be meaningfully ordered along a single continuum.

We shall treat a political party as equivalent to a given preference ordering over alternatives (i.e., each party representative will be expected to vote in accord with the same linear ordering). List PR is inadequate to provide faithful representation, because there are generally too few parties to represent the variety of preference orderings in the electorate, and voters are forced to choose the closest approximation to their own ordering (e.g., relying only on first preferences); the process of approximation can severely distort the outcomes. For example, consider a situation with three alternatives—*A*, *B*, and *C*—and three parties—X, Y, and Z—each associated with the modal preference orderings as follows: X:*ABC*, Y:*BCA*, Z:*CAB*.

Now consider that the population has the full range of preference orderings in the proportions shown in the example below:

Voter Preference Orderings	*Votes*	*Most Likely Party Choice*
ABC	23	X
BCA	20	Y
CAB	20	Z
CBA	15	Z
BAC	17	Y
ACB	5	X

In this example we also have shown the party that most closely approximates the preferences of voters with each preference ordering. If parties are given votes in proportion to their choice by voters, then X, Y, and Z have 28, 37, and 35 votes, respectively. No party has a majority on its own; their votes on issues result in the collective choices of *A* over *B* by 26, *B* over *C* by 30, and *C* over *A* by 44. This is in contrast to very different preferences of the voters, *B* over *A* by 4, *B* over *C* by 20, and *C* over *A* by 10. One of three decisions is made in the wrong direction; all of the voter margins are incorrect, and the majority preference ordering of the voters is *BCA*, while that of their chosen representatives is a cycle.

This simple example shows the inadequacy of list proportional representation. With additional political parties the system might work better, but there is no guarantee that the most representative set of parties will emerge; it is probably realistic to expect that there will be a few parties representing the modal preferences, just as in our example. Even if other parties are included, the choices and vote margins might still be distorted as long as some voters are forced to choose a party that does not perfectly represent them. These findings seem to suggest that proper representation requires as many different representatives as there are preference orderings in the population. Indeed, at least one knowledgeable political scientist suggested to us that the search for faithfully representative committees was apt to be fruitless for exactly this reason: He conjectured that if the society was large, only a large committee (presumably one representing most of the preference orderings in the population) would be able to represent it faithfully. For *m* alternatives, there are *m*! different possible preference orderings; for 12 alternatives, everyone in the United States could have a different preference ordering. Even if people had only ideologically consistent orderings (i.e., single-peaked preferences over the continuum defined by a particular linear ordering), there would still be 2,048 different preference orderings. If any such large number of preference orderings had to be represented directly in a committee, a system of faithful representation would be unworkable. Fortunately, we can show that faithful representation can always be achieved with a relatively small number of representatives (e.g., a maximum of 66 in the case of 12 alternatives), no matter how many voters there are.

Basic Theorems on Faithful Representation

McGarvey's (1953) interest in demonstrating the possibilities of vote paradoxes led him to show that any possible set of pairwise preferences over m alternatives (and thus in particular any cycle) can be reproduced with at most $m(m - 1)$ different preference orderings. Since the task of constructing vote paradoxes is of little more than technical interest, this result is almost unknown even in the social choice literature. However, it is an immediate corollary of McGarvey's theorem:

COROLLARY TO MCGARVEY'S THEOREM. *For any society whose members have preference orderings over m alternatives, there exists a set of at most m(m − 1) representatives whose majority choices between every pair of alternatives perfectly mimic the directionality of the pairwise majority choices of the society as a whole.*

We shall call a committee that mirrors the pairwise majority choices of a larger group an *order-representative* committee. McGarvey's result tells us that we can obtain an order-representative committee with at most $m(m - 1)$ representatives. Order representation is often relatively simple to satisfy. For example, if a society has a transitive ordering over alternatives, then a single individual with that set of preferences is order representative for that society. As McGarvey shows, even if the group majority preferences are not transitive, then one can generate a pair of preference orderings that produces the majority for any alternative over any other without affecting the results of any other binary choices. This can be seen by considering five alternatives, A through E, and the decision between A and B. If someone has ordering *ABCDE* and someone else has ordering *EDCAB*, then together they prefer A to B and cancel out each other's preferences (they are effectively indifferent) over all other binary choices.

This analysis contains the essential logic needed to extend McGarvey's results from order representation to faithful representation. McGarvey's argument can be easily extended to show that faithful representation (of any margins) is possible by weighting each member of the pair of preference orderings used to produce a binary choice (*ABCDE* and *EDCAB* in the example above) by half the margin; then, together, they produce the particular margin while all other representatives cancel out one another on this binary choice. This extension of McGarvey's Theorem indicates that a maximum of $m(m - 1)$ different preference orderings are sufficient to represent any group faithfully. We can, however, prove the stronger result that this same representation can always be achieved with half that many, or $m(m - 1)/2$, different preference orderings.

THEOREM 1. *A maximum of m(m − 1)/2 different (weighted) preference orderings (i.e., a set of m(m − 1)/2 committee members voting sincerely, each in accord with a distinct preference ordering) is sufficient to represent the margins of pairwise preferences over m alternatives that emerge from any* *group—that is, to faithfully represent the group.*

Proof. Two different forms of proof provide insight into this theorem, and so are worth some discussion.

