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THE LOGICAL FOUNDATIONS OF IDEOLOGY

by Bernard Grofman[1] *and Gerald Hyman*[2]

College students' assignments of probabilities to a set of 15 interrelated propositions are shown to be highly consistent with a formal model of belief systems which satisfies the axioms of probability theory and propositional logic and which is based on the operation of symmetric difference. The model allows us to impute subjectively perceived truth functional relationships between propositions in a belief system to actors on the basis of their subjective probability assignments to these propositions and their pairwise conjunctions. The model also enables us to derive measures of the distance between sets of beliefs; of the congruence and consistency of belief systems; and of the degree of polarization of belief systems.

THE ASSUMPTION of systemicity is the central assumption of science because unless we assume that process is systematic we abandon the attempt at understanding and comprehension: without system there is no order; without order there is no comprehension. There can be no science without the assumption of systemicity. That our inability to understand and therefore deal with chaos and randomness may be a function of our genetic construction (Tax, 1960; Geertz, 1965) and that we may therefore—and not because it accurately portrays empirical reality—require the assumption of systemicity, ironically lends to that central scientific assumption an almost religiously transcental quality. We cling to it with a faith almost approaching, one might say, religious fervor. This quality is not always even covert. The religious vocabulary in which the systemic assumption is stated is sometimes quite overt even among the more sophisticated scientists (McGuire, 1968, p. 143).

Usually, if we test it at all, we test the systemic assumption indirectly in terms of the hypotheses we can generate by making it. That is as it should be. But a more direct test has not been constructed. It would require that systemicity itself be the potential variable and, therefore, that the empirical background which exemplifies it be rendered irrelevant. The study of ideology, precisely because it is so vague, provides one such possible background. By ideology we mean here, the system of concepts in terms of which people order and comprehend their social system. It is precisely the assumptions of order, system and comprehension that we wish to test here.

Three properties are usually associated with the notion of system: (1) connectedness, the notion that the elements comprising a system are related one to the other so that changes of sufficient magnitude in one element will effect changes in the other elements of the system; (2) consistency, the notion that these connections are somehow logical or at least not antithetical to each other; and (3) coherence, the notion that the consistency of the connections defines, somehow, a natural web or relationships, i.e., a web at the cognitive level which makes natural sense of a web at the empirical level, naturally exists. In fact, these three properties—connectedness, consistency and coherence—are often taken to be the defining characteristics of the notion of system.

Studies of cognitive process have traditionally emphasized differing dimensions of judgment. These dimensions have included, among others, utility (Davidson & Suppes, 1957), morality (Lane, 1962), affect (Heider, 1946; Harary, 1959; Festinger, 1957) and probability of occurrence (McGuire, 1960a; 1960b; 1960c). A major concern has been the extent to which attitudes and cognitions are constrained, i.e., "the success we would have in predicting, given initial knowledge that an individual holds a specific attitude, that he holds further ideas and attitudes" (Converse, 1964, p. 207). McGuire, (1968, pp. 141–142, 161–162) more forcefully in his early work than in his later work

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(McGuire, 1968; 1960a; 1960b; 1960c), took as a working assumption that "people's conceptions are highly interconnected and that people maintain a high degree of internal coherence within this structure" (1968, p. 142), and he has contrasted what Abelson (1968) has termed the maximalist position with what he called the minimalist position of those who "seem to regard the human cognitive system as having the fine infrastructure of a bowl of oatmeal or an urn of marbles (McGuire, 1968, p. 143; Freedman, 1968; Converse, 1964; Luttbeg, 1968).

In this paper we shall confine our attention to one dimension of judgment, the probability dimension. We shall analytically separate the actor's perceptions of the way the world is, his world view, from his values about that world—what he likes and dislikes about it—and from his consequent views as to the way the world ought ideally to be in some best of all possible worlds, his social ideal. We shall deal only with the former. While the probability components of a cognitive system are in principle analytically separable from both its affective and normative components, the practicality of separating them empirically is rather more problematic.

Following Hall and Fagen (1956) we shall assume the maximalist position to be tested and we shall define a system as a set of elements and their attributes together with the relationships among and between the various elements and their various attributes. By a belief system—the belief component of a cognitive system—we shall mean here a set of propositions along with an actor's beliefs in them, as measured by his probability assignment to them, together with the relationships among and between propositions and beliefs. Note that, following McGuire (1960c), we use the term beliefs in a somewhat peculiar way: by an actor's belief in a given event or proposition—which events or propositions may be hypothetical—we simply mean his assessment of the probability of the proposition (event) being true. This assessment is, of course, subjective. Beliefs should not be confused with intensities of affect, nor should they be confused with the second order judgments which reflect the actor's degree of confidence in his probability judgments, nor, because of the confounding factor of an actor's preferences for risk or risk avoidance (Friedman & Savage, 1948), need they be equivalent to the odds at which the actor would accept a bet on the events taking place or the propositions being true. By belief we mean, quite simply, "the conviction of the truth or reality of a thing . . . " (*American College Dictionary*, 1957), and we propose to measure the depth of the actor's conviction by simply asking him.

Central to the research design of this paper is the notion that beliefs are not randomly held but are arranged, by definition, into systems of interrelated elements; and that changes in some elements of the system can be expected to eventually effect changes in other elements of the system. Our aim is to present a measurement model which will make these linkages among beliefs and changes therein subject to precise empirical measurement.

We shall denote individual propositions by lower case letters, and sets of propositions by capitals. We shall denote by $a_1, a_2, \cdots, a_n$, the propositions in a set A, which may be conjectured to be part of some belief system A'. To represent relationships among propositions as perceived by some given actor(s), we shall use the familiar logical operators $\rightarrow$, implies; $\leftrightarrow$, equivalent to; $-$, negation; $\vee$, or; $\wedge$, and. Similarly, we shall use the set theoretic operators $\cup$, union; $\cap$, intersection; $\in$ membership in; $\subset$, is included in; $\supset$, includes; to represent relationships among sets.

