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PROVING A DISTRIBUTION-FREE GENERALIZATION OF THE CONDORCET JURY THEOREM*

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We provide a proof for a result due to Grofman, Owen and Feld (1982), a distribution-free generalization of the Condorcet Jury Theorem (1785). In proving this result we show exactly what distribution of individual competence maximizes/minimizes the judgmental accuracy of group majority decision processes.

Key words: Condorcet; jury; distribution-free; reliability; preference; majority rule.

1. Introduction

While Condorcet is best known to present-day economists for his identification of the paradox of cyclical majorities[1] and his work on the logic of majority preference (in particular for being the first to propose the criterion for majority choice which now bears his name).[2] Condorcet's philosophical stance was that collective decision making was not a matter of preference aggregation but rather 'was a matter of the articulation of public reason' (Baker, 1967, p. 142). The principal

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[1] To translate into contemporary terminology, Condorcet was the first person to realize that the core of a majority voting game may be empty. (See Plott, 1976.)

[2] The Condorcet criterion: choose that alternative (if any) which receives a majority of votes against each and every other alternative in a paired comparison. (See Black, 1958, for historical details.)

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aim of his magnum opus, *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix* (1785), was to answer the question 'Under what conditions will there be a degree of probability that the majority decision of an assembly or tribunal is true...?' (Baker, 1967, p. 139), where 'true' had for Condorcet a special meaning. Condorcet posited some ideal standard (e.g., a social ordering) along which alternatives could be ordered from best to worst. The task for collective decision making as he saw it was to make the 'best' (i.e., 'truest') choice from among the available alternatives. (See Grofman, Owen and Feld, 1982; Nitzan and Paroush, 1985; Grofman and Owen, 1986a, 1986b; Shapley and Grofman, 1984; Young, 1986.)

In seeking to answer the question of how reliable were group judgments, Condorcet proved an important result which has come to be known as the 'Condorcet Jury Theorem' (Black, 1958; Grofman, 1975). To state that theorem and our own further new results, some notation will be useful:

Let us consider a group of size n confronting a dichotomous choice situation. Let:

p_i = judgment competence of the ith voter ($0<p_i<1$) in a dichotomous choice situation, i.e., the probability that the voter will make the correct (better) choice of the two available to him
n = number of voters in the group (for simplicity, n will generally be taken to be odd)
m = a majority = $(n+1)/2$ for n odd
$\bar{p}$ = average judgmental competence of voters in the group
p = judgmental competence of a voter in a homogeneous group
P_n = probability that at least a majority of voters will make the correct choice in a dichotomous choice situation, where n is the number of voters in the group

To obtain his basic result. Condorcet assumed:

(1) Voters' choices are independent of one another.
(2) Voters are homogeneous, i.e., $p_i=p=\bar{p}$ for all i.
(3) The group decision rule is simple majority.
(4) There are exactly two alternatives, only one of which is correct.
(5) The prior odds as to which of the two alternatives is the correct one are even.

Theorem 1. (*Condorcet Jury Theorem, Condorcet*, 1985): *If* $1>p>\frac{1}{2}$ *then* P_n *is monotonically increasing in n and* $\lim_{n\to\infty}P_n\to 1$; *if* $0<p<\frac{1}{2}$ *then* P_n *is monotonically decreasing in n and* $\lim_{n\to\infty}P_n\to 0$; *while if* $p=\frac{1}{2}$ *then* $P_n=\frac{1}{2}$ *for all n*.

Also

$$P_n=\sum_{h=m}^{n}\binom{n}{h}p^h(1-p)^{n-h} \tag{1}$$

The implication of this result is, for $p > \frac{1}{2}$, that 'vox populi, vox dei'. Indeed, the accuracy of a group majority judgment goes up rather rapidly with n. (See Grofman, 1975, 1978; Table 1).

The Condorcet Jury Theorem result, which was 'lost' for most of this century (see Black, 1958; Grofman, 1975 for details of its rediscovery) has, in the past two decades, been extended in a number of ways. In particular, Grofman (1978) generalized the Condorcet Jury Theorem by replacing Assumption 2 (voter homogeneity) with the assumption that voter competence is normally distributed with a variance equal to the binomial variance.[3] In this case we obtain expressions analogous to those in (1) with $\bar{p}$ replacing p. Grofman (1975) also proved some results about the relative accuracy (under a majority voting rule) of a large group of (homogeneous) 'blue-ribbon' group. Other extensions and related results may be found in Poisson (1837), Steiner and Rajaratnam (1961), Margolis (1976), Gelfand and Solomon (1973, 1974, 1977), Klevorick and Rothschild (1978), Grofman (1979, 1980) and Nitzan and Paroush (1980b, 1985). Feld (unpublished) has further generalized this result to apply to any distribution which is symmetric around its mean.

