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The Borda count in n-dimensional issue space*

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Abstract. We provide a natural extension of the Borda count to the n-dimensional spatial context, an algorithm to find the spatial Borda winner based on the notion of an inverse Borda count, the result that the Borda winner and the Condorcet winner coincide in unidimensional space when all alternatives on a line are feasible, results that show that in multi-dimensional space the Borda winner and the Condorcet winner (except under very implausible circumstances) will be distinct, and some results on the manipulability of outcomes under the Borda rule as a function of the domain of alternatives over which the Borda count is to be defined.

1. Introduction

In a majority-rule, spatial-voting game without a core, we expect most or all alternatives to be in the top cycle set (Schofield, 1978; McKelvey, 1979). Thus, absent a Condorcet winner, there may be no clear majority choice. One option in such a situation is to pick the alternative with the highest Borda count. Although some writers suggest the decision rule of using the Borda count as a supplementary criterion for choosing among a finite set of alternatives when there is no Condorcet winner (see, e.g., Black, 1958), the implications of using this criterion in a spatial context remain unexplored. The very definition of the Borda count might seem problematic when there is an infinite number of alternatives. This essay provides a natural extension of the Borda count to the spatial context, sufficient conditions for the Borda winner and the Condorcet winner to coincide in the unidimensional case, and some results on the manipulability of outcomes under the Borda rule as a function of the domain of alternatives over which the Borda count is to be defined. We also explore the relationship between the location of the Borda winner and an important central domain of the space, the *yolk* (McKelvey, 1986), the minimal sphere that intersects all median hyperplanes.

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2. The Borda count in the spatial context

The *Borda count*, named for Jean-Charles de Borda (1733–1799), who first proposed it (Borda, 1781), gives the total number of votes that each alternative would get if placed in turn against each of the other alternatives. That alternative with the highest Borda count is the *Borda winner*. There are two ways to calculate the Borda count in the finite alternative case. The standard method is for each voter to assign one point to that alternative for each alternative to which he prefers it and then to sum up (over the set of all voters) to obtain the total point score that each alternative receives. In other words, each first-place preference is worth $m-1$ points, a second-place preference is worth $m-2$ points, and so forth (and of course, a last-place preference is worth zero points). Alternatively, for each alternative we can count up the number of voters who prefer it (in a series of pairwise comparisons) to each of the possible other alternatives, and sum over the set of all alternatives to obtain the Borda counts for each alternative in turn.

There is a third way to define the Borda count, however, which will prove especially useful for present purposes.

Definition 1. The *inverse Borda Count* for a given alternative is the number of alternatives that beat that alternative in each voter's preference ordering, summed over all voters.

Clearly, the Borda winner is the alternative with the minimum inverse Borda count.

There are two natural ways to extend the Borda count to the case in which the number of alternatives to be compared against may be infinite.

As we have just described, we can define the Borda count for a given alternative as the number of alternatives that that alternative beats in each voter's preference ordering then summed over all voters. If the feasible set consists of the points in some space, we can simply take the *area* of the set of alternatives to which each voter prefers a given alternative as a measure of the size of the set of alternatives to which that voter prefers it, and then sum the resultant areas over all voters. If the space is unbounded, however, then this approach may not be well-defined. Instead, we can find the area of the set of alternatives that each voter prefers to a given alternative, and make use of an inverse Borda count as follows.

Definition 1′. For alternatives that are points in some multi-dimensional space, the inverse Borda Count of an alternative is the area of the set of alternatives that each voter prefers to it, summed over all voters.

Clearly, in the spatial context, as elsewhere, the Borda winner is the alternative with minimum inverse Borda count.

In the spatial context, the inverse Borda count avoids the problems of infinite areas. If voters have ideal points and convex indifference curves, then the only alternatives that a voter prefers to a particular alternative, x, are contained in the area enclosed by that voter's indifference curve through x.

When there is a Condorcet winner, of course, the Borda winner and the Condorcet winner need not coincide. For example, with five voters, if three voters have preferences xyz and two have preferences yzx, then x is the Condorcet winner but y is the Borda winner. If voters' preferences are single-peaked, of course, then there is also a Condorcet winner: the alternative corresponding to the median voter is the ideal point. Single-peakedness is not sufficient to guarantee that the Borda winner and the Condorcet winner will coincide – as the preceding example in which preferences are single-peaked with respect to the ordering xyz demonstrates. When we move from the finite-alternative case to the spatial context, though, single-peakedness is sufficient to guarantee the coincidence of the Borda winner and the Condorcet winner. The explanation for this coincidence, as we find later, sheds light on the way in which voters can manipulate the Borda winner by the addition or deletion of alternatives.

