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The accuracy of group majority decisions in groups with added members*

SCOTT L. FELD**

BERNARD GROFMAN***

1. Introduction

Analytic results indicate that many heads often can be better than one *even when none of the added heads is as individually competent as the first.* We define individual competence (accuracy) in a dichotomous choice situation as the probability p_i of making the 'correct' or 'better' choice.[1] Condorcet (1785) shows that if each person has equal competence, and if that competence, p_i, is greater than chance (.5), then the accuracy of majority choice increases monotonically towards infallibility with increasing numbers of voters. Owen, Grofman and Feld (1981) generalize the Condorcet Theorem to verify that as long as the *average* competence of a group is greater than chance, then the competence of the group majority increases towards one as the group becomes larger. However, the practical problem in collective decision making often is to ascertain whether adding a particular set of persons to the decision-making group will add or detract from the competence of the decisions made. Shapley and Grofman (1981) and Nitzan and Paroush (1980, 1982) show that we can obtain the maximum competence with a weighted voting rule, with the weights monotonically related to the individual competences (according to a simple function that they specify). However, the competence of particular persons often is unknown; and in any case, weighting according to competence often is impractical. Consequently, we follow Grofman (1975, 1978); Margolis (1976); Owen, Grofman and Feld (1981); and Grofman, Owen and Feld

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** Department of Sociology, State University of New York at Stony Brook, Stony Brook, NY 11794.
*** School of Social Sciences, University of California, Irvine, CA 92717.

(1982, 1983) in investigating the conditions under which adding certain persons adds to or detracts from the competence of a 'one person/one vote'majority.

Consider a situation in which a 'root' group with known majority competence is currently making the decisions, and the issue is to ascertain whether adding a supplementary group will increase or decrease the competence of the group. We assume that information about the average or majority competence of the supplementary group is available. We define the *average competence* of a group as the mean of the competences of its individual members. The average competence is also the competence of the group as a whole, deciding by a probabilistic decision rule (such that each person has an equal chance of being chosen to make the decision himself). A group's *median competence* is defined as the competence of the group's median (in competence) member. The *majority competence* is the competence of a group as a whole, deciding by majority rule.[2] In the subsequent discussion, we assume that both the root group and the supplementary group are *minimally competent* (that is, each group has a probability of making the correct choice by majority vote greater than or equal to .5). We begin with the important special case in which either the root group or the supplementary group consists of a single person. A practical instance of this case is when an individual investor is deciding whether or not to join an investment club which makes decisions by the majority vote, or when an investment club must decide whether or not it will permit a person to join the club – with the only basis of its decision being an evaluation of whether the addition will or will not increase the group's expected judgmental accuracy.

In the following discussion we neglect persuasion, conformity and synergistic processes in groups and assume that simple majority rule prevails; we do not distinguish between Type I and Type II errors, and we also neglect transaction costs, which obtain when any collectivity is expanded. Most of the examples that we consider are from relatively small groups; but small groups, such as executive committees, en banc panels of judges and, of course, juries, make many important decisions.

2. Adding the many to the one

We consider the situation in which a person considers joining a group. We begin by showing that a number of plausible propositions about the competence of the combined group are false. Then, we state a sufficient condition given which many of our earlier propositions will hold. From the person's perspective, the question is under what conditions the combined group will make more correct decisions than he will make alone. Specifically, he might expect that a group with greater average competence than he enjoys will improve the decisions, or that a group with greater overall competence (the competence of the majority) will improve on his decisions. In general, we show by counterexamples that these expectations are incorrect.

From the group's perspective, the question is whether the enlarged group makes more correct decisions than the original one. Group members might expect that their decisions will improve if the added person has greater than the average competence of the group, or if he has greater competence than the overall (majority) competence of the group. We show by counterexamples that these expectations are also false.

Among others, there are six plausible propositions about the relationship between individual and group competence.

Proposition 1: If the *average* competence of a group is *greater* than the competence of a person to be added, then the combined group has a *greater* majority competence than has that *person*.

Proposition 2: If the *median* competence of a group is *greater* than the competence of a person to be added, then the combined group has a *greater* majority competence than has that *person*.

Proposition 3: If the *majority* competence of a group is *greater* than the competence of a person to be added, then the combined group has a *greater* majority competence than has that *person*.

Proposition 4: If the *average* competence of a group is *less* than the competence of a person to be added, then the combined group has a *greater* majority competence than that of the *original group*.