It might appear that $P_n \gtreqless \frac{1}{2}$ as $\bar{p} \gtreqless \frac{1}{2}$. Actually the situation is more complex, as some simple examples will demonstrate. When the distribution of voter competence is not symmetric, then we can obtain rather 'perverse' results, at least for small values of n, as demonstrated by these examples from Grofman, Owen and Feld (1982).

For example, a group can have $\bar{p} < \frac{1}{2}$ and yet have $P_n > \frac{1}{2}$. Consider the following distributions of voter competence in 3 and 5 member groups.

(a) (0.72, 0.72, 0); $\bar{p} = 0.48$, yet $P_n = 0.5184$. (b) (0.8, 0.8, 0.8, 0, 0); $\bar{p} = 0.48$, yet $P_n = 0.512$. (c) (0.8, 0.9, 0.7, 0, 0); $\bar{p} = 0.48$, yet $P_n = 0.504$.

Similarly, a group can have $\bar{p} > \frac{1}{2}$ and yet have $P_n < \frac{1}{2}$. Consider (a) (1, 0.28, 0.28); $\bar{p} = 0.52$, yet $P_n = 0.4816$. (b) (1, 1.0, 0.2, 0.2, 0.2); $\bar{p} = 0.52$, yet $P_n = 0.488$.

Almost equally strange are the following examples:

(a) (0.72, 0.72, 0); $\bar{p} = 0.48$, yet $P_n = 0.5184$. (b) (0.8, 0.8, 0.8, 0, 0); $\bar{p} = 0.48$, yet $P_n = 0.512$. (c) (0.8, 0.9, 0.7, 0, 0); $\bar{p} = 0.48$, yet $P_n = 0.504$.
$\frac{1}{4}$) $\bar{p} = 0.5$ but $P_n = 0.4213$.

Nonetheless it is possible to prove a generalized form of the Condorcet Jury Theorem, a result first stated (but without proof) in Grofman, Owen and Feld (1982).

Theorem II. (*Distribution-Free Generalization of the Condorcet Jury Theorem*):
(1) *If* $\bar{p} > 0.5$ *then* $\lim_{n \to \infty} P_n \to 1$
(2) *If* $\bar{p} < 0.5$ *then* $\lim_{n \to \infty} P_n \to 0$
(3) *If* $\bar{p} = 0.5$ *then*

[3] Grofman (1978) also looked at the relationship between group size and the question of whether the group majority rule judgment is likely to be better than that of its most competent member.

$$1-e^{-1/2}<\lim_{n\to\infty} P_n<e^{-1/2}.$$

The proof of this result (and of other relevant lemmas) is given in the mathematical appendix to this paper.[4]

2. Discussion

Theorem II is important because in proving it (see Appendix) we find, for a fixed average competence, exactly what distribution of individual competence maximizes/minimizes the judgmental accuracy of group majority decision processes.[5] Theorem II also shows us that, for large n, except for the 'knife-edge' case $\bar{p}=\frac{1}{2}$, if we know $\bar{p}$ (indeed if we merely know whether $\bar{p}$ is less than or greater than $\frac{1}{2}$, we would know virtually all we needed to estimate group competence. Of interest, too, for its counterintuitive force, is Part (3) of Theorem II, which gives the quite paradoxical result that groups which are on average 'half-wits' ($\bar{p}=\frac{1}{2}$) can generate a group judgmental competence anywhere from 0.39 to 0.61 – a result which does not 'go away' with increasing n.

We believe that results dealing with the *reliability* of group decision making offer a useful complement to the more typical emphasis in the social choice literature on *preference* aggregation. However, even if one is uninterested in the question of judgmental competence, our results can be reinterpreted in the context of more general group choice processes, where p_i, instead of representing a group member's judgmental competence, is simply taken to be a given group member's probability of selecting a given alterative from a two-alternative set. Under this interpretation, the theorems in this paper simply deal with majority rule given stochastic preferences. In particular they tell us about the expected relationships between a group's majority choice and its members' mean preferences when individual choice is probabilistic in nature.

Appendix. Proofs of Lemmas required for Theorem II

Let $N=\{1,2,...,n\}$ be a finite set of decision-makers. We assume these n group members are faced with a simple yes-or-no decision; player i is assumed to have probability p_i of making a correct decision. If these probabilities are independent, then the probability that a majority (defined as m or more, where m is usually

[4] Parts (1) and (2) of the corollary may also, it has been suggested, be established from Chebyshev's inequality. If so, the proof is far from straightforward. In any case, part (3) of the corollary, which sets precise bounds on the limits at the knife-edge case, cannot be so derived.