We shall henceforth make the common simplifying assumption that preferences are Euclidean; that is, each voter ranks alternatives simply in terms of their distance from his or her ideal point. For simplicity, we also assume that the number of voters is finite and odd and neglect the essentially technical potential complications that ties might cause. In two or more dimensions a Condorcet winner is unlikely. However

Theorem 1. If we can array voters' ideal points along a line, and if all alternatives along the line are feasible, then the Borda winner and the Condorcet winner coincide at the median voter's ideal point.

Proof. Let $\overline{xv_i}$ be the distance from x to v_i. Given an alternative, x, and a voter, V_i, with ideal point, v_i, it is clear that V_i prefers x to all alternatives further away from v_i than is x; that is, if $\overline{v_i y} > \overline{v_i x}$, then V_i prefers x to y. Since all alternatives along the line are feasible, then x loses only to a y for which $\overline{v_i y} < \overline{v_i x}$. The line segment that contains such a y is of length $2\overline{v_i x}$. The inverse Borda count is the sum of these lengths over all voters, $\sum_{i=1}^{m} 2\overline{v_i x}$, where m is the number of voters.

Clearly, if we move x closer to any voter, then the inverse Borda contribution of that voter declines; when x is at the voter's most-preferred location, then the inverse Borda contribution of that voter is zero. The sum of the absolute distances to all the voters' ideal points is minimized at the median voter's ideal point; consequently, this point is the Borda winner. But the median voter's ideal point, of course, is also the Condorcet winner. Q.E.D.

We can specify a general rule for finding the Borda winner:

Theorem 2. In n dimensions, where all alternatives are feasible, the Borda winner is the point x, that minimizes $\sum_{i=1}^{N} (\overline{v_i x})^n$ (1)

Proof. Each voter, V_i, prefers all points around V_i within a radius of $\overline{v_i x}$ to a particular point x; that is an area or volume described by a sphere of radius r with $r = \overline{v_i x}$. The area/volume of such spheres is proportional to r raised to the nth power (for example, πr^2 for two dimensions). The area of these spheres determines the inverse Borda counts, and the point minimizing the sum of these areas is the Borda winner. Q.E.D.

Corollary 1 to Theorem 2. In two dimensions the Borda winner is the center of gravity, the mean on each dimension.

Proof. The mean is the point minimizing the squared deviations, so for $n = 2$, the corollary follows directly from Theorem 2.

Corollary 2 to Theorem 2. If voters' ideal points lie along a line, but alternatives are distributed evenly over an entire plane including the line, then the Borda winner is the alternative at the mean along the line.

Proof. Follows directly from Corollary 1.

In the spatial context, even if there is a Condorcet winner, the Borda winner and the Condorcet winner need not coincide, because the mean of the voters' ideal points and the median voter's ideal point need not be the same. Indeed the mean and the median can be very far apart.

Corollary 3 to Theorem 2. The Borda winner is always within the convex hull of the voters' ideal points, that is, among the set of Pareto optimal points.

Proof. A point minimizing a power function of distances must be inside the convex hull determined by those points. Q.E.D.

Theorem 3. In two dimensions the inverse Borda count increases monotonically with distance from the Borda winner. In particular, the inverse Borda count for any point at a distance e from the Borda winner is given by the Borda count of the Borda winner plus Πe^2.

Proof. In two dimensions the inverse Borda count for any point is given by

$$\sum_{i=1}^{N} \Pi(d_{ip})^2, \tag{2}$$

where d_{ip} is the distance from p to the ith voter's ideal point. (See Proof to Theorem 2.)

Note that

$$d_{ip}^2 = x_{ip}^2 + y_{ip}^2, \tag{3}$$

where x_{ip} is the distance along the x axis from p to the ith voter's ideal point, and y_{ip} is the distance along the y axis from p to the ith voter's ideal point. Now, we may rewrite Eq. (2) as

$$\Pi \left[\sum_{i=1}^{N} (x_{ip})^2 + \sum_{i=1}^{N} (y_{ip})^2 \right]. \tag{4}$$

We know that

$$\sum_{i=1}^{N} (x_{ip})^2 = \sum_{i=1}^{N} (x_{ip})^2 + N(x_{pb})^2 \tag{5a}$$

and

$$\sum_{i=1}^{N} (y_{ip})^2 = \sum_{i=1}^{N} (y_{ip})^2 + N(y_{pb})^2, \tag{5b}$$

where b is the Borda winner. In two dimensions the Borda winner lies at the mean of each dimension, and thus Eq. (5) is a version of a well-known identity involving the mean and variance of a distribution.

