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Why so much stability?

Research note
Partial single-peakedness: An extension and clarification

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Abstract

Niemi (1969), in an important but neglected paper, found that when orderings were drawn from a simulation based on the impartial culture, the greater the proportion of voter orderings that were single-peaked (a condition he called partial single-peakedness), the more likely was there to be a transitive group ordering. Niemi also found that the likelihood of transitivity increased with n, group size – approaching one as n grew large. Niemi's simulation was restricted to the case of three alternatives. Also, he provided no theoretical explanation for the results of his simulation. Here we provide a theoretical explanation for Niemi's results in terms of a model based on the idea of net preferences, and we extend his results for the general case of any finite number of alternatives, m, for electorates that are large relative to the number of alternatives being considered. In addition to providing a rationale for Niemi's (1969) simulation results, the ideas of net preferences and opposite preference we make use of have a wide range of potential applications.

1. Introduction

The 'paradox of cyclical majorities' whereby group majorities may be cyclical even though each individual voter has a linear preference ordering, has been a subject of intense theoretical interest to political scientists and social choice theorists because cyclical majorities appear to preclude the possibility of reasonable collective decisions (Black, 1958; Arrow, 1963; Riker, 1961, 1980, 1982). Theoretical considerations seem to dictate that cyclical majorities should be commonplace, especially as m, the number of alternatives, increases (Riker, 1982: 182–188; Bell, 1978; McKelvey, 1976; McKelvey and Wendell, 1976; Schofield, 1978; Kramer, 1973; Plott, 1967). Yet empirical observations of a wide variety of actual collective decision-making processes indicate that cyclical majorities are very rare. Thus, cycles do not appear to be a real problem for group decision making (Niemi, 1983),

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although some paradoxes may occur which go undetected.

The inconsistency between theory and practice has led researchers to ask 'Why so much stability?' (see esp. Tullock, 1981). One proposed answer is to suppose that individual preferences are restricted to certain possibilities that are consistent with an underlying (usually ideological) continuum. It is well known that if *all* individual preference orderings are 'single-peaked' with respect to the continuum defined by some linear ordering, then majority preferences will contain no cycles (Black, 1958). However, there are practically no political situations where everyone has single-peaked preferences over all alternatives. To have all voter orderings be single-peaked with respect to some given linear ordering is an *extremely* strong condition, since it requires that only 2^{m-1} of the m! possible orderings occur, where m is the number of alternatives. If we let $Q = \frac{2^{m-1}}{m!}$, we can see that Q rapidly goes to 0 as m gets large. In particular:

m:	3	4	5	6	7
Q:	.67	.33	.13	.04	.01

Common sense might suggest that when 'almost all' individual preference orderings are single-peaked with respect to the continuum defined by some linear ordering, then there will be a transitive majority preference ordering, or at least a clear majority winner. While 'common sense' is in this case correct (see below), it is also misleading. To require even that most orderings be single-peaked with respect to some given left-right ordering of alternatives is still a *very* strong requirement, and, contrary to common sense, having most preferences single-peaked is not at all necessary in order for the group majority to act *as if* their preferences were generated by an underlying single-peaked ordering, as we shall show below.

In an important but neglected paper, Niemi (1969) has shown that in simulations with three alternatives in which all linear orderings were a priori equiprobable (an assumption known as that of the 'impartial culture'), the greater the proportion of voter orderings which are single-peaked with respect to some continuum the more likely it is that there will be a majority winner, i.e., an alternative which is preferred to each of the other two.[1] Moreover, Niemi (1969) shows that, for any fixed proportion of orderings single-peaked, as n, the number of voters increases, so too does the likelihood of there being a transitive majority ordering. However, Niemi does not provide a clear theoretical rationale for his simulation outcomes, nor does he consider values of m, the number of alternatives, other than m = 3. The absence of a theory to underpin Niemi's results is somewhat troubling, since it can be shown that *if even one* individual has non-single-peaked preferences then there can be a paradox of cyclical majorities.[2]

Moreover, it is possible to have a transitive majority preference ordering and yet have *no* voters who are single-peaked with respect to the continuum defined by that voters preference ordering.[3]

In this paper we shall provide an analytic foundation for Niemi's (1969) simulation result and extend it to m > 3. We shall show that Niemi's (1969) results, in effect, derive from the fact that small perturbations from the impartial culture assumption will virtually guarantee transitive majority preferences. Niemi's (1969) results follow as a corollary to Theorem 1 in the next section which gives necessary and sufficient conditions for the existence of a transitive majority preference ordering for group preferences among a set of alternatives. This theorem is based on the idea of 'net preferences,' a term we define below.