We shall denote by $p(a_i)$ an actor's subjective belief in the truth of proposition a_i. If we confine ourselves to propositions a_i containing free unquantified variables, and expressing set theoretic statements such as $X \in Y$, then $p(a_i)$ may be interpreted as the percentage of Xs which are perceived by the actor also to belong to Y. For example, the actor might be asked to assess the probability that members of the John Birch Society support, i.e., are included in the class of those who support, school busing for the purpose of integration. In this case, $p(a_i)$ would be the percentage of John Birch members whom the actor believes also to be supporters of school busing for integration, i.e., $p(a_i)$ gives the likelihood of a randomly chosen Bircher also supporting busing. Similarly, we invite the reader to interpret $p(a_i \wedge a_j)$ as the percentage of xs for which the relationships $a_i(X \in Y)$ and $a_j(X \in Z)$ are believed by the actor to simultaneously hold. While a frequency interpretation of probabilities is the most common, a number of difficulties occur in coping with subjective probabilities of unique events (Nagel, 1939). Although we believe the difficulties are not insurmountable, we shall not attempt to deal with the subjective probability of propositions such as "Richard Nixon will be regarded by history as a good President," but shall confine ourselves to propositions which state set theoretic relationships of the kind that can be interpreted in percentage terms, such as that previously cited about John Birch Society members.

The questionnaire by which we intended to test the systemic assumption requested students to estimate the percentage of students at Stony Brook/Smith who (1) had smoked pot at least once, (2) favored legalized abortion, (3) favored the legalization of marijuana, (4) thought that progress in securing jobs and housing for blacks has been too slow, (5) were Jewish. Then they were asked to estimate the percentage of students at Stony Brook/Smith who shared two of these traits, e.g., "What percentage of Stony Brook (Smith) students do you think have smoked pot at least once and would also favor the legalization of marijuana." Students were instructed to fill out the questionnaire as precisely as they could and to think before they answered. Because of possible ambiguities in the wording of the questions about conjoint attitudes, attributes, the experimenter(s) explained that these questions asked for "the percentage of Stony Brook (Smith) students who were both ______ and ______." Note that three of the properties referenced involve affect, favor legalized abortion, favor legalized marijuana, progress for blacks too slow; one involves overt behavior, smoke pot at least once; and one involves an attribute, Jewish. This mix was deliberately used to demonstrate the potential range of applicability of the model.

TABLE 1
PROBABILITY MATRIX FOR SUBJECT 12

		Smoke pot at least once a_1	Favor legalized abortion a_2	Favor legalized marijuana a_3	Progress for blacks too slow a_4	Jewish a_5
Smoke pot at least once	a_1	70	65	40	45	60
Favor legalized abortion	a_2	65	89	60	55	65
Favor legalized marijuana	a_3	40	50	50	40	50
Think that progress for blacks too slow	a_4	45	55	40	68	40
Jewish	a_5	60	65	50	40	70

When the relationships among attitudes, attributes, are being examined, the questionnaire requires $k + \binom{k}{2}$ questions; in this case 15 ($k = 5$). In addition, students were asked whether they themselves had smoked pot at least once, etc. These questions were added to enable us to test the hypothesis that students with given attitudes, attributes, would overestimate those attitudes, attributes, in the general student population. The questionnaire totaled 20 questions; it required, on the average, including instructions, about 12 minutes to complete. The questionnaire was administered to 91 Stony Brook students in an introductory political science course, and to 58 Smith students in an introductory anthropology course.

For purposes of example, let us now consider a typical protocol—Stony Brook Protocol 12—which displays some, but not many, logical inconsistencies. Table 1 gives Subject 12's percentage responses in tabular form.

The diagonal entries of Table 2 give $p(a_i)$ for the generalized case.

The ijth entry gives $p(a_i \wedge a_j)$. Naturally, the matrix is symmetrical. Internal consistency in a belief system requires that, for all i,j:

I $p(a_i \wedge a_j) \leqq p(a_i)$

II $p(a_i \wedge a_j) \leqq p(a_j)$

III $p(a_i) + p(a_j) - p(a_i \wedge a_j) \leqq 100$ percent

TABLE 2

MATRIX OF RESPONSES FOR A GENERALIZED HYPOTHETICAL SUBJECT

		Smoke pot at least once	Favor legalized abortion	Favor legalized marijuana	Progress for blacks too slow	Jewish
		a_1	a_2	a_3	a_4	a_5
Sample pot at least once	a_1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}
Favor legalized abortion	a_2	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}
Favor legalized marijuana	a_3	a_{31}	a_{32}	a_{33}	a_{34}	a_{35}
Progress for blacks too slow	a_4	a_{41}	a_{42}	a_{43}	a_{44}	a_{45}
Jewish	a_5	a_{51}	a_{52}	a_{53}	a_{54}	a_{55}

Requirements I and II are obvious. Requirement III follows from the fact that $p(a_i \wedge a_j) + p(a_i \wedge -a_j) + p(-a_i \wedge a_j) + p(-a_i \wedge -a_j) = 100$ percent, but its implications for consistency can be quite strong. Looking at Fig. 1, we see that $p(a_2 \wedge a_3) > p(a_3)$; hence II is violated. In order to permit manipulations of this subject's protocol in accord with the model we shall subsequently develop, we may remove the inconsistency by setting $p(a_2 \wedge a_3) = p(a_3)$. We also note that $p(a_2) + p(a_4) - p(a_2 \wedge a_4) = 102$ percent > 100 percent, violating III. Here, we may remove the inconsistency by setting $p(a_2 \wedge a_4) = p(a_2) + p(a_4) - 100$. This redefinition preserves conformity to constraints I and II. The transformed logically consistent protocol is shown in Table 3. Note that consistency required the changing of only three entries out of the 20 nondiagonal entries, and only 1.4 percent (14/1010) of the possible changes in nondiagonal percentages.