Proposition 5: If the *median* competence of a group is *less* than the competence of a person to be added, then the combined group has a *greater* majority competence than that of the *original group*.

Proposition 6: If the *majority* competence of a group is *less* than the competence of a person to be added, then the combined group has a *greater* majority competence than that of the *original group*.

We demonstrate the incorrectness of each of these propositions by providing counterexamples. In general, the inadequacy of all of these propositions rests on their rootedness in an assumption that majority competence is somehow a monotonic function of average competence, such that average competence greater than chance implies majority competence greater than chance. Owen, Grofman and Feld (1981; see also Grofman, Owen and Feld, 1982, 1983) demonstrate that group average competence can be greater than chance and group majority competence less than chance and vice versa (for example, a group of size eleven, with average competence of .51, can have a majority competence of .48). They also show that majority competence may vary considerably for groups of the same size with the same average competence, but with different distributions of individual competences. For example, a group of three persons with an average competence of two-thirds can have a majority competence between .74 (for individual competences of .67, .67, and .67) and 1.0 (for individual competences of 0, 1, and -1); similarly, the majority competence of a large group with average competence of .5 can vary from .40 ($e^{-1/2}$), depending upon the distribution of individual competences. Earlier work (for example, Grofman, 1975) also shows that a group might increase its competence by adding in a member with lower competence than present group members enjoy (for example, adding any minimally competent person to an even-numbered group increases its overall competence, even if that person has a lower competence than do any of the existing group members). Similarly, a group sometimes can decrease its competence by adding in a person with a higher competence (for example, adding any nonminimally competent person to an even-numbered group decreases its overall competence, even if that person is more competent than is any other group members). These examples provide indications of how both intuition and approximations based on large groups sometimes may overlook particular cases. Here we present simple counterexamples to each of our six seemingly intuitively plausible propositions.

3. The individual perspective

Because the mean competence of a combined group is greater than the competence of a particular member if and only if the mean competence of the original group is greater than the competence of that person, we may restate Proposition 1 thus:

Proposition 1': The majority competence of a group is at least as great as its mean competence.

Counterexample: Consider a group (1, .35, .35, .35) with a mean competence of .5125, greater than the competence of an additional person with a competence of .51. The combined group of five persons has an overall competence of .508, which is less than the competence of the additional person (and less than its own mean competence of .512).

Because the median competence of the combined group is greater than the competence of the added person if and only if the median competence of the original group is greater than the competence of the added person, we may restate Proposition 2 thus:

Proposition 2': The majority competence of a group is at least as great as its median competence.

Counterexample: Consider a person with a competence of .65 and a group (0, .8, .8) with a median competence of .8, greater than the competence of the added person. The combined group of four has an overall competence of .632, less than the competence of the added person (and less than the median, .7225, of the combined group).

As we show, it is possible to combine a person with a group with an average or median competence *greater* than the competence of that person and obtain a combined group with a competence *less* than the competence of that person. Proposition 3 posits that a group with a majority competence greater than that of the added person necessarily leads to a combined group that is better than that person alone. This, too, is false.

Counterexample: Consider a modification of the example that we use for Proposition 2 – an added person with a competence of .63 and a group (0, .8, .8) with an overall competence of .64, greater than the competence of the added person. The combined group of four has an overall competence of .6224, less than the competence of the added person (and less than the majority competence of the original group).

4. The group perspective

The examples that we use for Propositions 2 and 3 also provide counterexamples for Proposition 4. However, the results are stronger with the same group (0, .8, .8) with a majority competence of .64 and a mean competence of .533 and an added person with a competence of .55, greater than the group's average competence. The majority competence of the combined group is .584, which is less than the majority competence of the original group. A group with competences (1, .4, .4) with median competence, .4, less than the competence, .51, of the added person provides a counterexample to Proposition 5. The combined group of four has a majority competence of .6142, which is less than the majority competence of the original group, .64. The example that we use for Proposition 2, a group with competency (0, .8, .8), with overall competence, .64, less than the competence, .65, of the added person offers a counterexample to Proposition 6 as well. The combined group of four has an overall competence of .632, less than the overall competence of the original group.

5. When all persons are minimally competent

Our counterexamples to the Propositions 1–6 demonstrate that persons and groups cannot generally assume that more heads are better, even if the added heads have greater (average/median/majority) competences. However, all of the counterexamples include one or more incompetent persons within the group (that is, persons with a competence less than .5). We now reexamine our six propositions for the case in which every person in the group is at least minimally competent. As we shall see, some (but not all) of the propositions will then hold.[3]

Theorem 1: If all persons are minimally competent, and if the *average* competence of a group is *greater* than the competence of a particular person in it, then the combined group has a greater majority competence than does that *person*.