First, reconsider our example. Let us try to faithfully represent this situation with preference orderings *ABC*, *BCA*, and *CAB*. It is easily seen that the margins of preference of *A* over *B* are the result of adding up the frequencies of all individual orderings preferring *A* over *B*, and subtracting the frequencies of all individual orderings preferring *B* over *A*, as shown below:

For $A > B$: $f(ABC) - f(BCA) + f(CAB) = -4$
For $B > C$: $f(ABC) + f(BCA) - f(CAB) = +20$
For $A > C$: $f(ABC) - f(BCA) - f(CAB) = -10$

Rows two and three are similarly based upon the computation of the margins of *B* over *C* and *A* over *C*, respectively. We have three independent equations and three unknowns that can be solved for the frequencies of each of these preference orderings; all three are necessary to reproduce these particular margins of preference. In this case the solution is

$f(ABC) = 8$,
$f(BCA) = 23$,
and $f(CAB) = 11$.

This example shows how faithful representation can be produced, and also (as per our previous discussion) that proportional representational procedures can distort decisions, even though faithfully representative weights do exist. As noted previously, in this example, weights derived from a list form of proportional representation give rise to cyclical majorities, even though the electorate as a whole has a transitive majority ordering.

In general, there is one equation for each of the $m(m - 1)/2$ binary choice margins, and so the equation set can be solved—with exactly as many equations as unknowns—so long as the equations are independent. Because it would be undesirable to give a representative a negative weight, if the solutions of the equations turn out to be negative, a representative assigned a negative weight can be replaced with one having exactly the opposite preference ordering who is given a positive weight. (By the *opposite preference ordering* we mean one which reverses each pairwise preference: e.g., the opposite of *ABC* is *CBA*.) Because of the independence requirement, not every set of $m(m - 1)/2$ preference orderings can be used to faithfully represent a group. On the other hand, the set of preference orderings that can be used to obtain faithful representation is not, in general, unique.

The second form of proof is by construction, using McGarvey's logic to construct a set of preference orderings (and frequencies/weights for each) that reconstruct a given set of margins. Let P_{ij} indicate orderings in which i is to the left of j, which we also denote as $i < j$. There are $m(m - 1)/2$ different preference orderings of the form

$$P_{ij} = [i+1, i+2, \ldots j], [i], [j+1, j+2, \ldots m], [i-1, i-2, \ldots 1]. \quad (1)$$

This apparently complex formula is actually a specification of a simple idea that can be illustrated for four alternatives: That is, $P_{1,2}$ = *2 1 3 4*; $P_{1,3}$ = *2 3 1 4*; $P_{1,4}$ = *2 3 4 1*; $P_{2,3}$ = *3 2 4 1*; $P_{2,4}$ = *3 4 2 1*; and $P_{3,4}$ = *4 3 2 1*.[6]

Each ordering and the negative of the next constitute a McGarvey pair; e.g., *2 1 3 4* and *4 1 3 2* (the reverse of *2 3 1 4*) cancel each other out, except for their both contributing to the margin of *1* over *3*. Thus, McGarvey pairs for the $m(m - 1)/2$ margins can be constructed from these $m(m - 1)/2$ preference orderings by using each one twice. This proof by construction provides an example of a set of representatives that can faithfully represent any set of margins.

Just as in the previous proof, the frequencies of each of these preference orderings necessary to reproduce the margins can be negative. Consequently, as above, any preference ordering that requires a negative weight would be replaced with its opposite with a positive weight. This is theoretically important, as we shall show below after we provide an extension to Theorem 1.

Theorem 1 says that one can faithfully represent the margins with $m(m - 1)/2$ weighted preference orderings. This procedure does not faithfully reproduce a 25-vote margin of A over B, however, this might have been 25 out of 2 million in the electorate, and be 25 out of 100 in the committee of representatives. Fortunately, only one additional preference ordering is required to reproduce the proportions as well as the margins themselves.

COROLLARY TO THEOREM 1. *A maximum of $m(m - 1)/2 + 1$ different weighted preference orderings is sufficient to represent the proportionate margins of pairwise preferences that emerge from any group—that is, to represent faithfully the group margins and their proportions of the total votes.*

Proof. The first proof of Theorem 1 can be extended to prove this corollary. We merely include one additional preference ordering and add one more equation specifying that the frequencies of all of these preference orderings add up to the total number of votes in the electorate.[7]

Our second proof of Theorem 1, the proof by construction, has further implications. It should be noted that every preference ordering given in the proof above is single peaked with respect to the ordering from *1* to *m* (*1* to *4* in the example). The opposite preference orderings are all single troughed with respect to the same ordering. (The opposite of a single-peaked ordering is always a single-troughed ordering with respect to the same continuum.) Consequently, this second proof constitutes a proof for an even stronger theorem:

THEOREM 2. *With respect to any linear ordering, a maximum of $m(m - 1)/2$ different (weighted) single-peaked and/or single-troughed preference orderings are sufficient to represent the group decisions faithfully over m alternatives.*

The Preferences of Members of Faithfully Representative Committees and Those They Represent

Since there are various faithfully representative committees, it would be useful to know whether the preference orderings of the members of such committees are necessarily similar to those of the represented population. The answer is there need be no overlap at all—a set of representatives can faithfully represent a group and yet have no overlap with the set of preference orderings of the group it represents.