Substituting Eq. (5) in Eq. (4) we obtain

$$\Pi \sum_{i=1}^{N} [(x_{ib})^2 + (y_{ib})^2] + N[(x_{pb})^2 + (y_{pb})^2]. \tag{6}$$

But this is just the inverse Borda count at the Borda winner, b, plus an expression that increases monotonically with d_{pb}, the distance from the Borda winner. That is,

$$\Pi \sum_{i=1}^{N} [(x_{ib})^2 + (y_{ib})^2]$$

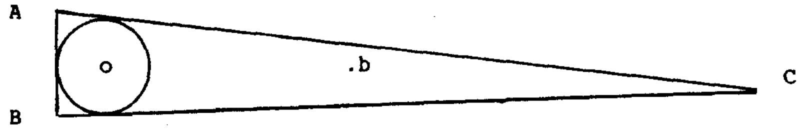

Figure 1. A three-voter example in which the Borda winner is far from the center of the yolk; o is the center of the yolk; b is the Borda winner

is merely the inverse Borda count for the Borda winner, b, and

$$(x_{pb})^2 + (y_{pb})^2 = e^2 = (d_{pb})^2. \qquad \text{Q.E.D.}$$

2.1 The Borda winner and the yolk

The yolk is the minimal sphere that intersects all median hyperplanes. We can think of the center of the yolk as a natural 'center' of the space. Ferejohn, McKelvey, and Packel (1984) show that outcomes under most reasonable agenda processes fall within four radii of the center of the yolk. (See also McKelvey, 1986; Feld et al., 1987.)

It would be nice if we could show that the Borda winner always lies near the center of the yolk. Grofman, Owen, Noviello, and Glazer (1987) argue that the Copeland winner, one natural generalization of the Condorcet approach to games without a core, almost always will be close to the center of the yolk. Unfortunately, the same is not true for the Borda winner. It may be very far from the center of the space and from the Copeland winner. To see that this is so, we need merely consider how we could add voters such that the yolk remains essentially unchanged but the Borda winner can move to any point in the convex hull. To move the Borda winner in any particular direction while leaving the yolk essentially unchanged, merely add two voters, one on each side of the center of the yolk along the specified direction. While the yolk may shrink, it must stay close to its original position. However, in two dimensions, Copeland values diminish with distance from the Copeland winner just as inverse Borda counts increase with distance from the Borda winner (Grofman *et al.*, 1987).

Figure 1 provides an example in which the Borda winner is arbitrarily far from the center of the yolk. The yolk is the inscribed circle, while the Borda point is the center of gravity.

2.2 The boundary of the space

So far we explore situations in which all alternatives in some space are feasible. Now we show that we can find a subset of the space in which the Borda winner

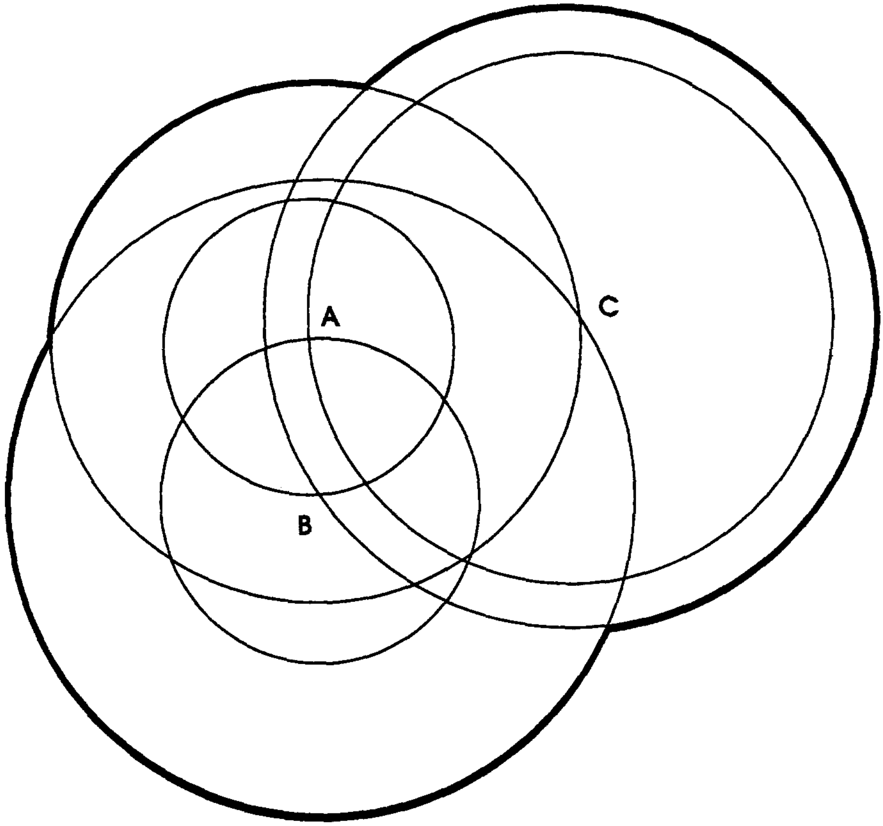

Figure 2. A three-voter illustration of the boundary of alternatives in a two-dimensional space

(and the Borda ranking of alternatives in the subset of alternatives) is the same as in the space as a whole.