We may equivalently state this theorem thus (see discussion of Propositiosn 1):

Theorem 1': If all persons are minimally competent, then a group's majority competence is at least as great as its mean competence.

Proof

Lemma 1: For a group of fixed size, all of whose members are minimally competent, increasing the variation in individual competences while holding constant the average group competence cannot decrease the group's majority competence.

Lemma 1 implies that the minimum majority competence of a group with given mean competence occurs when all of the individual competences are equal to the mean competence, and that the maximal majority competence occurs when all of the individual competences are equal to either .5 or 1.0.[4] But the overall competence of a group of equally competent members increases monotonically with size (Condorcet, 1785), and such a group (N > 2) must have its majority competence greater than that of any of its (equally competent) members; therefore, if Lemma 1 is true, then Theorem 1 must hold.

Proof of Lemma 1: Let us assume, for simplicity, that the group has an odd number of members. (The results are directly generalizable to even-number groups, simply by recognizing that an even-number group has the same overall competence as an odd-number group including itself and an additional person with competence .5.) Suppose that two persons in the group (for convenience denoted 1 and 2) have competences of p_1 and p_2, with $p_1 > p_2$. We denote the group's majority competence without these members as P_o. Let P_{m-} be the probability that the rest of the group has a bare minority correct, and let P_{m+} denote the probability that the rest of the group has a bare majority correct. The majority competence of the total group is just P_o plus the probability of the additional two members' votes changing a bare minority (of the reduced group) to a bare majority (of the complete group) minus the probability of the additional two members' votes changing a bare majority (of the reduced group) to a bare minority (of the complete group). In other words, the majority competence of the total group, which we denote P_N, is simply the probability of a correct choice for both the reduced group and the combined group, plus the probability that the total group will reverse an error of the reduced group, minus the probability that the total group will overturn a correct decision of the reduced group.

$$P_N = P_0 + (p_1 p_2)(P_{m-}) - (1-p_1)(1-p_2)(P_{m+}). \tag{1}$$

We can compare the value of P_N in equation 1 with the majority competence of a modified group in which person 1 (the more competent) has an added competence of $p_1 + K$, while person 2 (the less competent) has a reduced competence of $p_2 - K$. This modification does not change group average competence. The majority competence for the new group, P_N', is

$$P_N' = P_0 + (p_1 + K)(p_2 - K)(P_{m-}) - (1 - p_1 - K)(1 - p_2 - K)P_{m-}, \tag{2}$$

$$P_N' - P_N = (P_{m-})(p_2 K - p_1 K - K^2) - (P_{m+})(p_2 K - p_1 K - K^2). \tag{3}$$

We can rewrite equation (3) as

$$P_N' - P_N = (p_2 - p_1 - K)\, K[(P_{m-}) - (P_{m+})]. \tag{4}$$

We wish to show that $P_N' > P_N$.

By assumption, $p_1 > p_2$ and $K > 0$, so $p_2 - p_1 - K < 0$. From the assumption that all members are minimally competent, it follows that a bare majority correct is at least as likely as a bare minority correct; so, $P_{m+} \geq P_{m-}$. Consequently, equation (4) is the product of two negative terms and a nonnegative term, which must necessarily be nonnegative.[5] Hence, in a group whose members are all minimally competent, increasing the variation in competences (by adding competence to a more competent member and subtracting the same amount from a less competent one) cannot decrease group majority competence. Q.E.D.[6]

For the case of groups whose members are minimally competent, that a person improves on his judgmental accuracy by joining a group with higher *mean* than his competence does not imply that he will necessarily be better off as part of a group whose *median* competence is greater than his, since group mean competence may be either higher or lower than group median competence. However,

Theorem 2: If all persons are minimally competent, and if the *median* competence of a group *exceeds* the competence of a person, then the combined group has a greater majority competence than that of the *person*.

We may restate Theorem 2 thus:

Theorem 2': If all persons are minimally competent, then a group's majority competence is at least as great as its median competence.

Proof: We prove the second form of the theorem, following procedures similar to those used to prove Theorem 1. Again, for simplicity of exposition we let the size of the group be odd. The proof first involves demonstrating that a group including a bare minority of persons with competence of .5 and a bare majority of persons with competence of q is the group with the minimal competence that has a median of q. Next, we show that such a group has a competence greater than q. It then follows that all groups with a median competence q have overall competence of at least q.

