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A GAME THEORETIC APPROACH TO MEASURING DEGREE OF CENTRALITY IN SOCIAL NETWORKS *

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We present a new measure of degree of centrality in a social network which is based on a natural extension of the Banzhaf (1965) index of power in an N-person game.

1. Introduction

Beginning with the pioneering experiments of Alex Bavelas (1948, 1950), there has been considerable research interest in the issue of how group structure, in particular the pattern of (feasible) communication flows, affects various elements of group process. The usual method of research has been to impose various communication networks on groups and then to examine the consequences for group process. One area which has been the focus of a considerable amount of research is the study of how communication structures affect group members' perceptions of the existence of individuals who are seen to be engaged in a leadership role and/or facilitate or hinder the emergence of those patterns of behavior which are commonly labeled leadership.

We show in Fig. 1 (taken from Shaw 1954) some of the patterns

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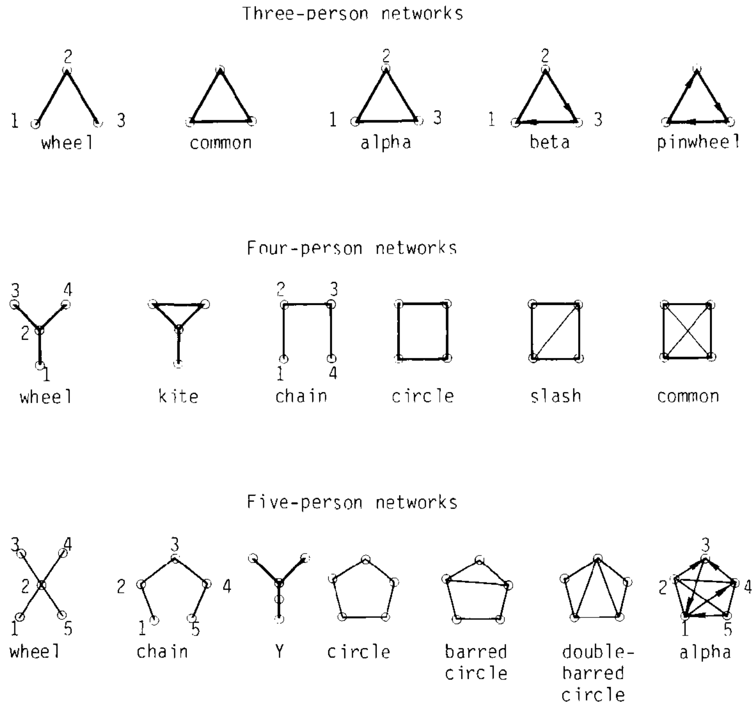

Figure 1. Some communication networks used in experimental investigations of group structure. Adapted from Shaw (1964).

which have most frequently been investigated. While most investigators have looked at communication networks, the graphs in Fig. 1 can be more generally interpreted in terms of any specified binary relationship. For the moment, however, we shall stick with graphs which will be interpreted in terms of communication structure.

In seeking to predict which individual or individuals would be seen to be acting as (or would emerge as) leaders and in studying the effect of communication networks on information flows, a number of authors (*e.g.* Leavitt 1951; Shaw 1954; Goldberg 1955; Shaw and Rothschild 1956; Cohen 1962) have made use of the idea of "centrality" in a graph-theoretic network. The hypothesis has been that the more central a position in a network, the more likely is the occupant of that position to emerge as (or be seen as) a leader. As Freeman (1977) has shown, the notion of "centrality" has a number of distinct meanings, although many of these give rise to convergent orderings of points for the case of the simpler network structures. Freeman (1979) reviews a number of different definitions of degree of centrality that have been offered in the literature. Freeman, Roeder and Mulholland (1979) provide some critical experiments in which different definitions give rise to different predictions, so as to test which of the various proposed notions of network centrality are most congruent with the leadership role.

In this paper we shall offer a new definition of degree of centrality, one based on a natural extension of a notion of power offered in the game theory literature: the Banzhaf (1965) power index. The measure we shall propose has some conceptual links to the notion of centrality as betweenness offered in Freeman (1977). Like the measures in Freeman (1977) and Freeman, Roeder and Mulholland (1979) the measure we propose will be applicable to both connected and unconnected graphs.