[5] It follows straightforwardly from Theorem II and its corollaries that a necessary condition for $P_n>\bar{p}$ is

$$\bar{p}>\left(\frac{n+1}{2n}\right)^{(n+1)/(n-1)}.$$

($(n+1)/2$ for odd n, $(n+2)/2$ for even n) will make a correct decision is

$$F(p_1, p_2, \dots, p_n) = \sum_{\substack{S \subset N \\ s \ge m}} \prod_{i \in S} p_i \prod_{i \notin S} (1-p_i), \tag{A1}$$

where the sum is taken over all sets $S \subset N$ such that s (the number of members of S) is at least m.

In the special case where all p_i are equal, we will have

$$F(p, p, \dots, p) = \sum_{s=m}^{n} \binom{n}{s} p^s (1-p)^{n-s}, \tag{A2}$$

which is the usual cumulative binomial expression (for at least m successes in n trials of a simple experiment).

We consider, now, the following problem: given that

$$\alpha = \sum_{i=1}^{n} p_i \tag{A3}$$

is fixed, how should the p_i be chosen so as to maximize F?

Apart from (A3), the p_i are subject to the natural constraints

$$0 \le p_i \le 1 \qquad i = 1, \dots, n. \tag{A4}$$

We therefore consider the Lagrangian

$$G(p_1, p_2, \dots, p_n, \lambda) = F(p_1, \dots, p_n) - \lambda \sum_{i=1}^{n} p_i \tag{A5}$$

with partial derivatives

$$\frac{\partial G}{\partial p_i} = \frac{\partial F}{\partial p_i} - \lambda \qquad i = 1, \dots, n \tag{A6}$$

and obtain the following:

Principle. A *necessary* (but not sufficient condition for $(p_1^*, \dots, p_n^*)$ to maximize F subject to the constraints (A3)–(A4) is the existence of a λ such that

$$\begin{aligned} &\text{(i)} && F_i(p^*) \ge \lambda && \text{if } p_i^* = 1, \\ &\text{(ii)} && F_i(p^*) = \lambda && \text{if } 0 < p_i^* < 1, \\ &\text{(iii)} && F_i(p^*) \le \lambda && \text{if } p_i^* = 0, \end{aligned} \tag{A7}$$

(where $F_i = \partial F / \partial p_i$).

To see what this means, we note first that the partial derivative F_i is given by

$$F_i(p_1, \dots, p_n) = \sum_{S} \prod_{j \in S} p_j \prod_{\substack{j \in S \\ j \ne i}} (1-p_j) \tag{A8}$$

where the sum is taken over all $S \subset N$ such that $i \notin S$ and $s = m - 1$.

Let X be the set of solutions to our problem, i.e. the set of all $(p_1, \ldots, p_n)$ which maximize F subject to (A3)–(A4). By continuity, X is non-empty and compact.

For given $p^* = (p_1^*, \ldots, p_n^*)$ satisfying (A3)–(A4), define a partition of N by

$$\begin{aligned} K_1(p^*) &= \{j \mid p_j^* = 1\}, \\ K_2(p^*) &= \{j \mid 0 < p_j^* < 1\}, \\ K_3(p^*) &= \{j \mid p_j^* = 0\}. \end{aligned} \tag{A9}$$

Let k_1, k_2, k_3 be the cardinalities of K_1, K_2, K_3 respectively. Let

$$q = \alpha - k_1, \tag{A10}$$

and note that, by (A3),

$$\sum_{j \in K_2} p_j^* = q. \tag{A11}$$

Define $Y(p^*)$ to be the set of all $(p_1, \ldots, p_n)$ satisfying

$$p_j = 1 \qquad \text{if } j \in K_1(p^*)$$

$$p_j = 0 \qquad \text{if } j \in K_3(p^*)$$

$$\sum_{j \in K_2} p_j = q \qquad 0 \le p_j \le 1$$

In other words, $Y(p^*)$ is obtained from p^* by all possible redistributions among the members of $K_2(p^*)$, leaving the competence of other members of N fixed at 0 or 1 (as the case may be).

We prove, next, that we need consider only points in which each p_i has one of the three values, 0, 1, and one other p.

