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PUBLIC CHOICE SOCIETY

Annual Meeting

May 3, 4, 5, 6,

Hotel Webster Hall
Pittsburgh, Pennsylvania

\$16.00	Single Room
\$20.00	Twin Room

Registration will be through Otto A. Davis
Public Choice Society
School of Urban and Public Affairs
207 MMCC
Carnegie Mellon University
Pittsburgh, Pennsylvania 15213

Registration Fee will be announced later.

Check-in time Wednesday, May 3
Meetings: Thursday, May 4, Friday, May 5 thru Saturday Noon, May 6

A Note on Some Generalizations of the Paradox of Cyclical Majorities.

Bernard Grofman *

Let us define a Condorcet (j, i) choice as a candidate which could receive at least j votes (j=1, n) in a paired contest against at least k-i-l (i=0, k-2) of the other k-l alternatives. It is easy to see that for j=m (where m is a majority) and for i=0, the Condorcet (j, i) choice is simply the familiar Condorcet choice. We may also readily verify that if a candidate is a Condorcet (j, i) choice for some given j_o, i_o, then it is also such a choice for all $i > i_o$, $j < j_o$.

It is well known that there need not exist a Condorcet choice, and thus that there need not exist a Condorcet (j, 0) choice for $j > m$. Is there always a Condorcet (j, 0) choice, $j < m$? More generally, for what values of j and i can we guarantee the existence of a C (j, i) choice? Let us consider this question for strong preference orderings. We shall assume n committee members (n odd) and k alternatives.

Theorem 1: If $\dfrac{(nk(k-1)/2)-k(j-1)}{k(k-2)} > n$, then there exists a Condorcet (j, 0) choice for $j < m$.

Proof: There are nk(k-1)/2 votes to be distributed when there are n voters and k alternatives. If every candidate receives fewer than j votes against at least one other candidate, then at minimum there are k matrix entries less than j; and in particular these k matrix entries contain at most k(j-1) votes. That leaves at least (nk(k-1)/2)-k(j-1) votes to be distributed among the remaining k(k-2) possibly nonzero cell (all diagonal entries are zero). But there cannot be more than n votes in any given cell. Hence if the quotient shown above were to exceed n this would contradict the absence of a C(j, 0) choice. Q.E.D..

If the above quotient does not exceed n, then there *may* be no C(j, 0) choice. We may readily show that the antecedent conditions of the above theorem are met for k=1, 2 for all n, and not met for $k \geqslant 3$ for all n. Hence, we obtain, *very* loosely speaking, a voting analogue to Arrow's "Possibility Theorem for Two Alternatives."[1]

Theorem 2: If $\dfrac{(nk(k-1)/2) - k(j-1)\ (i+1)}{k(k-i-2)} > n$, then there exists a C(j, i) choice.

The proof of this theorem is analogous to the proof given for Theorem 1. The antecedent conditions of this theorem are met if $j \leqslant m$ and i=k-2. Hence, the

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[1] Kenneth Arrow, *Social Choice and Individual Values,* 2nd Edition, New York: John Wiley, 1962.

remarkably unsurprising result that there will always exist a candidate who can receive a simple majority or less against at least one other candidate. On the other hand, for $j > m$, even $i=k-2$ is not sufficient to guarantee the existence of a $C(j, k-2)$ choice, i.e. a choice which can receive a special majority of j $(j > m)$ against even one other alternative. Moreover, even for $j < m$, only for $i \geqslant k-3$ does there necessarily exist a $C(j, i)$ choice.

Let us now consider what happens when we restrict ourselves to single-peaked preferences.

Lemma 1: Any subset of a single-peaked set of preference schedules is itself single-peaked.[2]

Lemma 2: Any set of single-peaked preference schedules contains a Condorcet choice.[3]

Theorem 3: For any single-peaked set of preference schedules there exist at least $m-j+1$[4] candidates who can receive at least j votes $(j \leqslant m)$ against at least $k-m+j-1$ other alternatives (including other members of the $m-j+1$ member set); i.e., there ixist at least $m-j+1$ distinct $C(j, j-m)$ choices.

Proof: By Lemma 2 above there will exist a Condorcet choice and we know such a choice will be unique. Hence, Theorem 3 obtained for that choice. Now consider the set of preference schedules obtained when the Condorcet choice is deleted. By Lemma 1 this set will still be single-peaked and hence by Lemma 2 there will be a new Condorcet choice for the set. But the new Condorcet choice will, by definition, be able to receive at least $m-1$ $(=n/2)$ votes against at least $k-2$ other alternatives. Thus, for $j=m-1$ there will exist $m-j+1$ $(=2)$ candidates who can receive at least j $(=m-1)$ votes against at least $k-m+j-1$ $(=k-2)$ other alternatives—to wit, the old and new Condorcet choices—and so on. Q.E.D.

[2] The proof of this lemma is trivial.

[3] For a proof see Duncan Black, *The Theory of Committees and Elections*, Cambridge University Press, 1958. Recall that we have stipulated n odd. For n even, the Condorcet Choice is that candidate, if any, who can receive at least $\frac{n}{2}$ votes in paired contest against each of the other k-1 other alternatives, and in the event two candidates satisfy this condition, is the canidate favored by the chairman.

[4] Defined only for $m-j+1 \leqslant k-1$.

Committee Decisions with Complementary Valuation by Duncan Black and R. A. Newing which has long been unavailable has now been reprinted by William Hodge and Company, its original publisher. It may be obtained from their Londan office for 1 20s

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Monographs

The Economics of Charity, by Thomas R. Ireland and David B. Johnson (1970), \$6.50.

The Political Theory of a Compound Republic, Vincent Ostrom (1971), \$6.95 Hard Cover, \$1.25 paperback.

These books may be purchased from the Center for Study of Public Choice, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061.