Table 4 shows the consistency of the Smith and Stony Brook students sampled vis a vis criteria I and II. The modal student in both Smith and Stony Brook did not violate criterion I or II. The mean student at both Smith and Stony Brook violated one or the other of these criteria in 16 percent of his or her responses. Over three-

TABLE 3

PROTOCOL FOR SUBJECT 12 REVISED TO ELIMINATE INCONSISTENCIES

		Smoke pot at least once	Favor legalized abortion	Favor legalized marijuana	Progress for blacks too slow	Jewish
		a_1	a_2	a_3	a_4	a_5
Smoke pot at least once	a_1	70	65	40	45	60
Favor legalized abortion	a_2	65	89	50	57	65
Favor legalized marijuana	a_3	40	50	50	40	50
Think that progress for blacks too slow	a_4	45	57	40	68	40
Jewish	a_5	60	65	50	40	70

quarters of the students sampled at both Smith and Stony Brook had fewer than 25 percent of his/her responses inconsistent with these criteria. In general, Smith and Stony Brook students were virtually identical in the extent to which they satisfied these

TABLE 4

DISTRIBUTION OF RESPONSES INCONSISTENT WITH THE CONJUNCTION OF REQUIREMENT I ($p(a_i \wedge a_j) \leq p(a_i)$); REQUIREMENT II ($p(a_i \wedge a_j) \leq p(a_j)$)*

Number of inconsistent responses	Absolute frequency (Relative frequency in percent)		Cumulative frequency in percent	
	Smith ($N = 58$)	Stony Brook ($N = 91$)	Smith	Stony Brook
0	19 (32.8)	28 (30.8)	32.8	30.8
1	9 (15.5)	15 (16.5)	48.3	47.3
2	5 (8.6)	9 (9.9)	56.9	57.1
3	5 (8.6)	10 (11.0)	65.5	68.1
4	4 (6.9)	3 (3.3)	72.4	71.4
5	4 (6.9)	4 (4.4)	79.3	75.8
6	2 (3.4)	6 (6.6)	86.2	82.4
7	1 (1.7)	2 (2.2)	89.7	84.6
8	1 (1.7)	2 (2.2)	91.4	86.8
9	1 (1.7)	3 (3.3)	93.1	90.1
10	0 (0.0)	3 (3.3)	93.1	93.4
11	0 (0.0)	1 (1.1)	93.1	94.5
12	1 (1.7)	2 (2.2)	94.8	96.7
13	1 (1.7)	2 (2.2)	96.6	98.9
14	0 (0.0)	0 (0.0)	96.6	98.9
15	1 (1.7)	0 (0.0)	98.3	98.9
16	0 (0.0)	1 (1.1)	98.3	100.0
17	0 (0.0)	0 (0.0)	98.3	100.0
18	0 (0.0)	0 (0.0)	98.3	100.0
19	1 (1.7)	0 (0.0)	100.0	100.0
20	0 (0.0)	0 (0.0)	100.0	100.0

* The maximum possible number of inconsistent responses is 20. The mean number of actual inconsistencies is 3.2 for both Smith and Stony Brook students.

TABLE 5

DISTRIBUTION OF THE NORMALIZED SUM OF CORRECTIONS NEEDED IN THE NONDIAGONAL MATRIX ENTRIES OF FIG. 2 TO INSURE CONSISTENCY WITH REQUIREMENT I $p(a_i \wedge a_j) \leqq p(a_i)$; REQUIREMENT II $p(a_i \wedge a_j) \leqq p(a_j)$*

Normalized sum of deviations from consistency in percent	Absolute frequency (Relative frequency in percent)		Cumulative frequency in percent	
	Smith (N = 58)	Stony Brook (N = 91)	Smith	Stony Brook
.00	19 (32.8)	28 (30.8)	32.8	30.8
.01– .10	3 (5.1)	14 (15.4)	37.9	46.2
.11– .25	12 (20.7)	13 (14.2)	58.6	60.4
.26– .50	3 (5.1)	13 (14.2)	63.8	74.7
.51– .75	7 (12.1)	5 (5.5)	75.9	80.2
.76– 1.00	2 (3.4)	3 (3.3)	79.3	83.5
1.01– 1.25	3 (5.1)	4 (4.4)	84.5	87.9
1.26– 1.50	0 (0.0)	2 (2.2)	84.5	90.1
1.51– 1.75	1 (1.7)	1 (1.1)	86.2	91.2
1.76– 2.00	1 (1.7)	0 (0.0)	87.9	91.2
2.01– 3.00	2 (3.4)	0 (0.0)	91.4	91.2
3.01– 5.00	1 (1.7)	0 (0.0)	93.1	91.2
5.01–10.00	1 (1.7)	2 (2.2)	94.8	93.4
10.01–20.00	1 (1.7)	3 (3.3)	96.6	96.7
above 20.00	2 (3.4)	3 (3.3)	100.0	100.0

* The maximum possible normalized sum of deviations from consistency is 100. The means of the actual deviations are 3.3 for Smith students and 2.5 for Stony Brook students.