Lemma 1. *Let $p^* \in X$, and suppose there exist $i, l \in K_2(p^*)$ such that $p_i^* \neq p_l^*$. Then $Y(p^*) \subset X$.*

Proof. Let us consider the expression (A8) for F_i. Letting $l \neq i$, we have

$$F_i = \sum_{\substack{S \\ l \in S}} \prod_{j \in S} p_j \prod_{\substack{j \notin S \\ j \neq i}} (1 - p_j) + \sum_{\substack{S \\ l \notin S}} \prod_{j \in S} p_j \prod_{\substack{j \notin S \\ j \neq i}} (1 - p_j),$$

where the first sum is taken over all S with $i \notin S$, $l \in S$, $s = m - 1$, and the second over all S with $i, l \notin S$, $s = m - 1$. We rewrite as

$$F_i = p_l \sum_S \prod_{\substack{j \in S \\ j \neq l}} p_j \prod_{\substack{j \notin S \\ j \neq i}} (1 - p_j) + (1 - p_l) \sum_S \prod_{j \in S} p_j \prod_{\substack{j \notin S \\ j \neq i, l}} (1 - p_j)$$

or, equivalently,

$$F_i = p_l \sum_S \prod p_j \prod (1 - p_j) + (1 - p_l) \sum_S \prod p_j (1 - p_j), \tag{A12}$$

where the first sum is taken over all S with $i, l \notin S$, $s = m-2$, and the second is taken over all S with $i, l \notin S$, $s = m-1$. In each case, the first product is over all $j \in S$, the second product over all $j \in N - S - \{i, l\}$.

We have, then (using an analogous expression for F_l)

$$F_i - F_l = (p_l - p_i)\left[\sum_S \prod p_j \prod (1-p_j) - \sum_S \prod p_j \prod (1-p_j) \right],$$

where the two sums are as in (A9), or equivalently

$$F_i - F_l = (p_l - p_i) H_{il}, \tag{A13}$$

where

$$H_{il} = \sum_{\substack{S \\ s=m-2}} \prod_{j \in S} p_j \prod_{\substack{j \notin S \\ j \neq i,l}} (1-p_j) - \sum_{\substack{S \\ s=m-1}} \prod_{j \in S} p_j \prod_{\substack{j \notin S \\ j \neq i,l}} (1-p_j), \tag{A14}$$

where the sums in (A14) are over all subsets $S \subset N - \{i, l\}$ with $m-2$ and $m-1$ elements respectively. We note, *inter alia*, that H_{il} depends on p_j, $j \neq i, l$, but *does not* depend on p_i or p_l.

Suppose, now, there is some pair of indices, $i, l \in K_2$ such that $p_i^* \neq p_l^*$.

By (A7-ii), we must have

$$F_i(p^*) = F_l(p^*)$$

or, by (A10)

$$F_i - F_l = (p_i^* - p_l^*) H_{il}(p^*) = 0.$$

However, $p_i^* \neq p_l^*$, so $H_{il}(p^*) = 0$.

As was pointed out, however, H_{il} is independent of p_i and p_l. Thus, for any t, the point $p'(t)$, given by

$$p_i'(t) = p_i^* + t$$

$$p_l'(t) = p_l^* - t$$

$$p_j'(t) = p_j^* \qquad \text{for } j \neq i, l$$

will also have $H_{il}(p') = 0$. For all t satisfying

$$\max\{-p_i^*, p_l^* - 1\} \leq t \leq \min\{p_l^*, 1 - p_i^*\},$$

point p' will satisfy the constraints (A3)–(A4). Moreover, the directional derivative of F, in the direction of increasing t, is $F_i - F_l$, and this will be 0 for all values of t. Thus, for all t,

$$F(p'(t)) = F(p^*).$$

Since p^* maximizes F, so does $p'(t)$. Thus, X contains not just p^*, but all points obtained from p^* by leaving the sum $p_i + p_l$ fixed, and all other p_j fixed at p_j^*.

Suppose some other index $k \in K_2$. For all except (at most) three values of t, we find that $p_i' \neq p_k'$, with both p_i' and p_k' different from 0. Since p' is also optimal, we can repeat the above argument, and find that the amount $p_i' + p_k'$ can be redistributed among i and k without losing optimality. Together with the compactness of set X, this means that $p_i^* + p_l^* + p_k^*$ can be redistributed among i, l, and k in any way (subject to constraints (A4)) without losing optimality.

Continuing in this way, and by induction, we come to the conclusion that the sum

$$\sum_{j \in K_2} p_j^* = q$$

can be redistributed among members of K_2 in any way, subject to (A4), without losing optimality, i.e., X contains the entire set $Y(p^*)$ as a subset.

Lemma 2. *Let* $0 < \alpha < m$. *Then the maximum of* F *can only be found at a point* $(p_1^*, \ldots, p_n^*)$ *of the form*

$$p_j^* = \begin{cases} 1 & \text{if } j \in K_1 \\ p & \text{if } j \in K_2 \\ 0 & \text{if } j \in K_3, \end{cases} \tag{A15}$$

where $0 < p < 1$, *and*

$$k_1 + k_2 + k_3 = n \tag{A16}$$

$$k_1 + k_2 p = \alpha. \tag{A17}$$

Proof. Suppose some p^*, not of this form, maximizes F. Then by Lemma 1, all points of $Y(p^*)$ maximize F, which means F is constant throughout Y. We claim this is not possible.

