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A MODEL OF CANDIDATE CONVERGENCE UNDER UNCERTAINTY ABOUT VOTER PREFERENCES

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Abstract—The paper considers two-candidate spatial voting games without a core where candidates have only imperfect knowledge of voters' preferences. We prove that a candidate maximizes his chances of victory by choosing a position as close as possible to that of the candidate who committed to his position first. We also determine the optimal position for this first candidate.

1. INTRODUCTION

The spatial model of politics considers voters who have preferences over positions that candidates may adopt, and candidates who choose positions that they think will attract majority support. Initial work with this model [see especially Downs (1957)] provided important insight into two-party politics. Further study revealed, however, a critical difficulty—equilibria are certain to exist only if voters' ideal points fall along a single continuum [such as a left–right dimension (Black 1958)] or if very strong symmetry assumptions are met (Plott 1967; Riker and Ordeshook 1973; Kramer 1973; McKelvey 1976, 1979; Schofield 1978). That is, the candidate who chooses a position first can always be defeated by a challenger, and the positions taken by the winning candidates may range over the whole policy space. Indeed, the policies could change dramatically from election to election. Political behavior, however, exhibits a degree of stability far greater than that predicted by theory—incumbents easily win reelection (particularly in Congressional and other legislative elections) and policy only rarely shows drastic changes when one set of officials replaces another.

One modification of the standard model to account for this difficulty is to assume that candidates aim to maximize something other than probability of victory. Kramer (1977) looks at a sequence of elections in which each candidate's objective is to win the election with as large a plurality as possible. Under these conditions the trajectory of winning positions is well-behaved and settles in a well-defined region, known as the minmax set.

Another approach, related to ours, is to introduce probabilistic voting. Coughlin and Nitzan (1981), Coughlin (1982), Enelow and Hinich (1984), Samuelson (1984) and Coughlin and Palfrey (1985) suppose that candidates wish to maximize their expected plurality, and that a voter is more likely, though not certain, to vote for that candidate whose position is closest to his ideal point. A natural interpretation is that voters are uncertain about the positions adopted by the candidates. Our assumption is that the candidates are uncertain about the preferences of the voters.

The main distinction between probabilistic voting and our assumptions is that in probabilistic voting the likelihood a voter will vote for a candidate has some elasticity with respect to how close the candidate locates to him. In contrast, if candidates are uncertain about preferences of voters, but a voter is certain to vote for the candidate nearest him, each voter's behavior shows a discontinuity, i.e. a small change in position can cause a large change in result. These differing assumptions lead to different conclusions. In particular, Coughlin and Palfrey (1985) prove that if candidate B views the position chosen by candidate A as fixed, then candidate B will choose a Pareto-optimal position. One implication of this is that, if candidate A happens to choose a position that is not Pareto optimal, then candidate B will choose a position that is distant from that of candidate A.

We shall prove, in distinction, that regardless of what position candidate A chooses, candidate B will choose a position that is very close to A's. This means that candidate B need not always

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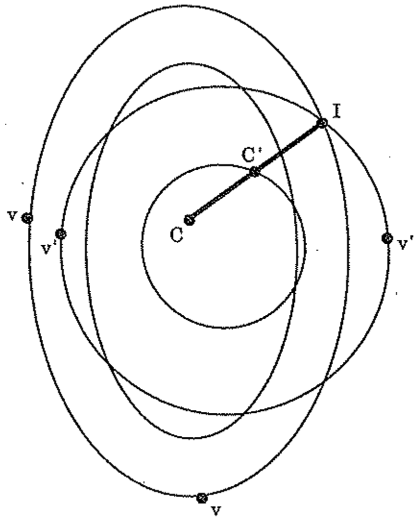

Fig. 1. Possible indifference curves for a voter.

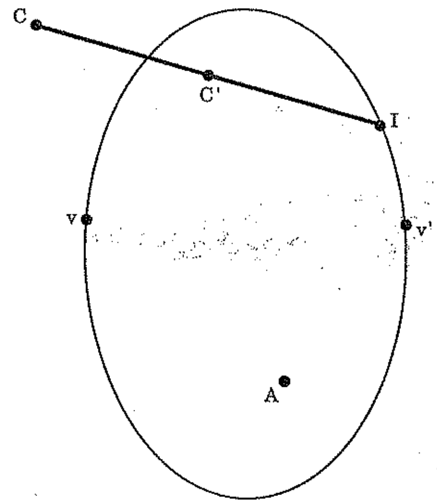

Fig. 2. Preference for intermediate points.

choose a Pareto-optimal position, and that in a long sequence of elections all candidates may choose positions that are not Pareto optimal.