Consider a society with a single individual with the preference ordering *3 6 5 1 2 4*. The committee shown below faithfully represents that society, but has no overlap with it.

Ordering	*Weight*
6 5 4 3 1 2	1
2 3 1 4 5 6	1
6 5 1 4 3 2	1
3 2 4 5 6 1	1
1 6 5 2 4 3	1
3 4 5 6 2 1	1
1 2 3 6 5 4	1

However, if there are only three alternatives, no committee can faithfully represent a society of a single person unless it includes that person's preference ordering.

The proof by construction for Theorem 2 showed how one can construct a set of preference orderings that can faithfully represent any group. If we have enough orderings to choose from, we can simply choose our committee so as to exclude from its members any with preference orderings in the society being represented.

COROLLARY TO THEOREM 2. *If a society has no preference orderings which are either single peaked or single troughed with respect to some given ordering, then that society can be faithfully represented by a committee whose preference orderings do not overlap those in the society.*

Proof. Use the second construction method used in the proof of Theorem 1, with the single-peaked ordering based on the single-troughed preferences not found in the society. Q.E.D.

This corollary can also be regarded as a special case of the more general theorem given below.

THEOREM 3. *If a society lacks at least $m(m-1)/2$ linearly independent preference orderings (i.e., orderings that generate linearly independent equations) from among the orderings held by any of its members, then it can be faithfully represented by a committee whose preference orderings do not overlap those in the society.*

Proof. Use the equational method used in the first proof of Theorem 1. (It may be necessary to give some members negative weights.) Q.E.D.

On pragmatic grounds, one would prefer a faithfully representative committee whose members did share preference orderings with the underlying electorate to one that did not. Fortunately, it is always possible to have a faithfully representative committee composed largely or entirely of preference orderings in the represented electorate.

THEOREM 4. *There is always a subset of no more than $m(m-1)/2$ of the set of preference orderings in a group that can be used to faithfully represent the group as a whole.*

Proof. If the society has less than $m(m-1)/2$ members, we are done. If not, set up $m(m-1)/2$ equations as in the first method used to prove Theorem 1, but using the complete set of n preference orderings in the group. Clearly, this set of equations has a solution, namely, that in which each ordering is given the weight equal to the number of group members holding the ordering. Of course, we have too many variables in our equation set. However, a well-known result in linear algebra tells us that since these equations do have a solution, there will exist at least one subset of no more than $m(m-1)/2$ of these variables that will be sufficient to determine the $m(m-1)/2$ margins,[8] since the rank of the $m(m-1)/2$ by n matrix specified by this equation set cannot exceed $m(m-1)/2$. Q.E.D.

It might appear that the faithfully representative subset of a group should, as much as possible, be representative of the full range of preferences of the group, but this may be unrealistic, because almost all orderings are held by some members of the society. Also, there is some reason to want a set of representatives whose preferences are more "coherent" than the probably widely divergent preference orderings held by the public. In particular, it would be nice to find a committee all of whose preferences are single peaked over some particular unidimensional continuum, since knowledge of this fact might simplify decision making in the group. We now turn to consider the relationship of single-peakedness to faithful representation.

Single-peakedness, Net Margins, and Faithful Representation

Knowledge of the paradox of voting, combined with the recognition of its empirical rarity, has frequently led researchers to suppose that there is coherence in individual preference orderings. Specifically, it has often been noted that if there is an underlying ordering of alternatives such that all individuals have single-peaked preference orderings with respect to the unidimensional continuum defined by that ordering, then the group as a whole has transitive majority preferences. Unfortunately for political theorists, few groups have individual preferences that are all single peaked. Majority decisions, however, may be transitive even though there are many individual preferences that are not single peaked (Feld and Grofman, n.d.; Niemi, 1969). We believe that the criterion of single-peakedness is too restrictive to be applicable in most actual situations, but that its weaker counterpart at the aggregate level, which we call the *Net Margin Condition,* is much more commonly found.

DEFINITION. *The Net Margin Condition is said to apply to a group's preferences over a set of alternatives, if and only if there is some linear ordering of alternatives such that for every i, j, and k arrayed* $i < j < k$ *with respect to that ordering,* $m(i,j) \leqslant m(i,k) \leqslant m(j,k)$, *where* $m(i,j)$ *is the margin by which alternative i is preferred to alternative j (negative if alternative j is majority preferred to alternative i).*

The definition may be made more meaningful with the diagram shown below. The alternatives are listed according to the linear ordering across the top and down the side. The entries in the matrix are just the margins $m(i,j)$. Those below the main diagonal are just the negative of the corresponding one above—e.g., $m(1,2) = -m(2,1)$; only those above the main diagonal are shown. The Net Margin Condition requires that the entries increase (or at least do not decrease) as one moves to the right or down within this half of the matrix of margins.