The necessary and sufficient conditions for consistent majority orderings given in Theorem 1, unlike single-peakedness and other similar domain restrictions, do not require that most of the voter's possible preference orderings not be present and also do not require radical symmetry assumptions. Theorem 1 leads us to argue that, for finite alternatives, in real-world contexts, transitive majority decision making is apt to be far more likely than has generally been thought to be true, because there will be sufficient single-peakedness to satisfy the net preference condition.

2. Results on net preferences

It is well known (see, e.g., Sen, 1966) that the presence of a majority cycle implies that there is at least one subset of three individuals that have cyclical preference orderings (either ABC, BCA and CAB; or BAC, ACB and CBA). It is further known (Sen, 1966) that

Lemma 1: The minimal conditions for avoiding any such cycles among triples can be separated into three (non-mutually exclusive) conditions, single-peakedness (SP),[4] single-trowedness (ST),[5] and polarization (P).[6]

Proof: To summarize these conditions and the proof that they are exhaustive, consider that to avoid any triples with cycles, one preference ordering from each of the two sets of cyclic preferences must be excluded.

The 'forward' cycle A > B > C > A, requires that some individual has each of the preference orderings ABC, BCA and CAB; if one of these preferences is absent then there cannot be a forward cycle. Similarly, a 'backward' cycle C > B > A > C, requires that some individual has each of the preference orderings CBA, BAC, and ACB.

Given the symmetry of the situation, it is sufficient to look at the cases where we are excluding ABC from the first set and one of the orderings from the second set. If BAC is excluded, then there are no orderings with C last and all orderings are single-peaked with respect to the continuum *ACB*. If

ACB is excluded, then there are no orderings with A first and all orderings are single-trowed with respect to the continuum *CAB*. If CBA is excluded, then there are no orderings with B in the middle, and all orderings have B either first or last (i.e., are polarized with respect to B); consequently, either B has a majority of first choices (B > A and B > C) or B has a majority of last choices (A > B and C > B); in either case, the group has transitive majority preferences.

To extend these results, we now introduce the concepts of 'opposite preference orderings,' 'net preferences,' and 'positive preference orderings.'

Definition 1: For an (individual) preference ordering ABC, over three alternatives, the *opposite* (individual) linear *preference ordering* is CBA, etc.

Definition 2: For every pair of opposite preference orderings over three alternatives, say ABC and CBA, the *net preference* is the frequency of ABC minus the frequency of CBA. The net frequency of x ABCs is equivalent to a net frequency of -x CBAs, etc.

Definition 3: The *positive net preference orderings* are those orderings whose net preference is positive.

Lemma 2: For three alternatives, the majority decisions of a group are transitive if and only if

a) the positive net preference orderings are single-peaked, or
b) one positive net preference ordering has a majority of the positive net preference orderings.

Proof: For three alternatives, there are exactly three different net preference pairings, ABC - CBA, BCA - ACB and CAB - BAC. The outcomes of votes are only dependent on the positive net preferences orderings because opposite preference orderings cancel each other out in any vote. One of each pair of orderings must have positive net preferences.

If it is the first of each pair, then either one ordering has the majority of net preferences or any two outnumber the third: (a) if one has the majority, then the group majority preference ordering is simply that ordering. The same argument holds if the second of each pair has the positive net preferences. (b) If one of the first and two of the second (or vice versa) have the positive net preferences, then the preferences are single-peaked, e.g., ABC, ACB and BAC are consistent with the BAC continuum. If one of these has the majority, then it is the consistent majority preference ordering. If not, then A > B (from ABC, ACB), B > C (from ABC, BAC) and A > C (from ABC, ACB and BAC). Similar results can be shown to hold for all such cases. Q.E.D.

Lemma 2 shows that it is not the proportion of single-peaked orderings, *per se*, that is important. We can have many individuals with non-single-peaked orderings as long as they are counteracted by enough opposite preferences that are single-peaked and/or overwhelmed by a single common uncounteracted preference.