2. The Banzhaf index of power

For illustrative purposes, we shall show how the Banzhaf (1965) index [1] is defined for a weighted voting game. Consider a set of N actors $\{(1, 2, 3, \ldots, n)\}$, where the ith actor has weight w_i, $0 < w_i \leq 1$ and $\Sigma w_i = 1$. Define the quota $q, 0 < q < 1$ as the number of weighted votes needed for a motion to carry. Define a coalition as *winning* if the sum of the weights of its members is equal to or greater than q. Define a

[1] Readers may be more familiar with another game-theoretic power measure, the Shapley–Shubik value (Shapley and Shubik 1954). Recent axiomatic approaches to the measurement of power in N-person games (see *e.g.* Shapley and Dubey, 1977) have shown the Banzhaf index and the Shapley–Shubik value to be mathematically very closely related. Each can be taken to be a special (extreme) case of more general class of power measures (Straffin 1977; see also Owen 1975). We have found the Banzhaf notion of the swing actor to be easier to work with in the network context than the Shapley–Shubik notion of the pivotal actor, although the Shapley–Shubik value can also be defined in terms of swings (see Straffin 1976). Other approaches are also possible (see *e.g.* Deegan and Packel 1978), but dealing with more than one power measure would only add unnecessary complications without affecting the basic points we make in the discussion below. For useful general discussion of power indices see Lucas (1974), Brams (1975) and Straffin (1976). Empirical applications are found in Lucas (1974), Brams (1975), Straffin (1977), Owen (1977), Grofman (1981), and Grofman and Scarrow (1979, 1981).

swing (*decisive vote*) for actor i to be a winning coalition containing i, from which i's defection would be crucial, *i.e.* would change the coalition from winning to losing.[2] We define actor i's relative Banzhaf power index, B_i as

$$B_i = \frac{\text{number of swings for actor } i}{\text{total number of swings for all actors}}. \tag{1}$$

If we wish to compare the power of actors in different networks the network-specific normalization imposed in equation (1) may not be appropriate. An alternative is a total power score. In any network with N actors there are potentially $(2^N - 1)$ winning coalitions. We define the total Banzhaf power index, B'_i, as

$$B'_i = \frac{\text{number of swings for actor } i}{2^N - 1}. \tag{2}$$

A simple illustration is helpful. Let us consider a three-person game with $w_1 = 0.4$, $w_2 = 0.4$, $w_3 = 0.2$; $q = 0.51$. There are eight possible coalitions (see Table 1).

Despite the fact that Actors 1 and 2 have weights twice that of Actor 3, in Banzhaf power terms they are all equal. Consider the same game, but with $q = 2/3$. Now, Actors 1 and 2 have Banzhaf relative power scores of 0.5, while Actor 3 has a power index of zero.[3] However, for Actors 1 and 3 absolute power scores are unchanged.

3. Applications of the Banzhaf index to communication networks

Now let us see how these ideas can be applied to communication networks. For communication networks, let us define a *winning coalition* as a path from actor j to actor k, for any actors, $j, k, j \neq k$. Define a *swing* (*decisive vote*) for actor i, to be a winning coalition containing i, from which i's defection would be crucial, *i.e.* such that without actor i, the remaining member of the coalition cannot construct from among their own members a path from j to k. Define the relative centrality of

[2] We could also have defined the Banzhaf index in terms of votes which turn coalitions from losing into winning. Because of symmetry, in general the two definitions will be equivalent.

[3] In the language of game theory, Player 3 is said to be a *dummy*.

Table 1
Banzhaf power index values for three actor weighted voting games *.

Actor	1	2	3	Weighted	Outcome
Weight	0.4	0.4	0.2	Votes in favor	(q = 0.51)
	Y	Y	Y	1	P
	(Y)	(Y)	N	0.8	P
	(Y)	N	(Y)	0.6	P
Votes	Y	N	N	0.4	F
	N	(Y)	(Y)	0.6	P
	N	Y	N	0.4	F
	N	N	Y	0.2	F
	N	N	N	0	F
Number of decisive votes (swings)	2	2	2		
B_i	1/3	1/3	1/3		
B'_i	2/7	2/7	2/7		

* Decisive votes (swings) are circled.

an actor with respect to a given network as

$$C_i = B_i = \frac{\text{number of swings for actor } i}{\text{number of swings for all actors}}. \tag{3}$$

Define the total centrality of an actor as

$$C'_i = \frac{\text{number of swings for actor}}{2^{N-4}(N+2)(N-1)}. \tag{4}$$

The reason for the denominator in expression (4) is that this is the maximum possible number of swings. In fact, for each k $(=1,\ldots,n)$, player i can belong to $\binom{N-1}{k-1}$ different sets with k players. Each such set can provide a path between any of $\binom{k}{2}$ pairs of players. All told this gives us

$$\sum_{k=2}^{N} \binom{k}{2}\binom{N-1}{k-1}$$

possible swings. Some algebra can then be used to show that this last sum is indeed equal to the denominator in expression (4).