Definition 2. The *boundary* of alternatives in a space is the union of all spheres drawn around one voter's ideal point through another voter's ideal point (see Figure 2).

Theorem 4. The Borda winner within the boundary of alternatives is the same as that within the entire space.

Proof. The boundary of alternatives is the set of alternatives that some voter prefers to some other voter's ideal point. Thus, the boundary includes all alternatives that any given voter prefers to any other voter's ideal point. Since each voter prefers every other voter ideal point (as well as his own) to anything outside of the boundary, because of convexity, all voters prefer all points within the convex hull to all points outside the boundary; that is, the area outside of the boundary is irrelevant in determining the inverse Borda count of any alternative within the boundary. Q.E.D.

Corollary to Theorem 4. When voters' ideal points are arrayed along a line, and all alternatives in the boundary of alternatives are feasible, the Borda winner and the Condorcet winner coincide at the median voter's ideal point.

Proof. We combine the result of Theorem 1 with that of Theorem 3. Q.E.D.

If we exclude from consideration alternatives within the boundary of alternatives, the Borda winner moves away from the excluded region. Eliminating alternatives reduces the impact of voters who would have preferred those alternatives to some others. Only voters near those alternatives prefer the excluded alternatives to others. In a single dimension, eliminating alternatives at one end moves the Borda point towards the preferences of those at the other end. Even confining ourselves to the convex hull can change the Borda winner.

Consider that there are 4, 2, and 1 voters with ideal points at x, y, and z, respectively, at 0, 10, and 15 along a line. The Condorcet winner is x, at 0, as is the Borda winner for the entire dimension of alternatives. If we limit alternatives to the convex hull of the ideal points (0 through 15, which is smaller than the boundary of alternatives), however, then the Borda winner will be at 10. This is because more voters' preferences are excluded from the bounded set of alternatives {−15 through 30} at the low end (4) than at the high end (3), so the Borda count moves away from the low end.

In general, if one wants to make it likely that alternative x will be Borda preferred to alternative y, one should introduce alternatives z, such that the margin of x over z is much greater than the margin of y over z. Notice that such alternatives may beat both x and y, lose to both x and y, or lose to x and beat y. What is important are the relative margins.

Because of the Borda winner's sensitivity to the particular set of alternatives being considered and compared, the Borda count is very easy to manipulate by strategic addition or deletion of alternatives (see Fishburn, 1974). Indeed, we believe that it should be used only under conditions where the set of alternatives is unambiguously defined (cf. Fishburn, 1982).

One of the commonly recognized advantages of the Borda count is that it provides a practical method for quickly choosing among a large set of alternatives. When the number of alternatives is too large, though, it is impractical to ask voters to rank-order all of them. But if, however, in a spatial context, voters' preferences among alternatives are a monotonic function of the distance of alternatives from voters' ideal points, then we can find the Borda winner by asking voters to identify their ideal points in a multidimensional space and then use the algorithm of Theorem 2. Of course, in the spatial context, as in the finite case, voters may have incentives to vote strategically.

3. Discussion

We show how the Borda winner always can be well-defined in the spatial context (using the ideas of the *boundary* of the space and of an *inverse Borda winner*). For a unidimensional space we provide conditions sufficient for the Borda winner and the Condorcet winner to coincide and show that they always will coincide if all alternatives are feasible (cf. Chamberlin and Cohen, 1978). For a multidimensional issue space, we show that even if there is a Condorcet winner (which, of course, is quite unlikely), the Borda winner and the Condorcet winner in general will be distinct (see Theorem 2). Also, the Borda winner and the center of the yolk need not be near one another. Finally, our results show how spatial manipulation of the Borda winner is possible.

For finite alternatives, Fishburn (1974), Young (1974), and Richelson (1980) give axiomatic characterizations of the Borda count and of a more general class of rules, scoring functions, of which the Borda rule is a member. Farkas and Nitzan (1979) give an alternative axiomatization, based on the finding that the Borda winner is also the alternative with highest average plurality. We believe that one can modify each of these characterizations to apply to the spatial case.

We can think of the Borda count as an 'equitable' outcome in choosing among finite alternatives (Black, 1958), because it maximizes 'average' plurality. Now that we show how to conceptualize the Borda winner in the spatial context, one direction for future research would be to examine the extent to which outcomes move away from a Condorcet winner and toward the Borda winner in spatial settings where equity concerns are emphasized (cf. Eavey and Miller, 1984; Eavey, 1987).

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