Lemma 2 states a result only for m = 3. We can extend this result to any m, by reformulating it as a condition on every triple of alternatives.[7]

Theorem 1: For any set of alternatives, the majority decisions of a group are transitive if and only if for every set of three alternatives, either

a) the positive net preference orderings are single-peaked, or
b) one positive net preference ordering has a majority of the positive net preference orderings.

Proof: If not, then there is a majority cycle among the three alternatives where the conditions do not hold (follows immediately from Lemma 2; see also Nicholson, 1965).[8]

Theorem 1 is a quite strong result relative to the standard results in the literature on single-peakedness (or single-trowedness or polarization) which require *all* voters to have preferences over every triple be SP (or ST or P). *The key feature of our results on net preferences is that we do not require that certain orderings be forbidden, merely that they be outvoted.*[9] Note also that Theorem 1 does not require that there be a single linear ordering of alternatives which each set of net preferences over three alternatives has in common.

3. Implications of net preferences

There are several important implications of Theorem 1 and of the idea of net preferences which have never previously been recognized:

Implication 1: If there is linear ordering such that for every set of three alternatives, there is a very small set of individuals, which set we shall label ϵ, who have (as a group) positive net preferences single-peaked over that ordering, while the rest of the society has preference orderings over these alternatives which essentially are random (i.e., drawn from the 'impartial culture'), then, with large numbers of individuals whose preferences are drawn from the impartial culture, there will exist a transitive majority preference ordering identical to the majority preference ordering held by the ϵ group.

Proof: With large numbers of individuals, all the net preferences of the group with random preferences will have an expected value of 0. Thus, for the combined group, the positive net preference orderings will be single-peaked with respect to the given linear orderings, since that is true for the ϵ fraction and the preferences of the larger society in effect cancel each other out. Consequently, the group majority decisions will be transitive. Q.E.D.

Of course, if there were *multiple* small ideological consistent groupings, but each with a different notion of what the relevant ideological continuum over triples were to ordered, there would be no guarantee that the divergent preferences of these multiple ideological groupings would aggregate into a transitive social preference ordering. However,

Implication 2: If there is one 'orienting' linear ordering, e.g., a left-right dimension such that for any pair of alternatives, i and j, the probability that a randomly chosen voter would see i as being 'to the left of' j was greater than $\frac{1}{2}$ (albeit perhaps only marginally) if and only if i was to the left of j in that orienting linear ordering, then group choice will be transitively ordered in accord with the orienting ordering.

Such an orienting ordering could occur if, for example, the news media consistently portray the political world (and all choices in it) in left-right terms. Even if citizens did not, as individuals, consistently see the world in these terms, as long as the media impact was sufficient to make the left-right ordering the most probable way that voters *in the aggregate* would see the world then, with near certainty (at least for choices by a large electorate among a small set of alternatives), group majority choices will be made *as if* all members of the group saw the world in common ideological terms.

Implication 3: Niemi (1969) found that the greater the proportion of voter orderings that were single-peaked the more likely was a consistent group ordering, and that this likelihood increased with n, group size – approaching one as n grew large. These results found in Niemi's (1969) simulation follow directly from properties of net preferences and the symmetry implied from the drawing orderings from the impartial culture – and are not really linked to the frequency of single-peaked orderings *per se*.

In drawing orderings from the impartial culture, as the proportion of orderings which are single-peaked with respect to some specified linear ordering increases, it becomes more likely that each and every one of the group's positive net preference orderings is single-peaked with respect to that ordering, since the remaining non-single-peaked orderings are likely to cancel each other out. As n gets larger, by the law of large numbers, the likelihood that the non-single-peaked preference orderings cancel or nearly cancel each other increases, approaching certainty is $n \to \infty$.

Since a minimum of $\frac{2^{m-1}}{m!}$ of all individual orderings will be single-peaked with respect to some one of the possible linear orderings, once the proportion of preferences which are single-peaked with respect to some linear ordering exceeds $\frac{2^{m-1}}{m!}$, then that ordering will tend to become the basis for a transitive group preference ordering when the remaining preference orderings are drawn from the impartial culture. For $m = 3$, $\frac{2^{m-1}}{m!} = .67$. Thus, in Niemi's (1969) simulation, when more than 67% of the group's preference orderings were single-peaked with respect to some linear ordering, as n increased, the probability of a transitive ordering approached one.