Over the set Y, we note that all members of K_1 are always right, and all members of K_2 are always wrong. The group decision will therefore be correct if *at least* $m - k_1$ of the members of K_2 are correct. This has probability

$$F(p) = \sum_{\substack{S \subset K_2 \\ s \geq m - k_1}} \prod_{j \in S} p_j \prod_{j \in K_2 - S} (1 - p_j).$$

We note that, by (A10), $k_1 \leq \alpha$, so we must have $k_1 < m$, and so $m - k_1 \geq 1$. We now distinguish two cases.

(a) $k_2 < m - k_1$. In this case, $F(p) = 0$ for all $p \in Y$, but this is clearly not a maximum so long as $\alpha > 0$ (since $p_j = \alpha/n$ for all $j \in N$ would give $F > 0$).

(b) $m - k_1 \leq k_2$. In this case, let $T \subset K_2$ be some set with $m - k_1$ elements. We have

$$q = \alpha - k_1 < m - k_1.$$

Define $\bar{p}$ by

$$\bar{p}_j = \begin{cases} 1 & \text{for } j \in K_1 \\ 0 & \text{for } j \in K_3 \cup K_2 - T \\ \dfrac{q}{m-k_1} & \text{for } j \in T. \end{cases}$$

Let $\bar{\bar{p}}$ be any point, other than $\bar{p}$, such that

$$\bar{\bar{p}}_j = \begin{cases} 1 & \text{for } j \in K_1 \\ 0 & \text{for } j \in K_3 \cup K_2 - T, \end{cases}$$

$$\sum_{j \in T} \bar{\bar{p}}_j = q,$$

$$0 \le p_j \le 1.$$

(There is some such $\bar{\bar{p}}$ since $0 < \bar{p}_j < 1$ for all $j \in T$.)

It can be seen that

$$F(\bar{p}) = \left(\frac{q}{m-k_1}\right)^{m-k_1}$$

whereas

$$F(\bar{\bar{p}}) = \prod_{j \in T} \bar{\bar{p}}_j.$$

By the inequality connecting the geometric and arithmetic means,

$$F(\bar{\bar{p}}) < F(\bar{p})$$

and so F is not constant throughout Y. The contradiction proves the Lemma.

Lemma 3. *If $\alpha \ge m$, then F is maximized by setting $k_1 \ge m$. If $\alpha < m$, then F is maximized by setting $k_1 = 0$, i.e. $K_1 = \emptyset$.*

Proof. If $\alpha \ge m$, it is easy to see that we get $F = 1$ by letting $k_1 \ge m$. This is clearly a maximum.

Suppose, next, that $\alpha < m$, but $K_1 \neq \emptyset$. Then $k_1 \le \alpha < m$, so we must have $k_2 > 0$ (as otherwise $F = 0$ which is clearly not a maximum).

Let $i \in K_1$, $l \in K_2$. Then $p_i = 1$ and $0 < p_l < 1$, so assuming p is optimal, we must have (by 7))

$$F_i \ge F_l.$$

Now, F_j is the probability that exactly $m-1$ players other than j be correct. Thus

$$F_i = \binom{k_2}{m-k_1} p^{m-k_1}(1-p)^{k_1+k_2-m}$$

and

$$F_l = \binom{k_2 - 1}{m - k_1 - 1} p^{m-k_1-1}(1-p)^{k_1+k_2-m}.$$

Thus, we have

$$\binom{k_2}{m-k_1} p^{m-k_1}(1-p)^{k_1+k_2-m} \geq \binom{k_2 - 1}{m - k_1 - 1} p^{m-k_1-1}(1-p)^{k_1+k_2-m},$$

which reduces to

$$\frac{k_2 p}{m-k_1} \geq 1$$

or

$$k_2 p \geq m - k_1.$$

By (A17), however, this gives us $m \leq \alpha$ which is a contradiction. Thus, for $\alpha < m$, then at the maximum, $K_1 = \emptyset$ as claimed.

By Lemma 2, then, we see that, in the 'complicated' case, $\alpha < m$, we have $K_1 = \emptyset$. Denote K_2 by K, K_3 by $N - K$, and so the optimum is obtained at a point of the form

$$p_j = \begin{cases} \dfrac{\alpha}{k} & \text{if } j \in K \\ 0 & \text{if } j \in N - K, \end{cases} \tag{A18}$$

where K has k elements. It remains to determine the value of k.