TABLE 6

DISTRIBUTION OF RESPONSES INCONSISTENT WITH REQUIREMENT III THAT $p(a_i) + p(a_j) - p(a_i \wedge a_j) \leqq 100$ PERCENT*

Number of inconsistent responses	Absolute frequency (Relative frequency in percent)		Cumulative frequency in percent	
	Smith (N = 58)	Stony Brook (N = 91)	Smith	Stony Brook
0	34 (58.6)	37 (40.7)	58.6	40.7
1	13 (22.4)	22 (24.2)	81.0	64.8
2	6 (10.3)	17 (18.7)	91.4	83.5
3	3 (5.2)	6 (6.6)	96.6	90.1
4	2 (3.4)	6 (6.6)	100.0	96.7
5	0 (0.0)	1 (1.1)	100.0	97.8
6	0 (0.0)	1 (1.1)	100.0	98.9
7	0 (0.0)	1 (1.1)	100.0	100.0
8	0 (0.0)	0 (0.0)	100.0	100.0
9	0 (0.0)	0 (0.0)	100.0	100.0
10	0 (0.0)	0 (0.0)	100.0	100.0

* The maximum possible number of inconsistent responses is 10. The mean distributions of actual inconsistencies are .724 for Smith students and 1.28 for Stony Brook students.

basic consistency requirements, and both student groups were reasonably consistent. Nonetheless, given the simplicity of these criteria, it is remarkable how many students violated one or the other of them at least once.

As can be seen from Table 5, however, students' deviations from consistency were usually relatively minor. For example, for almost 80 percent of the Smith students and for over 80 percent of the Stony Brook students, changes in fewer than one percent of the nondiagonal matrix entries (see Table 6) were needed to obtain patterns of responses consistent with criteria I and II. The consistency of the two samples was again quite similar, but we see here that Smith students appear slightly more consistent than those at Stony Brook, a finding which recurs in the case of the less obvious criterion, criterion III.

Again, both student groups are, on the whole, quite consistent. In particular, 40.7 percent of the Stony Brook students and 58.6 percent of the Smith students satisfy criterion III for all propositional pairings. If we look at the sum of corrections, normed to 100, needed in the nondiagonal matrix entries of Fig. 2 to insure consistency with the requirement that $p(a_i) + p(a_j) - p(a_i \wedge a_j) \leqq 100$ percent, we find that the means of the deviations are 0.09 for Smith students and 0.14 for Stony Brook students. Clearly both groups require, on the average, quite minimal numbers of corrections to their matrix entries to obtain consistency with criterion III. In short, college freshmen and sophomores are capable of generating responses which are predominantly consistent with our set of simple, axiomatically derived consistency requirements.

Consider, then, the set of propositions about which we have obtained student responses. If these propositions are perceived by the actor as related, i.e., if some are perceived as implied, or evoked by, or are perceived to imply or evoke others, then the actor's belief in one of these propositions ought logically to be constrained by his belief in other propositions which he perceives as related. If the actor's subjective probability assignment satisfies the usual probability axioms,

I $0 \leqq p(a_i) \leqq 1$

II $p(a_i \vee -a_i) = 1$

III $p(a_i \wedge -a_i) = 0$

IV $p(a_i \vee a_j) = p(a_i) + p(a_j) - p(a_i \wedge a_j)$

V $p(a_i \wedge a_j) = p(a_j|a_i)p(a_i)$

where $p(a_j|a_i)$ refers to the conditional probability of a_j given a_i, and if the actor is logical in the sense that his subjective beliefs about the logical relationships between propositions obey the rules of propositional logic, then we may readily prove a number of theorems which indicate the constraints imposed upon an actor's beliefs when some aspects of his belief system are assumed fixed.

For example, let us assume that $a_1 \rightarrow a_2$, i.e., the actor subjectively perceives the first proposition to imply the second, then $p(a_1)$ ought to be less than or equal to $p(a_2)$ and $p(a_1 \wedge a_2)$ ought to equal $p(a_1)$. Similarly if $a_2 \wedge a_1 \rightarrow a_3$ and a_1 and a_2 are perceived as unrelated, i.e., independent in the sense that knowledge about one proposition is not perceived by the actor as providing him with knowledge about the other, then $p(a_3)$ ought to be greater than or equal to $p(a_1) \times p(a_2)$. These and similar results, although independently derived by the senior author, were subsequently found in Reichenbach (1938).

Let us now consider a simple verbal example. Suppose that a man believes that all Americans who are not members of the John Birch Society are communist sympathizers and also believes that no member of the John Birch Society is a communist sympathizer. We can ask him his assessment of the percentage of Americans who are members of the John Birch Society and his assessment of the percentage of Americans who are communist sympathizers. Now, if the man is consistent in one clear sense of that term, then these two percentages should sum to 100 percent, the subjective probabilities should sum to one. As we shall see below, it is also possible to make inferences in the reverse direction, i.e., from actual subjective probability assessments back to subjectively perceived logical relationships among propositions. And we shall attempt to construct a measure that will enable us to estimate an actor's perceived psychological relationships among our five propositional elements.

Consider the following set theoretic operator, $\oplus$, the symmetric difference operator. If a_1, a_2 are sets of beliefs, then

(1) $a_1 \oplus a_2 \equiv (a_1 \vee a_2) - (a_1 \wedge a_2)$.

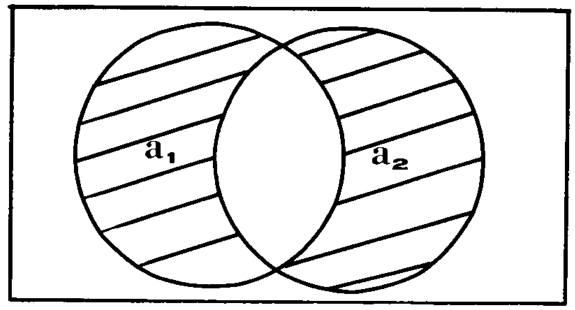

FIG. 1. $a_1 \oplus a_2$: The difference between two sets.

The operator $\oplus$ gives us the elements in the sets a_1,a_2 which are unique to them, exclusive of the elements they hold in common; it is equivalent in mathematical terms to the exclusive sense of the word "or" which is sometimes notated $\vee$. For sets a_1,a_2, $a_1 \oplus a_2$ may be regarded as a measure of the difference between them, given by the shaded area in Fig. 1.