Some illustrations will again be helpful. Consider first the three-person wheel pattern of Fig. 1. There are six winning coalitions (1, 2), (2, 1), (2, 3), (3, 2), (3, 2, 1) and (1, 2, 3). Note that (1, 2) is distinct from (2, 1) since the former is a coalition involving a path from 2 to 1, while the latter is a coalition involving a path from 1 to 2. If all connections are bidirectional, then by symmetry we need consider only half the possible cases. Since we wish to consider graphs both with bidirectional and unidirectional elements, we shall present all feasible coalitions even if symmetry would make it possible for us to reduce the set. In the first four of these coalitions both members are crucial; in the last two all three are critical. Hence the total number of swings for player 1 is four and the same is true for player 3, while for player 2 the total number of swings is six. Hence, $B_1 = 2/7, B_2 = 3/7, B_3 = 2/7$. While in this example values of C_i' are almost identical to those for B_i, in general $\Sigma_i C_i' \leq 1$, while $\Sigma_i C_i = 1$.[4]

We can show that the relative power of the central actor in a wheel pattern is a monotonically decreasing function of N but with an asymptote of $1/3$, while the relative power of any hub actor is roughly inverse to N. For $N = 4$ (with B_2 in the hub), we have twelve winning coalitions: (1, 2), (1, 2, 3), (1, 2, 4), (2, 1), (2, 3), (2, 4), (3, 2, 1), (3, 2), (3, 2, 4), (4, 2, 1), (4, 2), (4, 2, 3); and $B_1 = B_3 = B_4 = 1/5$; $B_2 = 2/5$. For $N = 5$ we have 20 winning coalitions, and $B_1 = B_3 = B_4 = B_5 = 2/13$; $B_2 = 5/13$. In general, for a wheel pattern there will be $(N-1)N$ winning coalitions, and a total of $4(N-1) + 3(N-1)(N-2)$ swings $(= 3N^2 - 5N + 2)$. Hence, treating Actor 2 as the hub, we will have

$$B_2 = \frac{N(N-1)}{3N^2 - 5N + 2} = \frac{N(N-1)}{(3N-2)(N-1)} = \frac{N}{3N-2}; \tag{5}$$

and thus for $i \neq 2$, $B_i = 2/(3N-2)$. It is obvious that $\lim_{N \to \infty} B_2 \to 1/3$, while $\lim_{N \to \infty, i \neq 2} B_i = 0$. In absolute power terms we have

$$C_2' = \frac{N(N-1)}{2^{N-4}(N+2)(N-1)} = \frac{N}{2^{N-4}(N+2)} \tag{6}$$

[4] Alternatively, we might wish to exclude from swings the initiating and terminating points of any communication and measure only "middleman" power. If we did this, then $B_1 = B_3 = 0$, and $B_2 = 1$. This idea of "middleman power" is closer in spirit to what is proposed in Freeman (1977). Which of these two measures is more appropriate will depend upon the group process being investigated. If Actor 2 wishes to communicate with Actor 1 or Actor 3, these actors have the power to deny Actor 2 his wish, and our basic approach to defining relative power (which credits both players with a swing in any pairwise linkage) would appear the best.

and for $i \neq 2$, the total centrality of all non-hub actors is

$$\sum C_i' = \frac{2(N-1)^2}{2^{N-4}(N+2)(N-1)} = \frac{N}{2^{N-5}(N+2)}. \tag{7}$$

Also, the ratio of B_2 to ΣB_i (or of C_2' to $\Sigma C_i'$) is simply $2(N-1)/N$.

The distinction between *total* power share and relative power ratio is important. It may be that actors are seen as central when their power is large relative to that of *any* other actor, even if it may not be especially large relative to the combined total of *all* other actors or to the maximum possible number of swings (*cf.* Freeman 1977: 39). However, for simplicity of exposition we shall focus on B_i (*i.e.* C_i) rather than on C_i' in the discussion that follows.