We have, indeed, for such p,

$$F = \sum_{s=m}^{k} \binom{k}{s} \left(\frac{\alpha}{k}\right)^s \left(\frac{k-\alpha}{k}\right)^{k-s}, \tag{A19}$$

and we are looking for that integer k, with $m \leq k \leq n$, which maximizes the expression:

$$F_{\max} = \max_{m \leq k \leq n} H(k, m, \alpha),$$

where

$$H(k, m, \alpha) = \sum_{s=m}^{k} \binom{k}{s} \left(\frac{\alpha}{k}\right)^s \left(\frac{k-\alpha}{k}\right)^{k-s}. \tag{A20}$$

In general, this maximum can be obtained from tables of the (cumulative) binomial distribution. For large m, n, this can be approximated by the Poisson or the normal distributions.

In carrying out this maximization, we find that:

Lemma 4.

(a) *For small α (i.e. α close to 0) H (and hence F) is maximized by setting $k=n$ (as large as possible). There is an exception for the case $m=1$.*

(b) *For large α (i.e. α close to m) H is maximized by setting $k=m$ (as small as possible).*

Proof

(a) We note $H(k,m,\alpha)$ is a sum of binomial terms

$$B\left(k,s,\frac{\alpha}{k}\right)=\binom{k}{s}\left(\frac{\alpha}{k}\right)^{s}\left(\frac{k-\alpha}{k}\right)^{k-s}$$

where $m\le s\le k$. If we increase k to $k+1$, we have a corresponding term

$$B\left(k+1,s,\frac{\alpha}{k+1}\right)=\binom{k+1}{s}\left(\frac{\alpha}{k+1}\right)^{s}\left(\frac{k-\alpha+1}{k+1}\right)^{k-s+1}.$$

Consider now the ratio of these two terms:

$$R(k,s,\alpha)=\frac{B(k,s,\alpha/k)}{B(k+1,s,\alpha/k+1)}.$$

Some algebra will simplify this to

$$R(k,s,\alpha)=\frac{k-s+1}{k-\alpha+1}\left(\frac{k+1}{k}\right)^{k}\left(\frac{k-\alpha}{k-\alpha+1}\right)^{k-s}.$$

Let α, now, approach zero. In the limit, we will have

$$R(k,s,\alpha)\to\frac{k-s+1}{k+1}\left(\frac{k+1}{k}\right)^{s}$$

or

$$R\to\left(1-\frac{s}{k+1}\right)\left(1-\frac{1}{k+1}\right)^{-s}.$$

Now, for $k>0$, and $s\ge 2$, we know that

$$1-\frac{s}{k+1}<\left(1-\frac{1}{k+1}\right)^{s}$$

and so

$$R(k,s,\alpha)<1$$

for all $k\ge s\ge 2$. Thus

$$B\left(k,s,\frac{\alpha}{k}\right)<B\left(k+1,s,\frac{\alpha}{k+1}\right)$$

and thus each of the summands increases with k. Since H is a sum of such terms (and $H(k+1, m, \alpha)$ has one more term than $H(k, m, \alpha)$) we must have

$$H(k, m, \alpha) < H(k+1, m, \alpha).$$

Hence H, and so F, is maximized by choosing k as large as possible, i.e. $k = n$.

(b) We note that, for $\alpha = m$,

$$H(m, m, m) = B(m, m, 1) = 1$$

whereas, for $k > m$,

$$H(k, m, m) < 1$$

thus $H(m, m, m) > H(k, m, m)$ for all $k \neq m$. Since only a finite number of k are possible, the continuity of these functions guarantees that, for α close to m,

$$H(m, m, \alpha) > H(k, m, \alpha)$$

for all $k \neq m$.