More generally, for any m, if all orderings are drawn from the impartial culture, as the number of orderings which are single-peaked exceeds $\frac{2^{m-1}}{m!}$, as n increases, the probability of a transitive ordering will approach one since $\frac{2^{m-1}}{m!}$ rapidly goes to zero as m increases. If m is larger than 3, if the electorate is large, even a *handful* of 'ideologically' minded voters can give rise to a consistent collective choice if the other voters are drawn from the 'impartial culture.' Indeed even for small n, if ϵ contains more than a few percentage points of the group's total membership, the probability of a consistent majority ordering is *quite* high. Analogously, if one particular single-peaked ordering is made more likely by dint of media focus on it, then, if other preferences are essentially randomly distributed à la the neutral culture, the median on the orienting ordering will be that which prevails.

We should also note that, even without the symmetry required by random drawing, if a significant number of orderings are single-peaked, single-trowed or polarized, then the effects on net preferences make it likely (but not inevitable) that there will be consistent majority orderings. Single-peakedness (e.g., with respect to *ACB*) implies few instances of ABC and BAC. Consequently, the positive net preferences will tend to be CBA, CAB and either ACB or BCA. These are single-peaked with respect to *ACB*. Single-trowedness is directly analogous. For a polarization ordering, (e.g., polarized with respect to B), there will be few ABC and CBA. Consequently, the positive net preference orderings will have very few ABC or CBAs; this makes it most likely that the larger of the other two positive net preferences will have a majority of the net preferences.

4. Discussion

For finite m, virtually all of the work showing the inevitability or near inevitability of cyclical majorities (and the certainty or near certainty of cycles involving all or almost all alternatives) is based on lessons drawn from the 'impartial culture.'[10] While such results are, of course, technically correct, in our view they have led to unjustified pessimism about the possibility of transitive majority orderings – at least for the finite alternative case. The impartial culture is simply an unreasonable assumption.

Our results show relatively weak conditions under which a consistent majority ordering can be expected. For example, if there is a *very* small but relatively coherent subset of the society, then the impartial culture assumption for the rest of the society, rather than producing generic instability, *on the contrary*, guarantees that the relatively coherent minority will prevail and impose its net preference ordering on the rest of society. Moreover, even that tiny subset need not have all (or even most) of its preferences single-peaked with respect to any given linear ordering.

Even more importantly, while single-peakedness (or single-trowedness or polarization) of *all* voters on all triples is totally unreasonable to expect, single-peakedness of the positive *net* preference orderings over all triples can occur with only a minimal societal coherence, i.e., minisculely more than could be expected by chance alone. If, for example, media and other elites discuss alternatives as if they were on a unidimensional continuum, then essentially all that is required for transitive group decision-making is that the mass electorate will be ever so slightly more likely to correctly identify the centrist candidate along the ordering imposed by that continuum than could be expected by chance alone.

Thus, we have shown how stable majority orderings can result even if all possible preference orderings are represented in the electorate.

NOTES

1. For three voters a majority winner (also known as a Condorcet winner) guarantees transitivity.
2. For m = 3 let (n – 1)/2 of the n voters (n odd) have preference orderings ABC and BCA respectively; and let one voter have preference ordering CAB. Then the majority has cyclical preferences A > B > C > A. Yet, with respect to the linear ordering *ABC*, only a single voter has non-single-peaked preferences.
3. Consider three voters with preferences ABCED, ABDEC, and ACDEB, respectively. These preferences give rise to the majority preference ordering ABCDE, yet *none* of the three voters has preferences which are single-peaked with respect to the linear ordering *ABCDE*, nor are all three preferences single-peaked with respect to *any* linear ordering we might propose.
4. Also known as the NW condition.
5. Also known as the NB condition and as single-cavedness.
6. Also known as the NM condition.
7. It also well known that if, for *every* voter, preferences are single-peaked over each possible *triple* of alternatives then there exists a linear ordering such that all voter preferences will be single-peaked over the set of all alternatives with respect to that ordering (see, e.g., Sen, 1966).
8. Miller (1970) has demonstrated this result, as have Gaertner and Heinecke (1978); Slutsky (1977) has also stated an equivalent result, although not in the terms used above. However, all of these authors have viewed the result as rather technical and esoteric, with little or no applicability to understanding real-world political dynamics. We believe this is a fundamental error.
9. For the positive *net* preferences orderings to be single-peaked requires that a majority of individual orderings be single-peaked, i.e., whenever the non-single-peaked orderings ap-