For purposes of comparison, let us calculate the power index values for various other patterns in Fig. 1. Let us look next at the chain. For $N = 3$, the chain is identical to the wheel and hence $B_1 = 2/7$, $B_2 = 3/7$, $B_3 = 2/7$. For $N = 4$, we have $B_1 = 3/16$, $B_2 = B_3 = 5/16$, $B_4 = 3/16$. For $N = 5$ we have $B_1 = B_5 = 4/30$, $B_2 = B_4 = 7/30$, $B_3 = 8/30$. For $N = 6$ we have $B_1 = B_6 = 1/10$, $B_2 = B_5 = 9/50$, and $B_3 = B_4 = 11/50$.

Some tedious but straighforward analysis reveals that, for a chain, the general formula is given by

$$B_{N-i} = \frac{2[(N-1)+(N-1-i)i]}{N^3+3N^2-4N/3} = \frac{6[(N-1)(i+1)-i^2]}{N^3+3N^2-4N} \tag{8}$$

for $i = 0,\ldots,(N-1)/2$ for N odd; and for $i = 0,\ldots,(N/2)-1$ for N even. The remaining values can, of course, be filled in by symmetry.

Now let us turn to graphs in which there are some one-way communication flows. Consider the three-person network labeled alpha in Fig. 1. There are nine winning coalitions: (1,2), (1,2,3), (1,3), (2,1), (2,3,1), (2,3), (2,1,3), (3,1) and (3,1,2). However, not all actors are decisive in each. Consider, for example, the winning coalition (1,2,3): Actor 2 is not essential, since the (1,3) coalition exists. Analogous results obtain for the (2,3,1) and (2,1,3) coalition. On the other hand, for the (3,1,2) coalition all actors are decisive, since no alternative path between 3 and 2 exists. When we count swings we obtain seven for Actor 1, six for Actor 2, and six for Actor 3. Hence Actor 1, who has bidirectional communication with both other actors, has the most power (centrality) in this communication network: $B_1 = 7/19$, $B_2 = B_3 = 6/19$.

Analogous results obtain for the three-peorson communication network labeled beta in Fig. 1. For this structure there are seven winning coalitions: (1, 2), (1, 2, 3), (2, 1), (2, 3), (2, 3, 1), (3, 1), and (3, 1, 2). All actors are decisive in each winning coalition, except for Actor 3 who is not decisive in the coalition (2, 3, 1); hence $B_1 = B_2 = 3/8$ (0.375) and $B_3 = 2/8$ (0.25). If we look at the five-actor network labeled alpha in Fig. 1, the relative centrality of the actor with the most bidirectional links is reduced and we find that there are 101 winning coalition for this five-actor network with $B_1 = 0.20$, $B_2 = 0.23$, $B_4 = 0.17$, $B_5 = 0.19$ and $B_5 = 0.21$.

This is a good point in our discussion to emphasize two features of the model we have been using. We have implicitly been assuming, first, that all winning coalitions are equally likely and, second, that all winning coalitions are equally important. There is nothing in the nature of the mathematics we have been using that constrains us to restrict ourselves to these assumptions. If, for example, we are dealing with, say, battlefield communications, then we might wish to look at only a restricted set of what we've been calling "winning coalitions," or to weight certain important communication linkages more heavily than others. Also, we might wish to drop the assumption of all coalitions being equally likely, if other features of the group or its members permit us to infer that some communication interactions will be more frequent than others. (For treatments of directly analogous issues in voting games, see Owen 1971; Straffin 1977; Merrill 1978, and Frank and Shapley 1981.)[5] For example we might wish to deal only with the

[5] Another alternative to the approach suggested above is also worth mentioning. In this modification we would calculate Banzhaf power scores for the winning coalitions *separately* for each (i, j) linkage and then *average* these index values for the entire set of (i, j) pairs – taking a weighted average if not all (i, j) pairs were to be regarded as being of equal importance – rather than looking at power share based on all winning coalitions as we did above. This modification would affect all the calculations we presented, although the differences will in general be minor. For example, for the three-person wheel (if we assign equal weight to the six possible two-way links), we would have $B_1 = B_3 = 17/60$ and $B_2 = 28/60$, rather than $B_1 = B_3 = 2/7$ and $B_2 = 3/7$. For the three-actor alpha network we would have $B_1 = B_2 = 0.36$ and $B_3 = 0.28$ rather than $B_1 = B_2 = 0.375$ and $B_3 = 0.25$. For the five-actor beta network we would have values of $B_1 = 0.23$, $B_2 = 0.23$, $B_3 = 0.17$, $B_4 = 0.20$, and $B_5 = 0.18$ as compared to the values of $B_1 = 0.20$, $B_2 = 0.23$, $B_3 = 0.17$, $B_4 = 0.19$ and $B_5 = 0.21$ obtained earlier. Our initial approach makes most sense in those cases where the likelihood that i will communicate with j can be taken to rise linearly with the number of paths through which such a communication is possible. The alternative approach we have just outlined may be preferred in those cases where the likelihood that i will wish to communicate with j is independent of the number/variety of communication channels between them.