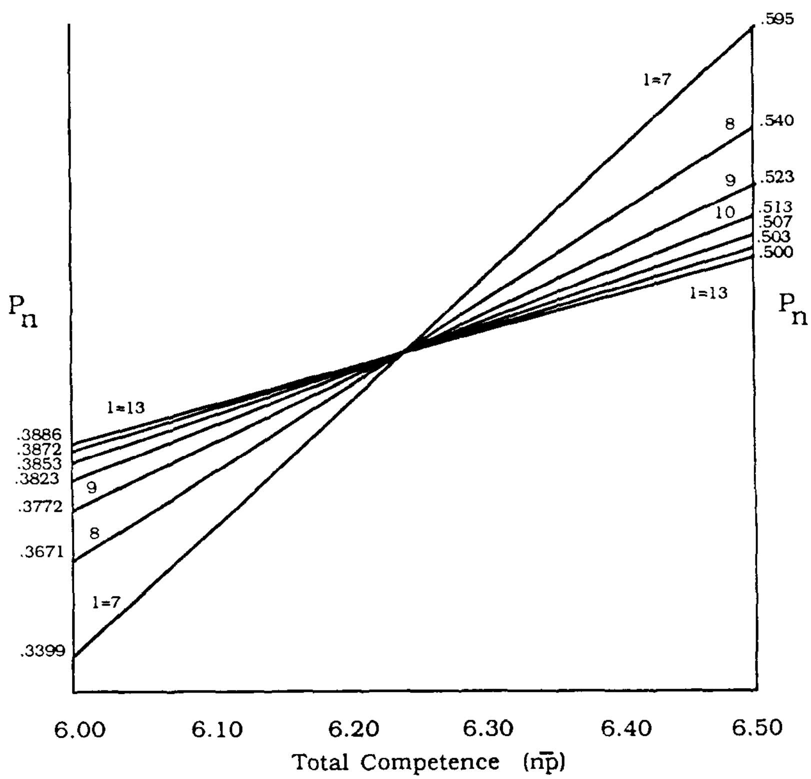

Fig. 1. The impact of concentrating competence on P_n, for $n = 13$.

Observation. Lemma 4 tells us that, for $m \geq 2$, there exists a number $\delta(m) < m$ such that, if $0 < \alpha < m - \delta(m)$ then it is optimal to choose $k = n$. For any m, there exists a number $\varepsilon(m) > 0$ such that, if $m - \varepsilon(m) < \alpha \leq m$, it is optimal to choose $k = m$. Clearly we must have $\delta(m) > \varepsilon(m)$, but otherwise it is not clear what the values of ε and δ must be, nor what happens in the remaining interval,

$$m - \delta(m) < \alpha < m - \varepsilon(m).$$

Experimental observation (by choosing a few values of m) seems to suggest that

(a) $\varepsilon(m)$ is relatively small - asymptotically it seems to approach the number $e - 2 = 0.718\ldots$

(b) $\delta(m)$ is only slightly larger than $\varepsilon(m)$, so that the interval $[m - \delta(m), m - \varepsilon(m)]$ seems to have length of the order of 0.1

(c) Throughout this interval, the optimal k seems to decrease from n to m, taking on all possible values over very small sub-intervals.

To see how this works, consider the case $n = 13$, $m = 7$. Fig. 1 shows the results of our calculations: $k = 13$ is optimal for $\alpha < 6.16$, while $k = 7$ is optimal for $\alpha > 6.30$. In between, there seem to be five small subintervals in which $k = 12$, 11, 10, 9, 8 are successively optimal. Again, consider the case $n = 5$, $m = 3$. It is easy to calculate that $k = 5$ is optimal for $\alpha < 2.117$, $k = 4$ is optimal for $2.117 < \alpha < 2.173$, and $k = 3$ is optimal for $\alpha > 2.173$.

For $n = 3$, $m = 2$, again, we note that $k = 3$ is optimal for $\alpha < 1.125$, while $k = 2$ is optimal for $\alpha > 1.125$.

In these cases, the pattern seems to be quite clear. We repeat, however, that this is only a conjecture - we have no proof for the general case of arbitrary m and n. However, the experimental results seem very suggestive.

Lemma 5. *If $\bar{p} = \frac{1}{2}$, for $k = m = (n+1)/2$ we find that*

$$F_{\max} = \left(\frac{\alpha}{m}\right)^m = \left(1 - \frac{1}{2m}\right)^m,$$

which, for large m is approximately equal to $e^{-1/2} = 0.61$. If $\bar{p} = \frac{1}{2}$, for $k = m = (n+1)/2$ we find that

$$F_{\min} = 1 - \left(\frac{a}{m}\right)^m = 1 - \left(1 - \frac{1}{2m}\right)^m,$$

which for large m is approximately equal to $1 - e^{-1/2} \cong 0.39$.

We may use the above Lemma to prove a critical additional Lemma, which shows for a fixed total group competence (which we may arbitrarily denote $\bar{p}n$), how to find the distribution which will maximize (minimize) P_n, the accuracy of the group's majority rule decision process.

Lemma 6. (*Maximization/Minimization of Group Majority Rule Competence, Sub-*

ject to a Fixed Average Competence Constraint):

For fixed $\bar{p}n$, P_n is maximized

(a) *if $n\bar{p} \geq (n+1)/2$, by setting a majority of the p_i's to one.*

(b) *if $(n+1)/2 \geq n\bar{p} \geq (n/2) - 0.2$, by setting $p_i = 0$ for $(n-1)/2$ members of the group and $p_j = \bar{p}(2n/(n+1))$ for the $(n+1)/2$ remaining members of the group.*

(c) *If $n\bar{p} \leq (n/2) - 0.4$, by setting $p_i = \bar{p}$ for all i.*

Similarly, P_n is minimized

(a) *if $n(1-\bar{p}) \geq (n+1)/2$, i.e., if $1 > \bar{p}(2n/(n-1))$, by setting a majority of the p's to zero.*