Of course, this result is well-known for *deterministic* voting if candidates cannot cross over each other's position (cf. McKelvey 1986). However, no general proof is known to us for the case of uncertainty.

Our analysis further complements previous approaches in two ways. In contrast to many (though not all) models of candidate competition, our results about the challenger's decision applies both when candidates aim to maximize their plurality, and when they aim to maximize their probability of victory. In general, the two are different objectives and it is not always true that maximizing one entails maximizing the other (see Hinich and Ordeshook 1971).

Unlike most authors [including Calvert (1985), to whom our work is closest in spirit, although independently derived], we show that our results hold not only in the neighborhood of the equilibrium solution, but everywhere: the challenger will wish to locate next to the incumbent even if the incumbent's position is not optimal. Similarly we show that the incumbent will prefer to move from *any* point that is not a Pareto optimal one to a point in its neighborhood that is Pareto superior.

2. THE OPTIMALITY OF A TWEEDLE-DUM RESPONSE TO TWEEDLE-DEE

Let a voter support that candidate whose announced positions would, if enacted, bring the voter the greatest utility. Each voter's preferences can be represented by a set of closed, convex, indifference curves, such that a point along any ray is less desirable the further away it is from the voter's ideal point.† This implies that all points preferred to a specified point lie in a convex set that includes the voter's ideal point. The standard spatial model with circular indifference curves is a special example of the preferences we consider.

We assume that two candidates run in each election. Except where otherwise noted, we shall speak of a first-mover, the incumbent who chooses some position, I, first, and cannot alter it after discovering the position of the challenger. The second-mover, the challenger, chooses a position, C, after having learned of the incumbent's position. Neither candidate has perfect information about the preferences of all the voters. This uncertainty can relate not only to the position of each voter's ideal point, but also to the form of the voter's utility functions, or to the exact configuration

†Even though choice may be deterministic in nature, it may be that not all of the elements that enter the voter's decision are known to the investigator. It may therefore be desirable to model voter choice as probabilistic in nature (as is commonly done).

of their indifference curves about their ideal points. For example, in Fig. 1 a candidate may not know if the voter's indifference curve through some point, say I, has the shape of curve Ivv, or instead the shape of curve Iv′v′.

We shall prove that, in this context of uncertainty, the challenger maximizes his chances of victory by adopting a position that is only infinitesimally distant from the incumbent's This holds for *any* position, Pareto optimal or not, chosen by the incumbent.

Let the incumbent choose a position at point I, and suppose that the challenger chooses a position at point C, which is a finite distance away. Let point C′ lie on the segment IC, as illustrated for two dimensions in Fig. 1.

Suppose that an arbitrary voter prefers point C over point I. Then by the convexity of the voter's preferences, it follows that he also prefers point C′ to point I. That is, if a voter prefers the challenger over the incumbent when the challenger is at point C, then the voter will also prefer the challenger when the challenger is at a position C′ that lies between points C and I. This result holds for *any* set of convex indifference curves a voter may have: since the candidate does not know for certain a voter's preferences, the challenger will not decrease his chances of gaining that voter's support by moving from point C to point C′ on the segment CI.

The converse, however, does not hold. This is illustrated in Fig. 2 for a voter with an ideal point at A, and whose indifference curve containing point I is curve Ivv′: since point C′ lies inside Ivv′ while point C lies outside curve Ivv′, the voter prefers C′ to I, but prefers I to C.

Look at some point C chosen by the challenger. Would the challenger be better off picking C′, on the segment CI, closer to I? For any voter there are two cases; either the voter prefers C to I or he does not. If the voter does prefer C to I, then any point C′ on the segment CI will also be preferred to I, so by shifting to C′ the challenger does not lose this vote. If the voter does *not* prefer C to I, then point C′ may or may not be preferred to I, but the challenger can lose no votes in moving to C′ (and may, in fact, gain votes if some voters prefer C′ to I but not C to I). We can repeat this argument with the challenger now located at point C′. Since this argument holds for any voter, and thus for all voters, it follows that for any position C that the challenger may choose, he can increase his chances of victory by shifting from that position along the segment CI in a direction closer to I. He therefore maximizes his chances of victory by choosing a position that lies within an infinitesimal distance of the position of the incumbent.† Q.E.D.