Now, consider:

(2) $p(a_1 \oplus a_2) \equiv p(a_1 \vee a_2) - p(a_1 \wedge a_2)$.

We can readily see that:

(3) $p(a_1 \oplus a_2) = p(a_1) + p(a_2) - 2p(a_1 \wedge a_2)$.

Let us define, for notational convenience:

(4) $p(a_i \oplus a_j) \equiv p \oplus (a_i,a_j)$.

We may readily establish that $p \oplus (a_1,a_j)$ is a distance metric, i.e.,

I $p \oplus (a_i,a_j)$ 0 and $p \oplus (a_i,a_j) = 0$ if and only if $a_i \leftrightarrow a_j$

II $p \oplus (a_i,a_j) = p \oplus (a_j,a_i)$

III $p \oplus (a_i,a_j) + p \oplus (a_j,a_k) \geq p \oplus (a_i,a_k)$.

The symmetric difference operator gives us, in effect, the measure of commonality of subjective meaning for two propositions, the extent to which they are subjectively seen to imply and be implied by the same things. The greater their commonality, the more they are seen to have common implications and antecedents; the fewer their differences, the greater their degree of "andness," and the smaller their degree of "orness," the less they are seen as incongruent. In those cases where there is complete commonality between a_i and a_j, where the subjective implications, evocations, of

TABLE 7
THEORETICAL $p \oplus$ VALUES FOR EIGHT SELECTED HYPOTHESES

Hypothesis relating a_i and a_j	Value of $p \oplus (a_i, a_j)$ associated with h_k being true
h_1 $a_i \leftrightarrow a_j$	0
h_2 $a_i \rightarrow a_j$	$p(a_j) - p(a_i)$
h_3 $a_j \rightarrow a_i$	$p(a_i) - p(a_j)$
h_4 a_j	$1 - p(a_i)$
h_5 a_i	$1 - p(a_j)$
h_6 $- a_i \rightarrow a_j$	$2 - p(a_i)\, p(a_j)$
h_7 $a_i \rightarrow a_j$	$p(a_i) + p(a_j)$
h_8 $a_i \wedge a_j \rightarrow a_j \wedge a_i$	$p(a_i) + p(a_j) - 2p(a_i \wedge a_j)$

the two propositions are identical, $p \oplus (a_i,a_j)$ will be zero; where there is no commonality between them and where they are negations of one another in their subjective implications, $p \oplus (a_i,a_j)$ will be maximal, in this case equal to one (see Table 7). Other authors who make use of the symmetric difference operator in this context are Restle (1959), and Bruner, Shapiro, and Taguiri (1959).

Column 2 of Table 7 gives the values which $p \oplus (a_i,a_j)$ ought to take on if a_i and a_j are perceived to be related as specified in column 1, and conversely for column 1. Table 7 is exhaustive of the eight basic pure truth functional relationships between two propositions; the remaining eight may be generated as negations of the hypotheses in the table. Of the values of $p \oplus (a_i,a_j)$ for the 16 possible cases, four are of particular interest.

(5a) $(a_i \leftrightarrow a_j) \leftrightarrow (p \oplus (a_i,a_j) = 0)$

(5b) $(a_i \rightarrow a_j) \leftrightarrow (p \oplus (a_i,a_j) = p(a_j) - p(a_i))$

(5c) $(a_j \rightarrow a_i) \leftrightarrow (p \oplus (a_j,a_i) = p(a_i) - p(a_j))$

(5d) $(a_i \leftrightarrow -a_j) \leftrightarrow (p \oplus (a_i,a_j) = 1)$.

We may use these relationships to define subjective equivalence, implication and negation. Thus, by ascertaining an actor's subjective probability assignments to a_i,a_j and $a_i \wedge a_j$, we may then impute to him the logical relationship between a_i and a_j to which his computed value of $p \oplus (a_i,a_j)$ most nearly corresponds. Thus in the limiting cases, if $p \oplus (a_i,a_j) = 0$, then we shall say a_i is subjectively equivalent to a_j; if $p \oplus (a_i,a_j) = p(a_j) - p(a_i) \neq 0$, then we shall say a_i subjectively implies a_j; if $p \oplus (a_i,a_j) = 1$, then we shall say that a_i and a_j are subjective negations of one another. Analogously, we may use the symmetric difference operator to define subjective logical independence, since if two propositions are subjectively independent then $p \oplus (a_i,a_j)$ ought to take on the value $p(a_i) + p(a_j) - 2p(a_i)\, p(a_j)$. We shall refer to the hypothesis that two propositions are subjectively independent as h_I.

In effect, then, by knowing only an individual's direct probability assignments and simple pairwise probability assignments among propositions, in theory we can impute to him an entire belief system complete with perceived relationships among propositional elements. Moreover, for each truth functional relationship possible between two propositions, we can construct a measure of the extent to which the experimental value of $p \oplus (a_i,a_j)$ approximates any such hypothesized relationship. For convenience, we wish a measure which varies between 0 and 1; it should be 0 if and only if the hypothesized relationship between propositions is perfectly exemplified by the experimental value of $p \oplus (a_i,a_j)$, and it should be 1 if and only if the experimental value of $p \oplus (a_i,a_j)$ is as far from its hypothesized value as it can be, given the experimental values of $p(a_i)$, $p(a_j)$ and $p(a_i \wedge a_j)$. Let $p \oplus (h_k,a_i,a_j)$ be defined as that value of $p \oplus (a_i,a_j)$ which obtains when the relationship h holds between propositions a_i and a_j. Note that the values of $p \oplus (h_k,a_i,a_j)$ for h_1 through h_8 are given by column 2 of Table 7. A measure which satisfies our requirements is

(6) $$d_{h_k}(a_i,a_j) \equiv [p + (a_i,a_j) - p + (h_k,a_i,a_j)]/\max \langle p + (h_k,a_i,a_j),\ 1 - p + (h,a_i,a_j)\rangle.$$

We may readily verify that $d_{h_i}(a_i,a_j) \equiv p \oplus (a_i,a_j)$.