Matrix of Margins

	1	2	3	4	..
1		$m(1,2)$	$m(1,3)$	$m(1,4)$	...
2			$m(2,3)$	$m(2,4)$	...
3				$m(3,4)$	...
.					
.					

Single-peakedness at the individual level implies the Net Margin Condition at the aggregate level. This can be seen by the implication of single-peakedness that everyone who prefers *1* to *2* must also prefer *1* to *3*, so the number of people voting for *1* over *3* must be at least as great as the number voting for *1* over *2*.

It is easily seen that the Net Margin Condition can hold while only a single individual has preferences that are single peaked with respect to the linear continuum along which margins are to be ordered; consider that all but one member of a group have preferences that are not single peaked, half of them having preferences opposite those of the other half, thus cancelling one another out on all binary decisions. (If *m* is large enough, we can easily satisfy this condition.) The one other member determines all of the margins; if that one has single-peaked preferences (i.e., if he has a transitive ordering of his pairwise preferences), then the margins are defined by his single-peaked preferences and therefore must satisfy the Net Margin Condition.

Moreover, the Net Margin Condition does not require any of the individuals to have single-peaked preference with respect to the linear continuum along which margins are to be ordered. For example, the Net Margin Condition for the seven-member committee discussed above obtains for the ordering *3 6 5 1 2 4*, even though none of the members of the committee have single-peaked preferences with respect to that ordering.

The Net Margin Condition permits a special form of faithful representation as expressed in Theorem 5:

THEOREM 5. *A group can be faithfully represented by a committee composed entirely of individuals with single-peaked preferences, if and only if group preferences satisfy the Net Margin Condition. In particular, for groups satisfying the Net Margin Condition, a maximum of* $m(m - 1)/2 + 1$ *different weighted single-peaked preference orderings is sufficient to faithfully represent the group margins and the proportions of the total votes cast on each pairwise choice.*

Proof. We have already proven that single-peaked preferences for all individuals in the society imply the Net Margin Condition for the group. In this situation, single-peaked preferences of the faithfully representative committee imply that the Net Margin Condition applies to the committee margins. Since the committee faithfully represents the group, the committee margins are those of the group, so the group must obey the Net Margin Condition. The other direction is proven by construction, as shown in the Appendix. We show that we can construct a set of single-peaked preference orderings that can faithfully represent any group whose margins satisfy the Net Margin Condition.

The Net Margin Condition is sufficient but not necessary for there to be a transitive majority ordering.[9] Thus, a group may have a transitive majority preference ordering but not be faithfully representable exclusively with single-peaked preference orderings, because the group fails to satisfy the Net Margin Condition (cf. Feld and Grofman [1986]).

Theorem 5 is important in that it indicates that a group having few single-peaked preferences with respect to any ordering can, if the Net Margin Condition is met, be faithfully represented by a committee whose preferences are all single peaked with respect to some given ordering. We believe that elected representatives are much more likely to hold "ideologically coherent" positions (on, say, a left-right dimension), and thus to have single-peaked preferences, than are their constituents. Empirical studies show that American voters hold a wide variety of preference orderings, including many that are not single peaked. We suggest that the large differences between voters and their representatives in the diversity and degree of ideological consistency of each group's preference orderings provide no obstacles to faithful representation.

The January 1980 National Election Study conducted by the University of Michigan Center for Political Studies provides an empirical example of a situation where the aggregate margins of preference satisfy the Net Margin Condition, even though there were voters who had every possible preference ordering over a given set of candidates. In the next section, we show one way that this electorate could have been faithfully represented by a committee composed entirely of representatives having single-peaked preference orderings.

An Empirical Example

In 1980, the University of Michigan Center for Political Studies undertook a panel study of voters preceding and immediately following the 1980 presidential election. On the first wave of that study, in January 1980, individual respondents were asked to indicate their feelings towards each of the candidates by indicating a number from 0 to 100 (where 100 is the most positive feeling). These data allowed us to determine a preference

Table 1. Proportions Favoring the First Candidate of Each Ordered Pair

	Candidates			
Candidates	Kennedy	Carter	Ford	Reagan
Kennedy	—	36.1	36.5	40.5
Carter		—	48.4	53.7
Ford			—	54.6
Reagan				—

Note: Proportions below the main diagonal are omitted: values would be 100 minus the proportions in the corresponding cell above.

ordering over the candidates at that time. The survey included a long list of candidates; for the present purposes, we confine our attention to the four best known candidates: Carter, Kennedy, Ford, and Reagan. Most voters indicated that they recognized the names of all these candidates.

Based upon our own knowledge of this situation, we determined that if there was an ideological (left-right) continuum, it would have to be Kennedy, Carter, Ford, and Reagan, from left to right respectively. There are a total of 24 different strong preference orderings over four alternatives. All individuals who indicated ties or did not give an answer for one candidate were excluded to simplify the analyses and interpretations. As we show elsewhere (Feld and Grofman, 1985), every one of the 24 possible preference orderings was represented in the sample of voters, and almost half (47%) of the preference orderings were not consistent with single-peakedness on the given hypothesized continuum.