subsets of minimal winning coalitions. The Freeman (1977) approach looks at betweenness in terms of the set of geodesics. Geodesics in graphs can be thought of as the natural analogue to minimal winning coalitions in game theory. The Banzhaf measure is based on decisiveness in all winning coalitions rather than being restricted to the set of minimal winning coalitions. The Deegan–Packel (1978) index of power is based on only minimal winning coalitions, but we shall not pursue here the development of an alternative game theoretic measure of centrality analogous to the Deegan–Packel power index rather than the Banzhaf index.

4. Other graph-theoretic applications of the Banzhaf index

Banzhaf scores can be derived for any binary relation once we specify what is to count as a "winning coalition." One natural application of Banzhaf scores is as measures of relative status in a hierarchic (or partly hierarchic) structure. Let P be a binary relation such as "exercises direct supervisory responsibility over" or "can issue direct orders to." For each ordered pair (i,j) of members of the organization $(i \neq j)$, let us define a winning coalition for (i,j) as a path from i to j, *i.e.* a chain by which i can indirectly (or directly) exercise supervisory responsibility over (give orders to) j. The group members who comprise this path will be said to be the members of this winning coalition. As before, we define a swing (decisive vote) for an individual in a winning coalition as a situation in which the individual's removal turns the winning coalition into a losing one, *i.e.* breaks the chain of command connecting i and j.

To the extent that organizational status comes from supervisory responsibilities, we can measure relative degree of supervisory scope at various hierarchy levels in organizations with different spans of control and flatness. Consider, for example, a 13-person organization structured as in Fig. 2 (left-hand diagram). There are 21 winning coalitions. The top-level boss is decisive in 12, each of the second-level managers is decisive in 7, and the third-level workers are each decisive in only 2. Hence, $B_1 = 0.24$, $B_2 = 0.14$ $(\times 3 = 0.42)$, and $B_3 = 0.04$ $(\times 9 = 0.36)$. Now consider another 13-person organization, structured as in Fig. 2 (right-hand diagram). For this organization there are 22 winning coalitions. The top-level boss is decisive in 12, each of the second-level

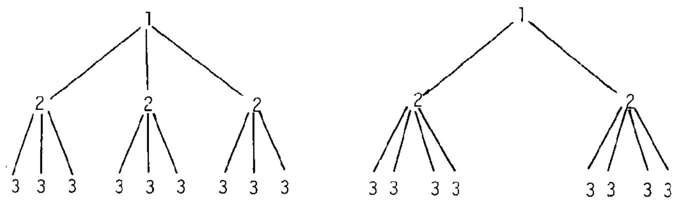

Figure 2. Two 13-person organizations.

managers is decisive in 11, and the 10 workers are each decisive in 2 winning coalitions each. The middle-level managers each have power scores of 0.20 compared to the top-level boss's power score of 0.22 and the workers' power scores of roughly 0.04 each. This may seem like a somewhat strange result, but it merely reflects the fact that, as we have drawn the organizational chart in Fig. 2 (right-hand), the top-level boss cannot transmit orders directly to the workers, while the middle-level managers can. Thus, the model reflects, perhaps quite realistically, the "power" of those supervisors (*e.g.* shop foremen) in direct supervision over large numbers of workers. Of course, if we modify our model so as to differently weight the importance of commands, treating commands issued to higher levels of management as more important than commands issued merely to workers, or if we change the organization chart to allow top management to bypass middle-management in setting policies which control the workers, then the Banzhaf measure will show an increase in the power of top management relative to that of middle management.

With appropriate "realistic" modifications (*e.g.* as to weightings as to the importance of different types of communication linkages), we believe the Banzhaf scores can provide a more theoretically useful measure of supervisory status than simply counting the number of workers whom an individual directly (or indirectly) supervises or looking at the distance matrix (Harary, Norman and Cartwright 1965: 189–191) or looking at status levels (Kemeny and Snell 1962: 104–105). We also believe the above examples demonstrate why top-level bosses who want to exercise power (1) prefer command structures in which there are *multiple routes* by which orders from the top can be funneled down to lower levels of the organizational hierarchy, and (2) want the ability to issue orders directly to subordinates, bypassing intermediate hierarchy levels (*cf.* Harary, Norman and Cartwright 1965: 273). In tree structures such as those of Fig. 2, middle-level managers serve as "bottlenecks" who block communication and control lines, and gain power in so doing! (*cf.* Freeman 1977: 36.)