By means of $d_{h_k}(a_i,a_j)$ we can measure the fit between an individual's views as to the logical relationship between two propositions as he might state it when asked directly, and the relationship between them that could be inferred from the data provided by his probability assignments. More

TABLE 8

	a_i	$-a_i$
a_j	a	b
$-a_j$	c	d

importantly, from probability data alone we can infer psychological connections between propositions. In order to see this link in more familiar terms, let us express experimentally obtained subjective probabilities in terms of the entries in a 2 × 2 contingency table (Table 8).

Where $a + b + c + d = 1$ and where

$$(7)\quad p \oplus (a_i,a_j) = p(a_i) + p(a_j) - 2p(a_i \wedge a_j) = [(a + c) + (a + b) - 2a]/100 = (b + c)/100.$$

We may restate the relationships of Table 7 in contingency table terms (Table 9).

Similarly, we may show that

(8a) for $ad \geqq bc$:

$$d_{h_i}(a_i,a_j) = 2(ad - bc)/(ab + ac + b^2 + bd + c^2 + cd + 2ad)$$

0 iff $ad = bc$, 1 iff $b + c = 0$

(8b) for $ad \leqq bc$:

$$d_{h_i}(a_i,a_j) = 2(bc - ad)/(a^2 + ab + ac + bd + cd^2 + 2bc)$$

0 iff $ad = bc$, 1 iff $a + d = 0$.

To see how these techniques work in practice, let us turn once again to the responses generated by Subject 12. Table 10 gives the b cell values of the 2 × 2 contingency table derived from Table 3. Analogous tables for c and d values are readily generated. The value in the a cell of the contingency table relating a_i and a_j is, of course, given by the ijth value of the matrix of Table 3.

In order to determine the optimal representation of protocol 12 in truth functional terms, we would generate matrices analogous to that of Table 11 giving the $d_{h_k}(a_i,a_j)$ values for each h_k, and would then determine for each ij combination the h_k for which $(1 - d_{h_k}(a_i,a_j))$ attained its maximum. We performed these operations for the relationships of protocol 12. The results are shown

TABLE 9

THEORETICAL VALUES OF d_{hk} FOR EIGHT SELECTED HYPOTHESES*

	Hypothesis relacing a_i and a_j	$d_{hk}(a_i, a_j)$	
h_1	$a_i \leftrightarrow a_j$	$(b + c)/100$	0 iff $b + c = 0$ 1 iff $a + d = 1$
h_2	$a_i \rightarrow a_j$	$c/(a + c + d)$	0 iff $c = 0$ 1 iff $a + d = 0$
h_3	$a_j \rightarrow a_i$	$b/(a + b + d)$	0 iff $b = 0$ 1 iff $a + d = 0$
h_4	a_j	$c/(a + c)$	0 iff $c = 0$ 1 iff $a = 0$
h_5	a_i	$b/(a + b)$	0 iff $b = 0$ 1 iff $a = 0$
h_6	$-a_i \rightarrow a_j$	$d/(b + c + d)$	0 iff $d = 0$ 1 iff $b + c = 0$
h_7	$a_i \rightarrow -a_j$	$a/(a + b + c)$	0 iff $a = 0$ 1 iff $b + c = 0$
h_8	$a_i \wedge a_j \leftrightarrow a_j \wedge a_i$	0	

* Formulas shown for $p \oplus (h_k,a_i,a_j) \leqq 1 - p \oplus (h_k,a_i,a_j)$.

in Fig. 2 for implication and independence only.

From Fig. 2 we see that no traits are perceived as psychologically independent attributes, and a_3 is seen as subjectively implying both a_1 and a_4 as well as a_2 and a_5. Similarly a_1 is seen as subjectively implying a_2 while a_1 and a_5 are mutually implicatory, i.e., appear with equal probability. The relationships between a_4 and a_5, a_2 and a_4, etc., appear to be something other than equivalence, implication or independence.

Analogous tables and figures can, of course, be generated for the other 90 Stony Brook and 58 Smith students but since the addition of another 148 anonymous, indi-

TABLE 10

b CELL VALUES IN PERCENT FOR SUBJECT 12

	a_1	a_2	a_3	a_4	a_5
a_1	0	24	10	23	10
a_2	5	0	0	11	5
a_3	30	39	0	28	20
a_4	25	32	10	0	30
a_5	10	24	0	28	0

TABLE 11
IMPLICATORY RELATIONSHIPS (h_2) IMPUTED TO SUBJECT 12 ON THE BASIS OF THE VALUE $1 - d_{h_2}(a_i,a_j)$

		Smoke pot at least once a_1	Favor legalized abortion a_2	Favor legalized marijuana a_3	Progress for blacks too slow a_4	Jewish a_5
Smoke pot at least once	a_1	1.00	.93	.67	.68	.89
Favor legalized abortion	a_2	.75	1.00	.61	.64	.75
Favor legalized marijuana	a_3	.86	1.00	1.00	.86	1.00
Think that progress for blacks is too slow	a_4	.70	.84	.69	1.00	.60
Jewish	a_5	.89	.93	.80	.58	1.00

vidual protocols and maps replicating those of Subject 12 seems hardly calculated to move the frontiers of knowledge very far, composite tables and figures for Smith and Stony Brook students, respectively, seem rather more enlightening and useful. These are generated by taking mean values for individual matrix entries, a procedure justifiable only given a high concordance in psychological networks between individuals which, indeed, we find.