Despite the large proportion of preference orderings that were not single-peaked, the margins of pairwise preferences conformed perfectly to the partial ordering expected to arise from single-peaked preferences, thus satisfying the Net Margin Condition, as shown in Table 1.

According to Theorem 5, if the Net Margin Condition is satisfied, a group can be faithfully represented by $[m(m-1)/2]+1$ single-peaked preference orderings. In this case, the margins in Table 1 can be reproduced by the following frequencies of seven single-peaked preference orderings, as can be determined by following the algorithm described in the Appendix:

36.1% of *KCFR*
.4% of *CKFR*
4.0% of *CFKR*
7.9% of *CFRK*
5.3% of *FCRK*
.9% of *FRCK*
45.4% of *RFCK*

This provides an empirical example of an electorate not limited to single-peaked preferences that can nevertheless be faithfully represented by a committee whose members all have single-peaked preference orderings.

Procedures to Provide Faithful Representation

As Chamberlin and Courant (1983, p. 718) point out, although there is a sizeable comparative politics literature on the PR vs. plurality debate, "social change theorists have paid almost no attention to methods of selecting representative bodies." This article, like theirs, seeks to remedy that surprising omission. Since it is always possible to faithfully represent the preferences of voters, the question is how this might practically be done. Unfortunately, as we showed with respect to the list form of proportional representation, current electoral systems do not necessarily provide faithfully representative legislatures; they either fail to choose a correct set of legislators or fail to weight committee members properly to produce faithful representation of the population. The same problems apply to the single transferable vote method of PR (STV) (Sugden, 1984) and to the form of PR proposed by Chamberlin and Courant (1983), although Sugden (1984) shows that if all voters have single-peaked preferences, STV will be order representative. The Chamberlin-Courant method may pick a committee that is not even order representative. Consider a seven-member society with preferences among three alternatives, as shown below:

Number of Voters		
3	1	3
A	*B*	*C*
B	*A*	*B*
C	*C*	*A*

The Chamberlin-Courant rule for a committee of size 2 would pick a committee with four votes for *A* and three weighted votes for *C*. This makes *A* dictator. However, as is obvious (because of symmetry), the preference ordering for society is *BAC*.[10]

The same information used to generate an outcome under STV[11] or under the Chamberlin and Courant procedure—namely a complete rank ordering of alternatives—may be used to generate a faithfully representative committee by treating each potential representative as "representing" a given preference order and assigning weights in such a way as to faithfully represent the voters. Also, because faithful representation is just that, if voters are asked to submit a rank ordering of alternatives to be used for purposes of generating a faithfully representative committee, they have no strong reason to lie about their true preferences, even if they were thinking about sophisticated voting in the sense of Farquharson (1969), since the sequence in which alternatives will eventually be paired is unknown.

If there are $m(m-1)/2$ independent representative orderings, then there will always be a faithfully representative committee. If there are more than $m(m-1)/2$ independent orderings in the electorate, then there will always be multiple ways to faithfully represent the voters; if there are multiple faithfully representative committees, then we suggest that the one whose frequencies of preference orderings are most highly correlated with those in the population should be preferred. In any case, we can produce a committee that is at least as representative as that which will be produced by any other voting procedures.

It should be noted that faithfully representative committees can have very reasonable size, especially in comparison to the size of populations and the number of distinct orderings possible. We have indicated that faithful representation does not depend upon the size of the population; for m alternatives, faithful representation can be assured with a maximum of $m(m-1)/2$ weighted representatives, and frequently with fewer than that. Table 1 shows how the size of the faithfully representative committees rises with the number of alternatives. Sometimes, however, we will be able to obtain faithful representation with less than $m(m-1)/2$ committee members, as is shown in Table 2.

Table 2 supports our earlier claim that while every individual in a population the size of the United States could have a unique preference ordering over twelve alternatives, the population could be faithfully represented with a maximum of 66 representatives.

One inevitable problem in any form of representation is the possibility that the representatives will have to choose among alternatives that were not anticipated

Table 2. Maximal Size of a Faithfully Representative Committee

Preference Orderings and Committee Size	Number of Alternatives (m) 3	6	9	12
Number of preference orderings ($m!$)	6	720	362,880	479,001,600
Number of single-peaked preference orderings (2^{m-1})	4	32	256	2,048
Maximal size of faithfully representative committee ($m(m-1)/2$]	3	15	36	66

when they were selected. This is, of course, a problem for any representative system, no matter how its representatives are chosen. Certainly, if the new alternatives have no consistent relationship to earlier ones, there can be no assurance that a committee chosen for its faithful representation over some given set of alternatives will faithfully represent the electorate in considering those new alternatives. However, most new issues are related to preexisting ones, and faithful representation may be preserved under some likely conditions.