6. Conclusions

We hope to have demonstrated how an approach borrowed from *N*-person game theory can provide a conceptually elegant key to understanding structural properties of social networks. We hope that others will be stimulated to investigate potential empirical applications of the approach we have proposed and to refine and modify our measurement techniques as necessary.

References

Banzhaf, John F., III
1965 "Weighted voting doesn't work: a mathematical analysis." *Rutgers Law Review 19:* 317–343.
Bavelas, Alex
1948 "A mathematical model for group structure." *Applied Anthropology 7:* 16–30.
1950 "Communication patterns in task oriented groups." *Journal of the Acoustical Society of America 57:* 271–282.
Brams, S.J.
1975 *Game Theory and Politics.* New York: Free Press.
Cohen, A.M.
1962 "Changing small group communication networks." *Administrative Science Quarterly 6:* 443–462.
Deegan, John, and Edward Packel
1977 "A new index of power for simple *N*-person games." Paper presented at the Annual Meeting of the Public Choice Society, New Orleans.
Frank, Frank A., and Lloyd Shapley
1981 *The Distribution of Power in the U.S. Supreme Court.* Rand.
Freeman, Linton C.
1977 "A set of measures of centrality based on betweenness." *Sociometry 40:* 35–41.
1979 "Centrality in social networks." *Social Networks 1/3:* 215–240.
Freeman, Linton C., Douglas Roeder and Robert R. Mulholland
1979 "Centrality in social networks: II experimental results." *Social Networks 2:* 119–141.
Goldberg, S.C.
1955 "Influence and leadership as a function of group structure." *Journal of Abnormal and Social Psychology 51:* 119–122.
Grofman, Bernard N.
1981 "Fair apportionment and the Banzhaf index. *American Mathematics Monthly 88:* 1–5.

Grofman, Bernard N and Howard Scarrow
1979 "Iannucci and its aftermath: the application of the Banzhaf criterion to weighted voting systems in the State of New York." In Steven Brams, Andrew Schotter and Gerhard Schwodiauer (eds.), *Applied Game Theory.* Vienna: Physica-Verlag.
1981 Weighted Voting in New York. *Legislative Studies Quarterly VI/2.*

Harary, Frank, Robert Z. Norman and Dorwin Cartwright
1965 *Structural Models: An Introduction to the Theory of Directed Graphs.* New York: John Wiley and Sons.

Kemeney, John G. and J. Laurie Snell
1962 *Mathematical Models in the Social Sciences.* New York: Ginn.

Leavitt, Harold J.
"Some effects of certain communication patterns on group performance." *Journal of Abnormal and Social Psychology 46:* 38–50.

Lucas, W.F.
1974 "Measuring power in weighted voting systems." *Case Studies in Applied Mathematics.* Mathematics Association of America: 42–106.

MacKenzie, Kenneth D.
1966 "Structural centrality in communications networks." *Psychometrika 31:* 17–25.

Merrill, S.
1978 "Citizen voting power under the electoral college: a stochastic model based on state voting patterns." *SIAM Journal of Applied Mathematics 34:* 376–390.

Owen, Guillermo
1971 "Political games." *Naval Research Logistics Quarterly 18:* 545–554.

Owen, Guillermo
1975 "Multilinear extensions and the Banzhaf values." *Naval Research Logistics Quarterly 22:* 741–750.

Owen, Guillermo
1977 "Evaluation of a presidential election game." *American Political Science Review 49:* 947–953.

Shapley, L.S., and P. Dubey
1977 "Mathematical properties of the Banzhaf power index," Rand Paper P-6016, The Rand Corporation, Santa Monica, California.

Shapley, L.S., and M. Shubik
1954 "A method for evaluating the distribution of power in a committee system." *American Political Science Review 48:* 787–792.

Shaw, Marvin, E.
1954 "Group structure and the behavior of individuals in small groups." *Journal of Psychology 38:* 139–149.

Shaw, Marvin E., and G.H. Rothschild
1956 "Some effects of prolonged experience in communication nets." *Journal of Applied Psychology 40:* 281–286.

Straffin, Philip D., Jr.
1976 "Power indices in politics." Module in Applied Mathematics, Mathematical Association of America, Cornell University.
1977 "Homogeneity, independence, and power indices." *Public Choice 30:* 107–118.