pear there must be at least as many as there are opposities. This is not a strong condition. Recall that by chance alone, if all linear orderings on a triple are equally likely we would expect 2/3rds of all orderings to be single-peaked with respect to some linear alignment.

In principle, a *different* set of voters can have single-peaked preferences over each triple. Thus, in practice, not many voters need to have single-peaked preferences over the *entire* set of alternatives for the positive net preference orderings to be single-peaked.

Note also that we can replace single-peakedness with single-trowedness in the above result without changing anything.

10. A relevant exception is Kuga and Nagatani (1974); however, their results are considerably less general than those given here.

REFERENCES

Arrow, K.J. (1963). *Social choice and individual values*, 2nd Edition. New York: Wiley.

Bell, C.E. (1978). What happens when majority rule breaks down: Some probability calculations. *Public Choice* 33(2): 121–126.

Black, D. (1958). *The theory of committees and elections*. Cambridge: Cambridge University Press.

Condorcet, Marquis de (1785). *Essai sur l'application de l'analyse à la probabilité des decisions rendues à la pluralité des voix*. Paris.

Gaertner, W., and Heinecke, A. (1978). Cyclically mixed preferences: A necessary and sufficient condition for transitivity of the social preference relation. In H.W. Gottinger and W. Leinfellner, *Decision theory and social ethnics: Issues in social choice*, 169–186. Dordrecht: Reidel.

Gehrlein, W.V., and Fishburn, P.C. (1976). The probability of the paradox of voting: A computable solution. *Journal of Economic Theory* 13(1) (August): 14–25.

Kramer, G.H. (1973). On a class of equilibrium conditions for majority rule. *Econometrica* 41: 285–297.

Kuga, K., and Nagatani, H. (1974). Voter antagonism and the paradox of voting. *Econometrica* 42(6) (November): 1045–1067.

McKelvey, R.D. (1976). Intransitivities in multidimensional voting models and some implications for agenda control. *Journal of Economic Theory* 12: 472–482.

McKelvey, R., and Wendell, R. (1976). Voting equilibria in multidimensional choice spaces. *Mathematics of Operations Research* 1: 144–158.

Miller, N. (1970). *A necessary and sufficient condition for transitive majority orderings*. Unpublished manuscript.

Nicholson, M.B. (1965). Conditions for the 'voting paradox' for committee decisions. *Metroeconomica* 7: 29–44.

Niemi, R.G. (1969). Majority decision-making with partial unidimensionality. *American Political Science Review* 63: 488–497.

Niemi, R.G. (1970). The occurrence of the paradox of voting in university elections. *Public Choice* 8: 91–100.

Niemi, R.G. (1983). Why so much stability?! Another opinion. *Public Choice* 41: 261–270.

Niemi, R.G., and Weisberg, H.F. (1968). A mathematical solution for the probability of the paradox of voting. *Behavioral Science* 13: 317–323.

Plott, C. (1967). A notion of equilibrium and its possibility under major rule. *American Economic Review* 57: 787–806.

Riker, W.H. (1961). Voting and the summation of preferences: An interpretive bibliographic essay of selected developments during the last decade. *American Political Science Review* 55: 900–911.

Riker, W. (1980). Implications from the disequilibrium of majority rule for the study of institutions. *American Political Science Review* 74(1): 432–446, and Reply, 456–458.

Riker, W. (1982). *Liberalism against populism: A confrontation between the theory of democracy and the theory of social choice*. San Francisco: Freeman.

Schofield, N. (1978). Instability of simple dynamic games. *Review of Economic Studies* 45(3): 575–594.

Sen, A.K. (1966). A Possibility theorem on majority decisions. *Econometrica* 34: 491–499.

Sen, A.K. Social choice theory. In K.J. Arrow and M. Intriligator (Eds.), *Handbook of mathematical economics*, forthcoming.