One possibility is that the underlying choice continuum is preserved by the new alternatives such that each of the new alternatives fits along the continuum in a way that preserves the Net Margin Condition. If the committee members recognize the position of a new alternative along the continuum, and include it there among their own single-peaked preferences, the resulting vote margins cannot diverge from those in the electorate by more than the vote margins of the two preexisting bounding alternatives along the continuum. For example, if W fits in the continuum such that the group meets the Net Margin Condition for $ABCDWE$, then the margins for $m(A,W)$ will be between $m(A,D)$ and $m(A,E)$, and the margin $m(D,W)$ must be between $m(D,E)$ and $m(E,E) = 0$. Thus, if the representatives properly recognize the position of the new alternative along the continuum, they can continue to approximate faithful representation.

Another possibility is that preferences over the new alternatives will be related to preferences over the old alternatives in some (approximately) linear fashion. For example, if individuals who prefer A to B are more likely to prefer A to W than are individuals who prefer B to A, then, to the extent that the preferences over some new alternative are determined by preferences over preexisting alternatives, the margins by which that new alternative is preferred to each of the old ones will be linear combinations of the previous observed margins. If roughly the same relationships hold for all the committee members, then the faithfully representative committee will have the same margins as the electorate over the new alternatives, and continue to be faithfully representative.

If the set of alternatives used to select the faithfully representative committee reflects the range of issue dimensions about which future decisions will be made, then it is reasonable to expect that the committee may be able to represent the electorate faithfully on choices over new alternatives by processes such as those identified above. Furthermore, the problem of continued faithful representation when new issues arise is certainly no greater under a legislature/committee chosen for faithful representation than for any other system of representation. In particular, under procedures for faithful representation, voters have no more reason to lie about their true preferences than they would in a direct democracy.

Conclusion

This paper has laid out the basic characteristics of faithful representation and some considerations for systems designed to produce it.[12] Our intention has been to demonstrate the importance of faithful representation and its practicability in principle. Implementation would require further consideration of various potential problems, especially the determination of a particular set of alternatives upon which representation should be based. However, as noted previously, if faithful representation is guaranteed, then voters have no good reason to lie about their true preferences. Moreover, if we find a committee that will, for a given set of issues or issue dimensions, be faithfully representative of the voters in the sense we have identified, then we also have some reason to believe this committee will as a whole be in sufficient sympathy with the views of the electorate so as to be likely to arrive at decisions that are in substantial accord with societal (perhaps ideally informed) preferences, even on new items for which we could not have an a priori certainty of faithful representation.

We have shown that faithful representation can be achieved with committees of relatively small size, and that it is always possible to faithfully represent a group having any variety of preferences with a relatively small committee, all of whose members have single-peaked or single-troughed preferences with regard to any particular ordering we might specify. Furthermore, if the Net Margin Condition is satisfied, we can find a faithfully representative committee, all of whose members have single-peaked preferences. We suggest that the actual functioning of a democracy often approximates the latter situation.[13] Representatives are much more likely than the underlying electorate to have single-peaked preferences; yet, they can fairly represent the choices of the majority of the electorate, even though that electorate is not single peaked. Consequently, large electorates may have only slight tendencies towards single-peakedness, and yet be faithfully represented by their single-peaked representatives. Analysis of legislative voting finds most votes can be scaled along a single dimension (see, e.g., Poole and Daniels, 1985), while the candidate preferences of the electorates in 1980 showed that single-peaked individual preferences were relatively uncommon. We have shown how these apparent inconsistencies in the preferences of elites and masses can, in principle, nonetheless, be fully compatible with faithful representation.

Appendix: Proof of Theorem 5

THEOREM 5. *A group can be faithfully represented by a committee composed entirely of individuals with single-peaked preferences if and only if group preferences satisfy the Net Margin Condition. In particular for groups that satisfy the Net Margin Condition, a maximum of $m(m-1)/2+1$ different weighted single-peaked preference orderings is sufficient to faithfully represent the group margins and the proportions of the total votes cast on each pairwise choice. One ordering less than that is needed to reproduce the margins without concern for proportions.*

Proof. Proof of the "only if" part of Theorem 5 is given in the text. The proof of the "if" part of the theorem consists of a procedure by which one can find a committee composed of $m(m-1)/2+1$ single-peaked preference orderings that perfectly reproduce the margins and the total votes in the electorate. We can show that the margins alone can be faithfully reproduced with one less preference ordering.

Consider that the alternatives are ordered from l to m along the continuum.

Since the margins satisfy the Net Margin Condition, we know that if $N(i,j)$ is the number favoring j over k, and if $i < j < k$, then $N(i,j) \leqslant N(i,k) \leqslant N(j,k)$; i.e., within the half matrix above the diagonal, the numbers go up as one moves to the right and down the matrix. We shall pay no attention to entries on or below the diagonal.

We can proceed by accounting for each of the numbers in the matrix sequentially from the smallest to the largest. The smallest number is $N(1,2)$. We begin with exactly $N(1,2)$ of the preference ordering $1\ 2\ 3\ \ldots\ m$. Then, we subtract $N(1,2)$ from all of the rest of the numbers in the matrix, to obtain a new matrix. The next smallest number in that new matrix will be in the $N(1,3)$ cell. We use exactly that many of the $2\ 1\ 3\ 4\ \ldots\ m$ preference ordering. Then we subtract $N(1,3)$ from all of the remaining cells to obtain a new matrix.