The composite data for Stony Brook students, represented in Fig. 3, show implicatory relationships for six pairs of

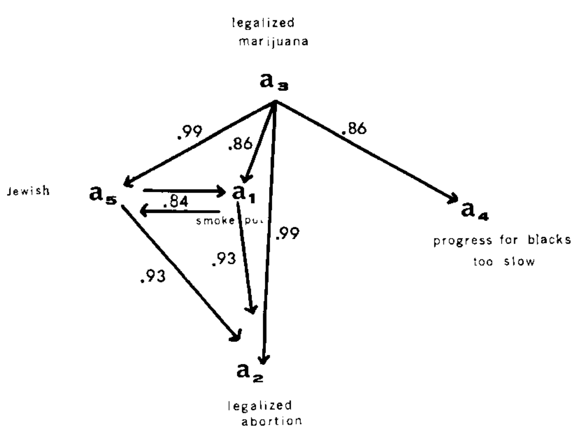

FIG. 2. Ideological map for Subject 12.*

* In terms of implication and independence, cutoff value of .8.

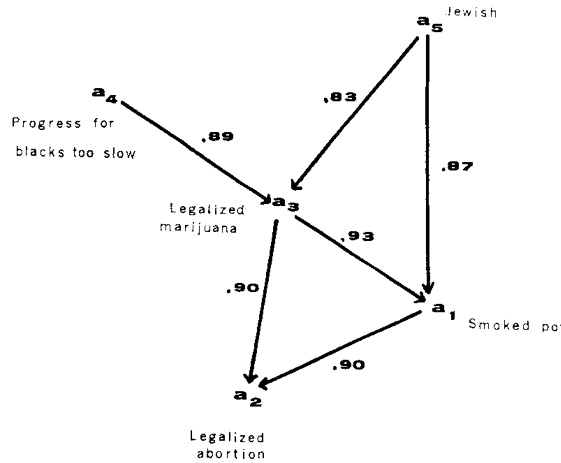

FIG. 3. Composite ideological map for Stony Brook students ($N = 91$).†

propositions and independence relationships for none; while Fig. 4, which gives composite data for the set of Smith students queried, shows implicatory relationships for four pairs of propositions, and independence relationships for none; with only one overlap in implicatory relationships between the two data sets.

We might also note that, unlike those for Stony Brook, the Smith data show being Jewish not to be a part of the psychological belief network. In fact, had we reduced the cutoff value to .7, being Jewish would have been defined, for Smith students, as independent of all of the other four traits.

These findings for our Smith and Stony Brook students are suggestive of possible shared cognitive networks in these student communities. One problem with the data, for example, is response set biases: some subjects achieved consistency in part by giving largely identical responses for all percentages, thus producing a response pattern with the appearance of mutually implicatory attitudes; others gave responses so close to 100 percent, introducing h_4 and h_5 as additional confounding hypotheses, that judging between h_1, h_2 and h_3 became virtually impossible, etc. Nevertheless we do provide a technique for construction of testable models of individual—and under the right conditions, corporate—belief sys-

† In terms of implication and independence, cutoff value of .8.

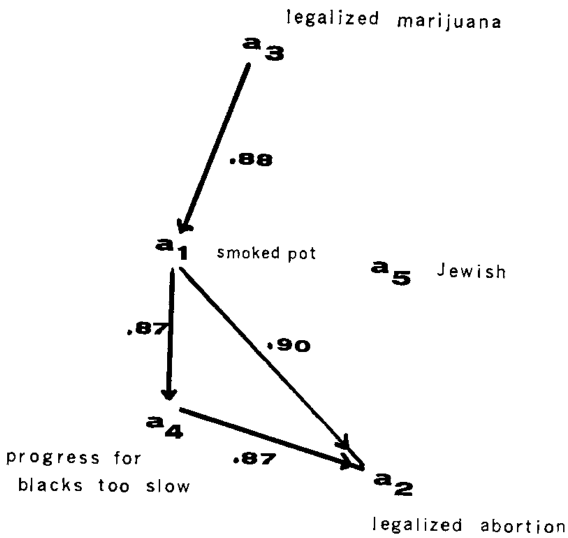

FIG. 4. Composite ideological map for Smith students (N = 58).*

tems which, in the absence of competing techniques or models, seems reasonably interesting if not immediately useful.

Let us consider now the notion of the consistency of a belief system. We may differentiate at least four meanings which can be attached to the term: (1) the actor's beliefs are consistent with scientific truth, the facticity of the real world. (2) The actor's beliefs are logically consistent. The beliefs satisfy the axioms of propositional logic and the axioms of subjective probability; this is the sense in which we were using consistency earlier. (3) The actor's beliefs are systematically interrelated, they hang together. (4) The actor's beliefs are not randomly generated and are stable at least in the short run. This type of consistency is related to the notion of construct reliability common in the sociological literature.

We shall label these types of consistency, respectively: scientific consistency, coherence; logical consistency; connectedness, systemicity; and reliability. These types of consistency are not independent of each other. For example, if the actor's beliefs are scientifically consistent, they are consistent in all of the other senses, at least under most commonly held epistemological and ontological views. Indeed, each type of consistency implies consistency of a lower type.

We may readily develop measures of the extent to which a belief system exemplified a particular level of consistency (Grofman & Hyman, 1973). We shall present one such measure for the connectedness of a system of beliefs, a measure which varies between 0 and 1.

To measure type 3 consistency, connectedness (systemicity), we make use of the measure defined on the hypothesis "a_i and a_j are perceived of as independent," that is, we let

$$(9)\quad \text{Connectedness}(A') \equiv [\textstyle\sum a_i \sum a_j \, d_{h_i}(a_i, a_j)]/m(m-1), \quad a_i \neq a_j .$$

We may do so since independence, as we have defined it, may be regarded as the paradigmatic case of nonsystemicity.