(b) *if $(n+1)/2 > n(1-\bar{p}) \geq n/2 - 0.2$, by setting $p_i = 1$ for $(n-1)/2$ members of the group and $1 - p_j = (1-\bar{p})(2n/n-1))$ for the remaining $(n+1)/2$ members of the group.*

(c) *if $n(1-\bar{p}) \leq (n/2) - 0.4$ by setting $p_i = \bar{p}$ for all i.*

Proof. The first part of Lemma 6 is merely a restatement of Lemmas 2, 3 and 4 in a slightly different notation (recall that $\alpha = \bar{p}n$). Equivalent minimization results can be obtained, by symmetry, by appropriate interchanging of 1's and 0's, and these give us the second part of Lemma 5. The parameters of 0.2 and 0.4 are only approximate (see discussion above).

There is a small set of values of $\bar{p}$ not covered by Lemma 5. For those values, P_n is maximized by dividing all competence equally among exactly k members of the group where k runs from $(n+1)/2$ to n as we approach the limiting bounds of the expressions in (b) and (c) above. We show how this works for the case $N = 13$ in Fig. 1 in the text. Exactly analogous results obtain for the minimization case. Of course, as n increases, the bounds of the righthand inequalization in expression (b) and (c) converge toward $\bar{p} = \frac{1}{2}$. For a given $\bar{p}$ value, in general, the closer a distribution is to the maximizing (minimizing) extreme case, the higher (lower) is the P_n value it gives rise to. □

We would expect that some of the perverse examples of a mismatch between $\bar{p}$ and P_n given in the first section of the text are phenomena of small numbers. Indeed, $\bar{p} < \frac{1}{2}$ while $P_n > \frac{1}{2}$ (or $\bar{p} > \frac{1}{2}$ while $P_n < \frac{1}{2}$) requires $\bar{p}$ closer and closer to $\frac{1}{2}$ as n gets larger. To see this we shall first establish Lemma 7:

Lemma 7(a): *A necessary condition for $P_n > \frac{1}{2}$ is that*

$$\left[\bar{p}\left(\frac{2n}{n+1}\right)\right]^{(n+1)/2} > \frac{1}{2}. \tag{A20}$$

Lemma 7(b): *A sufficient condition for $P_n > \frac{1}{2}$ is that*

$$\left[(1-\bar{p})\left(\frac{2n}{n+1}\right)\right]^{(n+1)/2} < \frac{1}{2} \tag{A21}$$

Proof. Let us look at the conditions maximizing P_n. Eq. (A20) is merely a restate-

ment of the distribution requirements of Case (b) in Part I of Lemma 5. A similar look at the minimization conditions in Part II of Lemma 5 suffices to establish the sufficiency assertion in Eq. (A18).
Because (A2) and (A3) are monotonic in n, we may look at the lowest odd value of n greater than 1, $n=3$, to establish that a necessary condition for $P_n > \frac{1}{2}$ is that $\bar{p} > 2/\sqrt{3} = 0.471$; while a sufficient condition for $P_n > \frac{1}{2}$ is that $\bar{p} > (\sqrt{3}-2)/\sqrt{3} = 0.529$. In other words, as n increases, the conditions on $\bar{p}$ necessary and sufficient for *both* $P_n > \frac{1}{2}$ and $P_n < \frac{1}{2}$ will converge toward $\bar{p} = \frac{1}{2}$ (from above or below). This suggests that as n gets large, if $\bar{p} > \frac{1}{2}$ (if $\bar{p} < \frac{1}{2}$ then $P_n < \frac{1}{2}$); but it also suggests the paradoxical finding that $\bar{p} = \frac{1}{2}$ is compatible with both $P_n > \frac{1}{2}$ and $P_n < \frac{1}{2}$.

Now we can complete our proof of Theorem II.

Proof. Let us first demonstrate (1). We need only look at the worst cases (minimizing distributions) from Lemma 5. The desired result is no problem for case (a). In case (b) of Part 2 of Lemma 5 we minimize by setting a bare minority of the group's p_i values to 1 and the remaining group members' values to $(1-\bar{p})\ (2n/(n-1))$. But, then, the whole group will be correct whenever at least one of the $(n+1)/2$ members of that group is correct. This occurs with probability

$$1 - \left[(1-p)\left(\frac{2n}{n-1}\right) \right]^{(n+1)/2},$$

which is monotonically increasing in n. On the other hand in case (c) of Part 2 of Lemma 5, we minimize by setting $p_i = \bar{p}$ for all i. But for this case we know from Theorem I that $\lim_{n\to\infty} P_n \to 1$, since $\bar{p} > \frac{1}{2}$. □

The proof of Part (2) is essentially identical, only looking at the maximization cases of Theorem II rather than those involving minimization.
Part (3) follows directly from Lemma 4. □

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