As we proceed at each step subtracting numbers from the cell entries in the preceding matrix, we obtain a new matrix with a sequence of zeroes on the left side of some rows above the main diagonal and with a sequence of nonzero entries, which we shall denote with Ys, on the right side of most rows above the main diagonal. We show below an example of such a matrix for a twelve-alternative voting situation:

	1	2	3	4	5	6	7	8	9	10	11	12
1	X	0	0	0	0	0	0	0	0	0	Y	Y
2		X	0	0	0	0	0	0	Y	Y	Y	Y
3			X	0	0	0	0	0	Y	Y	Y	Y
4				X	0	Y	Y	Y	Y	Y	Y	Y
5					X	Y	Y	Y	Y	Y	Y	Y
6						X	Y	Y	Y	Y	Y	Y
7							X	Y	Y	Y	Y	Y
8								X	Y	Y	Y	Y
9									X	Y	Y	Y
10										X	Y	Y
11											X	Y
12												X

In general, to find out at each subsequent step which preference ordering is to be used and with what weight, we identify at each step the *corner cell* or cells in the new matrix. A *corner cell* is one with zeroes above and to the left of it. In the matrix above the cells (4,6), (2,9), and (1,11) are corners. If the coordinates of the corners are denoted by (i_1, j_1) for the one with the lowest j value, (i_2, j_2) for the corner with the next lowest j value, etc., then one may specify the preference ordering that should be added to the set as follows:

$j_1-1, j_1-2, \ldots i_1 \qquad j_1, j_1+1, \ldots j_2-1,$
$i_1-1, i_2-2, \ldots i_2 \qquad j_2, j_2+1, \ldots j_3-1$
$i_2-1, i_2-2, \ldots i_3$, etc. $\qquad j_3, j_3+1, \ldots j_4-1$, etc.

In the example of the matrix above, we have the sequence *5,4; 6,7,8; 3,2; 9,10; 1,11,12,* as the preference ordering. It is clear that any such preference ordering is single peaked. Furthermore, it should be clear that this preference ordering contributes to the frequency of i_1 over j_1 and above, the frequency of i_2 over j_2 and above, the frequency of i_3 over j_3 and above, etc.; i.e., it contributes to exactly those nonzero cells that are yet to be accounted for and to no others.

One uses as many instances of this preference ordering as the lowest of the numbers in any of the corners; i.e., we give the preference ordering *54678329101112* the weight that is the lowest of the cell values in cells (4,6), (2,9), and (1,11).

The procedure continues until all of the matrix entries are accounted for. This takes exactly $m(m-1)/2$ preference orderings, because there are that many cells to account for.

Note that this is sufficient to reproduce all the numbers in the original matrix.

If one wanted also to reproduce the number of voters opposed, one would subtract the number of votes already accounted for from the size of the group and add in that many of the preference ordering $m, m-1, \ldots 1$. That method serves to reproduce the exact number in favor and opposed in each pairwise choice, and so reproduces the desired proportions. Thus, to account for proportions we require $m(m - 1)/2 + 1$ weighted preferences. Q.E.D.

This procedure is not as complex as the abstract statement above might suggest.

A complete example with four alternatives can be used to illustrate the method: Consider the following matrix based on pairwise choices of 60 voters:

Alternatives	1	2	3	4
1	X	15	20	25
2		X	22	32
3			X	40
4				X

To create this matrix requires

frequency	*preference ordering*
15	*1 2 3 4*
5	*2 1 3 4*
2	*2 3 1 4*
3	*3 2 1 4*
7	*3 2 4 1*
8	*3 4 2 1*

and, since this accounts for only 40 votes, we also need 20 of the *4 3 2 1* preference ordering in order to total 60.

If one replaces the 15 orderings of *1 2 3 4* and the 20 orderings of *4 3 2 1* with 5 orderings of *4 3 2 1*, then the margins are faithfully reproduced, but the proportions are not. If one uses all seven different preference orderings, then both the margins and the proportions are perfectly reproduced. In general, if one were only interested in reproducing the margins, one could include either $1, 2, \ldots m$ or $m, m-1, \ldots 1$, whichever was weighted more, because these orderings are opposite and thus cancel each other out (e.g., 20 orderings of $1, 2, \ldots m$ and 15 orderings of $m, m-1, \ldots 1$ produce the same margins as 5 orderings of $1, 2, \ldots m$ alone).

Notes

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1. They then go on to distinguish this view from one suggesting that representatives shall not reflect societal preferences, but what those preferences would be if every member of the society was "ideally informed, ideally expert, and ideally clear about his own interests" (Rogowski, 1981, cited in Chamberlin and Courant, 1983, p. 719). However, in either view "it is direct democracy (either actual or ideal) that is used as a measuring rod" (Chamberlin and Courant, 1983, p. 719).

2. In a multistage process of candidate selection (e.g., winner-take-all primaries or the U.S. electoral college), the failure to represent vote margins at each stage can result in the choice of a candidate who would not be selected by the (party) electorate as a whole. Indeed, in sequential decision processes, faithful representation at each stage is necessary to assure an even order of representation overall.