We may make use of Eq. (9) to determine the connectedness of our subjects' protocols. The mean connectedness value of Stony Brook protocols was .38; for Smith protocols it was .42. While a value of 1.00 would indicate perfect connectedness, the mathematical properties of $d_{hI}(a_i, a_j)$ are such that even values as low as .30 still reflect a reasonably high level of connectedness, since values near one are obtained only when both cells on one of the diagonals of the contingency table are close to zero. Still another sense of inconsistency, related to types 2 and 3, is involved in the notion that two propositions are inconsistent if they are perceived by the actor as being contradictory. This kind of inconsistency we shall call congruence, and its lack incongruence. It is clear that the paradigmatic case of congruence is $a_i \leftrightarrow a_j$ and of incongruence $a_i \leftrightarrow a_j$. It should also be clear that if sets of propositions are either highly congruent or highly incongruent, they will be highly systematically related, that is, highly connected.

CONCLUSIONS

It is useful to compare our approaches to other more traditional approaches to the analysis of cognitive systems. Unlike Abel-

* In terms of implication and independence, cutoff value of .8.

son and Rosenberg's (1958) justly famous model of "symbolic psycho-logic," and unlike most Likert-scale based techniques (Campbell et al, 1960), or the work of balance theorists (Heider, 1946; Harary, 1959), our approach is not restricted to dichotomous or even polychotomous attitudes, but provides ratio scale measurement. We believe this to be an important advantage. As McGuire has emphasized "people do not consider propositions to be simply true or false but to have different gradations of assent" (McGuire, 1968, p. 157). Moreover, while our model is restricted to probability judgments and cannot be extended to directly deal with affect, it can deal indirectly with judgments with affective dimensions when those are assessments of the affective components of other people's attitudes, e.g., "What percentage of ______ do you think like/dislike ______?" It may be that asking an actor to make such derivative judgments can be used indirectly to reveal affective dimensions of the actor's own attitudes. Such an approach may prove quite useful for the study of attitudinal stereotyping offering an improvement over the usual checklist method. Also, we believe, although we have not rigorously demonstrated, that the usual assumptions of Guttman scaling (MacRae, 1957) can be subsumed as a special case of our model: one in which all propositions are assumed to be chainwise implicatory. Restle (1959) has proven a related result. Finally, measures of association derived from our d_{h_i} measures may be used as components of factor analyses of belief or other cognitive dimensions.

Like most techniques for attitudinal analysis, with the notable exception of content analysis, our model requires specially prepared questionnaire data. Moreover, the data that are required are unlike that normally gathered. This provides a considerable limitation to our method's widespread use—at least until such time as its advantages have been clearly demonstrated. Also, as previously noted, the level of sophistication required to generate the requisite probabilities is high, perhaps unrealistically so. It might be possible, however, to lower our requirements and enlarge our potential sample base by asking for probability responses on a Likert-like scale, e.g., very likely, likely, about 50-50 chance, unlikely, very unlikely, and then converting this data into metricized or quasimetricized form suitable for use in our model, e.g., providing an example in percentage terms of how these category responses should be used. We could create a certain likely response range centering around each scale category (Muller, 1971). In any case, we have, at minimum, already established that college students have no difficulty responding to questions couched in terms of percentages and give responses which generally satisfy the interconnectedness and consistency constraints of our model.

There are two other important difficulties in applying our model to empirical data, but both difficulties are shared by most other approaches to the study of attitudes. On the one hand we have no clear way of initially determining the appropriate universe of propositions whose systematic properties we wish to examine; we have no nonarbitrary way of determining the cognitive boundaries within which we can expect coherence. On the other hand we have no way of weighing the importance to the actor of one belief or element relative to others; we have no measure of salience. The first difficulty may be resolved in large part by extensive pretest. Also, more use might be made of free association procedure. The subject could, for example, be given certain propositions and asked to generate other propositions that he feels are related to the given one (McGuire, 1968, p. 65). The second difficulty, too, may not be insuperable and might be ameliorated by combining our technique with some scaling technique for ascertaining belief salience. Taking this approach would enable us to compare beliefs in terms of both salience and centrality. We concur however with Shepsle (1971, p. 792) that "this theoretical void, i.e., a measure of (political) salience, needs desperately to be filled."

We see as the chief advantages of our approach its ability to provide a simple, unified, operationalizable and mathematically quite powerful way to measure and distinguish among such important aspects of individual belief systems as their consistency and connectedness; its suitability for graph theoretic mapping of individual belief systems; and its potential use in the measurement of differences in beliefs among sets of actors, i.e., for the measurement of the belief dimension of ideologies.

We believe the potential applications of our model are many:

(1) In experiments on attitude change. For example, in experiments designed to test the strain to consistency hypothesis and to distinguish between the various ways actors may attain consistency, e.g., minimizing connectedness vs. maximizing congruence, McGuire (1968, p. 143), for example, has argued that "people do not simply minimize inconsistency, but . . . they also seek maximum interconnectedness in their belief system."

(2) In studies of coalition formation and ideological conflict. For example, Axelrod's (1969) theory of minimum winning coalitions utilizes a notion of ideological connectedness which might be formalized along lines suggested by our model.

(3) In small group studies. For example, the link between commonality of group beliefs, as formalized by our model, and group satisfaction and morale might be investigated.

(4) In studies of mass-elite and elite-elite relations. For example, the commonality of political belief systems between elites and masses as well as between sections of the elite might be examined.

In short, we believe our model to be high in potential, yet we recognize that its advantages and disadvantages will become apparent only after a considerable amount of other work. Whether that work is worth the effort is, of course, debatable. In any case we think our model, at least on the theoretical if not the practical level, has given a harder, more precise form than that traditionally given to the notion of system and to two of its three components: interconnectedness and consistency.

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