The first part of the proof is simple. Reducing the competence of any person in the group, while leaving the others' competences the same, can only diminish the group's overall competence. Begin with any group with median competence q. We can order the individual competences in the group from the least to the most competent. If we reduce the competences of a bare minority to .5, while we reduce the competence of the rest to the median, the group will have a median q of minimal majority competence.

It is almost as simple to show that a group consisting of all .5s and qs has an overall competence at least equal to q. We accomplish this task by using induction, building up from a group with one member with competence q, and thus with majority competence of q, by showing that adding in pairs of persons with competence .5 and q, respectively, cannot diminish the overall group competence.

In Theorem 1, as the adding of a minimally competent person with higher average competence than the group increases group accuracy, so adding a person of competence q can only increase the group's majority competence, because the group mean must be less than q. But adding a .5 person to an even-number group leaves the competence unchanged. Thus, the new group with the .5 and the q persons added must have a greater majority competence than the older group, even though its median remains q. Since we begin with a group with a majority competence of q, the desired result follows. Q.E.D.

Theorem 3: If all persons are minimally competent, and if the *majority* competence of a group is *greater* than the competence of an additional person, then the combined group has a greater majority competence than has that *person*.

Proof: Consider any group with a majority competence greater than that of the added person. Now consider a modified group obtained by reducing the competences of some of the persons in the group, so that the new average competence is exactly equal to the competence of the added person. Theorem 4, which we prove later, tells us that the combined group competence of the modified group plus the added person, which is less than the competence of the original group plus that person's, is greater than the competence of the modified group, which in turn is equal to the individual competence. Q.E.D.

Thus, people are better off entrusting decision making to a group that they join whose members have either average or median or majority competence greater than their own. However, as we show momentarily, a *group* can become worse off by adding a person with a competence greater than the group average.

Proposition 4: If all persons are minimally competent, and if the *average* competence of a group is *less* than the competence of an added person, then the combined group has greater majority competence than the original *group*.

Proposition 4′ appears quite reasonable, but we show a counterexample.

Counterexample: Consider a group (.5, 1, 1) with mean .83 and an added person with a competence of .9. The original group has an overall competence of 1.0, but the combined group has a competence of .975, less than that of the original group.

Proposition 5′: If all persons are minimally competent, and if a group's median competence is less than the competence of an added person, then the combined group has a greater overall competence than the original group.

Again, we demonstrate the falsity of the proposition with a counterexample.

Counterexample: Consider a group (.5, .5, 1) with median .5, and an added person with a competence of .6. The combined group has a majority competence of .725 compared with the original group with a majority competence of .75.

An added person can harm a group's competence if his competence is greater than the group's average or median competence. However, group majority competence inevitably improves if an added person has a greater competence than the group's majority competence, as this version of Proposition 6 demonstrates.

Theorem 4: If all persons are minimally competent, and if the group's majority competence is *less* than the added person's competence, then the combined group has greater *majority* competence than that of the original *group*.

Proof: The proof rests on the idea that, for groups in which all persons are minimally competent, the greater the plurality the more likely is the group to be correct (Grofman, Owen and Feld, 1982), and that the probability that a group is correct is just the weighted average of the probabilities that each plurality is correct (weighted by the expected frequency with which that plurality will occur). Consequently, the probability that a plurality of one voter is correct is the smallest number in that weighted average, and so it must be less than P_N. Let the individual competence be denoted P_j. We have

$$\frac{P_{m+}}{P_{m-}} < \frac{P_N}{1-P_N} . \tag{5}$$

But by assumption,

$$\frac{P_N}{1-P_N} < \frac{p_j}{1-p_j}, \tag{6}$$

$$(P_{m+})(1-p) < (P_{m-})p, \tag{7}$$

so that the probability of changing a bare majority in the original group to a bare minority in the combined group is less than the probability of changing a bare minority in the original group to a bare majority in the combined group; the combined group's majority competence must then be greater than the original group's majority competence.

In sum, these results show that a person's decisions necessarily improve if he combines with a group with greater average competence, or with greater median competence, or with greater majority competence than his own, and that a group's competence improves by adding in a person with greater than the group's majority competence. But a group's majority competence does not necessarily improve by adding in a person with greater-than-group average competence or median competence.