The result obtained above also applies if the challenger's objective is to maximize his expected plurality rather than his probability of victory. A challenger who chooses a position at some point on the line segment CI instead of at point C may attract the support of an additional voter, but can never lose the support of any voter by doing so. Thus, by moving from point C to a point on the segment CI the challenger would increase his expected plurality.

Note that the argument about proximity applies not only for two dimensions, but for any number of dimensions, as long as voters' indifference surfaces are convex. The implication of the above discussion is that if the incumbent's position is fixed and known, and the challenger is not certain about the location of voters' indifference curves, the challenger maximizes his probability of winning by choosing a position as close as possible to that of the incumbent.‡

A challenger may be unable to adopt a position close to the incumbent's; the challenger, for example, may have already committed himself otherwise in a primary race. In the absence of such

†The reader may be wondering whether this result proves too much. It seems to imply that the challenger can always defeat the incumbent. But we have said nothing of the kind. Under uncertainty, given our convexity assumption, the challenger's probability of defeating the incumbent will not be more than 0.5. This is because, in general, if motion from the incumbent's position in one direction gives the challenger victory, motion in the opposite direction would lead to his defeat and vice versa. (This statement is true so long as the voters' utility functions are differentiable, unless the incumbent's position happens to coincide with some voter's ideal point; and for other points, except for a set of directions with Lebesgue measure zero.) Thus, very close to the incumbent's position, only about half of the positions lead to victory for the challenger.

There is a further point, moreover, namely that the incumbent, *qua* incumbent, seems to have an additional advantage. It is not clear how this should be modelled, but what seems to happen is that a substantial number of voters (the *diehards*) will support the incumbent no matter what position the challenger takes. What this would mean is that the challenger needs considerably more than 50% of the non-diehards to win the election.

‡Of course, "as close as possible" is not meaningful given a continuum of possible points. We posit that, if the challenger comes too close to the incumbent's position, he will be accused of "me-too-ism." Thus, he will look for a position just far enough to avoid this accusation, and no farther.

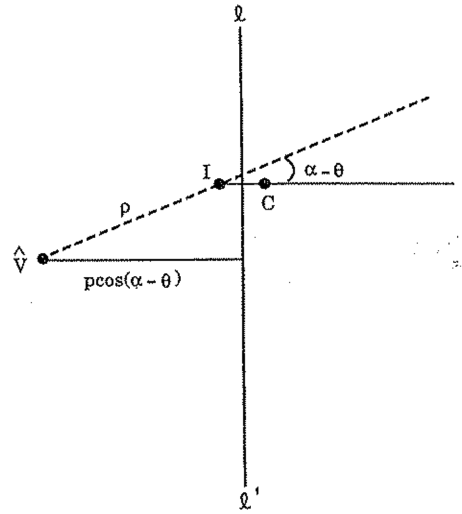

Fig. 3. Choice of optimal direction for the challenger.

constraints, however, our results imply that the political system will not see dramatic changes in policy. Though it is true that a challenger with perfect information could defeat the incumbent by choosing a position somewhat distant from the incumbent's the challenger will find it best to choose a nearby position.† And this will hold true even if the preferences of voters are almost exactly known; the slightest degree of uncertainty leads to Tweedle-dum–Tweedle-dee politics.

3. THE CHALLENGER'S OPTIMAL DIRECTION AND THE INCUMBENT'S OPTIMAL POSITION

A full solution of the model would determine the optimal positions for both candidates. Clearly, the optimal position for the candidate who must choose his position first is determined in part by the position that the challenger will take in response. We are unable to find general solutions, but an example can prove instructive. We shall consider the problem in two stages. First, we solve for the position the challenger should adopt for any given position by the first candidate. Second, given the response function of the challenger, we determine the vote-maximizing position for the first candidate.

Where exactly should the challenger place himself? Granted that he should be close to the incumbent, we would like to find an optimal direction for him, that is the optimal slope for the line IC. (The challenger would then place himself an infinitesimal distance away from point I in this direction.) A precise answer requires specifying the challenger's uncertainty about the voter's preferences. For simplicity, consider circular indifference curves in two dimensions. The only uncertainty concerns the location of the voters' ideal points. If the expected location of a voter's ideal point is at $\hat{v}$, with coordinates $(\hat{v}_1, \hat{v}_2)$, then we suppose that with some positive probability the actual ideal point will have coordinates $(\hat{v}_1 + R\cos(\theta), \hat{v}_2 + R\sin(\theta))$. The value of θ can range from 0 to 2π with a uniform density; R is a positive random variable with a density function, $f(R)$, such that $f'(R) < 0$. These assumptions imply that the variation in a voter's ideal point is symmetric about his expected ideal point, and that large changes are less likely than small changes.