3. We use the term *faithful representation* differently than it is used elsewhere in the social choice literature (e.g., Murakami, 1968).

4. For a given ordering of alternatives—say *bcdae*—*single-peaked preferences* with respect to that ordering are those where each voter's utility reaches a peak at some given alternative and then declines as one moves left or right along the ordering away from that most preferred alternative. For example, if a voter had his or her peak at *c*, for the preferences to be single peaked we must have *cPd*, *dPb*, *cPa*, and *aPe*, where *P* denotes "is preferred to." Thus, for voters with peak at *c*, single-peaked preferences could be *cdbae*, *cdabe*, *cdaeb*, *cadeb*, *caedb*, and *cadbe*. In like manner, *single-troughed preferences* for a given ordering of alternatives along a line are those where each voter has a least-preferred alternative, and utility increases for that voter as one moves right or left along the line from that alternative.

5. Faithful representation does not imply there is a transitive social ordering, but merely that committee members make the same pairwise choices (and by the same margin) as the society as a whole. If societal majority preferences are intransitive, then the same will also be true for the majority preferences of any faithfully representative committee.

6. Alternatively, if we begin with the *1 2 3 4* ordering, we can get the requisite orderings by first moving the *1* successively further down to *2 1 3 4*, *2 3 1 4*, and *2 3 4 1*; then we move the *2* down to *3 2 4 1*, *3 4 2 1*; then the *3* is moved down to *4 3 2 1*. The first and last of these processes are the

same (only reversed), so the first is omitted in the final list of six preference orderings:

2 1 3 4, 2 3 1 4, 2 3 4 1, 3 2 4 1, 3 4 2 1, 4 3 2 1.

7. If a set of weights that reproduces the margins has fewer voters than are in the general population, then one can increase the votes in the committee to the number in the electorate simply by adding equal numbers of one of the preference orderings and its opposite; this adds to the total number of votes cast without changing any of the margins. However, if the committee has more votes than the electorate as a whole, there may be no way to reproduce the proportions without giving some preference orderings negative weights—one cannot substitute a positive weight for an opposite preference ordering in this situation.

8. We may have to use negative weights if the opposite preference ordering is not found in our society and our equational technique assigns a negative weight. However, since there are multiple preference sets to choose from, we usually expect to be able to find a solution with only positive coefficients.

9. A proof of sufficiency is obtained by combining results in Feld and Grofman (1985, 1986). For a counterexample to necessity, consider this eight-voter, three-alternative example:

5 *ABC*
2 *BCA*
1 *CAB*

This society has a transitive majority preference ordering of *ABC*. There is, however, no linear ordering for which the Net Margin Condition is satisfied. There are three possible cases:

ABC	*BAC*	*BCA*
$m(A,B) = 4$	$m(B,A) = 4$	$m(B,C) = 6$
$m(A,C) = 2$	$m(B,C) = 6$	$m(B,A) = -4$
$m(B,C) = 6$	$m(A,C) = 2$	$m(C,A) = -2$

As we see, in none of these cases is the Net Margin Condition satisfied.

10. In their article, Chamberlin and Courant (1983, p. 718) identify two key criteria for representation: (1) representativeness of deliberations, and (2) representativeness of decisions. Chamberlin and Courant (1983) offer a method for selecting representatives (and the weights that each representative is to have in deliberations that are to take place under a weighted voting rule) that is especially attentive to the first feature, representative deliberations. We have provided a method that is based on the second feature, representative decisions. Thus, our work and that of Chamberlin and Courant (1983) can be seen as complementary. However, if there is a conflict between representative deliberations and representative outcomes, we believe that faithful representation is considerably more important than symbolic representation of the sort that the Chamberlin-Courant method seeks to provide.

It is often possible to find committees that are arguably optional with respect to both criteria: representative deliberations and representative decisions. However, sometimes the two criteria will conflict—i.e., the intersection of the Chamberlin-Courant set and the set of faithfully representative committees will be empty. In the case of conflict we believe representative decisions ought to have priority.

We might also note that the Chamberlin and Courant (1983) definition of representative deliberations is not the only one that might be used. We could use as our litmus test the choice of a minimal k-cover for the set of voters (Harary and Cartwright, 1965). A k-cover is a committee such that, for every voter, some member of the committee is no worse than that voter's kth choice. Nelson Polsby (personal communication, October 10, 1985) has suggested that the members of the APSA Council be chosen in such a fashion that this group will constitute a *1*-cover for the political science profession. He has also attempted to formulate selection criteria for the council so as to achieve this end.

11. In STV elections, voters are required to rank order candidates. In some jurisdictions using STV (Wright, 1986), a full ballot must be cast for a ballot to be valid (i.e., a complete preference ordering must be given by each voter). Even if less than a full ballot is cast, a voter's indifference among alternatives that are not rank ordered (and their location at the bottom of his preference ordering) is not unreasonable to assume.

12. It would also be useful to consider various types of approximation to faithful representation, but we leave that task to a subsequent paper.

13. Recall that the Net Margin Condition may be satisfied even though most of the orderings in the group are not single peaked.

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