5. Combining groups

These findings hold implications for combining groups in general. First, the findings indicate that two groups can combine for an overall competence less than the average competence of either of them (see the counterexample for Proposition 4′), and two groups can combine for an overall competence less than the median competences of either of them (see the counterexample for Proposition 5′). Second, if there are any incompetent persons, a combined group can be less competent than either of its components; that is, this proposition is false:

Proposition 7: The combination of two groups has a *majority* competence *at least as great* as the majority competence of the *less competent group*.

The counterexample to Propositions 3 and 6 are sufficient to prove that this proposition is false. In addition, we can illustrate its falseness in the situation in which two groups, each containing more than one person, are combined. Specifically, a group can be combined with itself over and over, with lowered overall competence. Consider the group of three persons with competences of 0, .76, and .76. It has a mean competence of .51, a median competence of .76, and a majority competence of .578. If this group is combined with itself (to obtain a six-member group with competences 0, 0, .76, .76, .76, and .76 respectively), it has a majority competence of .544. If the same group is added in over and over again, then the group's majority competence often goes down, as Table 1 shows. This is a clear example of how we can combine competent groups to form less competent groups.[7]

That Theorems 3 and 4 are true suggests, however, that if all persons are minimally competent, then a combined group composed of a person and a group generally must be at least as competent as one of its parts. We suggest:

Proposition 8: If all persons are minimally competent, then the combination of two groups has a majority competence *at least as great* as the majority competence of the *less competent group*.

Unfortunately, we are unable to provide a general proof for this proposition.

Cases also occur in which the addition of *less* competent members, even those with an average or majority competence less than chance, will increase a group's majority competence. In particular, these two propositions are false.

Proposition 9: If the *average* competence of a supplementary group is less than chance (.5), then adding the supplementary group will diminish the majority competence of the combined group from that of the root group.

Counterexample: Consider a root group consisting of a person with a competence of .51 and a supplementary group of three persons with competences of 0, $\frac{20}{27}$ (.74), and $\frac{20}{27}$ (.74), respectively, and average competence of .49. The combined majority competence is then .512, a slight increase from the competence of the root group.

Proposition 10: If a supplementary group's *majority* competence is less than chance (.5), then adding the supplementary group will decrease the majority competence of the combined group from that of the root group.

Table 1. The majority competence for a group with individual competences (0, .76, .76) duplicated 1 to 10 times

# Times duplicated	*N*	Majority competence
1	3	.578
2	6	.544
3	9	.558
4	12	.548
5	15	.556
6	18	.551
7	21	.557
8	24	.554
9	27	.559
10	30	.557

One might suspect that the problem is that although the supplementary group in the counterexample to Proposition 9 has less than chance average competence, it has a greater than chance (.55) majority competence. Perhaps, as stated in Proposition 10, whenever the supplementary group's majority competence is less than chance, then adding the supplementary group cannot increase the root group's majority competence. However, consider the same root group as in the preceding example, now with a supplementary group of three persons, with competences of 1, .29, and .29, respectively, and a consequent majority competence of .496. The combined majority is .52, greater than the root group's majority competence, contrary to Proposition 10.

NOTES

1. We posit some ordering of alternatives along which the 'correctness' of alternatives can be judged, even if only on a *post hoc* basis.
2. For groups with even numbers, majority competence is calculated by assuming that when the group arrives at a tie vote it is correct half the time. Somewhat different results would obtain if we assume that a tie was a no vote with a rerun till a nontie resulted.
3. For an even-number group this is treated as the competence of the $\frac{N}{2}$th most competent member.
4. We agree with Margolis (1976) that there are many decisions in which some persons are more likely to choose the wrong answer than the right answer (as is evident in at least some questions on most any True/False examination), and we believe that we must recognize and understand those situations. But, we also believe that there are many decisions in which we can assume that all of the relevant decision makers are at least minimally competent (that is, $p_i \geq .5$), and it is useful to analyze such situations, even if the results do not apply more generally.
5. $P_{m+} > P_{m-}$ unless $P_o = .5$.
6. If we drop the requirement that all members are minimally competent, then the result does not hold. Sattler (1966) proposes it as true in its unrestricted form, but Grofman, Owen and Feld (1982) show it to be false by counterexample. They do not give the lemma stated here.
7. Of course, as we continue to replicate, the general limit result in Owen, Grofman and Feld (1981; see also Grofman, Owen and Feld, 1982) must hold, and majority competence increases with increasing group size, since the average competence of the group is greater than one half.

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