Suppose the challenger chooses a position in the direction θ from I, that is, his position is given by the coordinates $(C_1, C_2) = (I_1 + \delta\cos\theta, I_2 + \delta\sin\theta)$ (see Fig. 3). Now a voter will prefer I over C if the voter's ideal point lies on the same side (as I) of the perpendicular bisector of IC, line ll'. Define $G(z)$ as

$$G(z) = \tfrac{1}{2} + \mathrm{sgn}(z) \int_0^{\pi} F(|z|\sec\phi)\,d\phi, \tag{1}$$

†If the challenger locates himself too far from point I he will be certain to lose—at least if I is a reasonably central point in the space of voter ideal points.

where

$$F(z) = \int_{-\infty}^{z} f(t)\,\mathrm{d}t$$

and

$$\operatorname{sgn}(z) = \begin{cases} 1 & \text{if } z > 0 \\ 0 & \text{if } z = 0 \\ -1 & \text{if } z < 0. \end{cases}$$

Thus, $G(z)$ is the probability that $\hat{v}$ actually lies to the left of a vertical line whose distance from $\hat{v}$ is $|z|$ units. Because of the circular symmetry of the distribution, it is also the probability that v lies on one side of a line whose perpendicular distance from $\hat{v}$ is $|z|$ units.

Suppose now that the vector $\hat{v}$I has magnitude ρ and direction ($\cos\alpha$, $\sin\alpha$), and that the vector ÎC has direction ($\cos\theta$, $\sin\theta$). Then the perpendicular distance from $\hat{v}$I to ll' is $\rho\cos(\alpha-\theta)$, where a positive value means that $\hat{v}$ lies on the same side of ll' as I, and a negative value that it lies on the same side as C. The probability that the voter's ideal point, v, will be on the same side of ll' as I is, as shown in Fig. 3, $G(\rho\cos(\alpha-\theta))$.

We may label this as P_v, the probability that the voter's ideal point is at a location which makes him prefer I to C. Consider, for illustration, the case of three voters. We wish to find the probability that at least two of the three voters X, Y and Z will prefer I to C, or the probability that I defeats C. Assuming that the expected ideal points $\hat{X}$, $\hat{Y}$ and $\hat{Z}$ are independently distributed, we obtain

$$P(\text{I wins}) = P_X P_Y + P_X P_Z + P_Y P_Z - 2P_X P_Y P_Z, \tag{2}$$

where

$$P_X = G_X(\rho_X \cos(\alpha - \theta)), \tag{3a}$$

$$P_Y = G_Y(\rho_Y \cos(\beta - \theta)) \tag{3b}$$

and

$$P_Z = G_Z(\rho_Z \cos(\gamma - \theta)). \tag{3c}$$

The variables α, β and γ are the directions of the three vectors $\hat{X}$I, $\hat{Y}$I and $\hat{Z}$I, and ρ_X, ρ_Y and ρ_Z are the magnitudes of the vectors. The functions in equations (3a–c) simply give the probability that a voter prefers I to C.

Equations (2) and (3a–c) can be used to find the optimal position of the challenger, and thus also the probability that the first candidate will win for any position he may choose. We can solve the first candidate's maximization problem numerically; Fig. 4 presents some of the results.

Let the expected ideal points of the three voters be at (0, 0), (0, 1) and (1, 0). Let $f(R)$ be normally distributed with a mean of 0 and a standard deviation (SD) of 1 for each of the three voters. We find that the incumbent maximizes his chances of victory by locating at the point (0.294, 0.294); his probability of victory then is 45% (note that in the absence of uncertainty about the voters' preferences, a challenger could defeat him with probability 1). The arrows in the figure show the optimal direction for the challenger's position. For example, if the incumbent is at point A, then the challenger maximizes his chances of victory by locating at a point near A, in the direction of the vector AB.

Figure 5 shows solutions when the candidates are less uncertain about the preferences of one of the voters. In particular, for the voter with an expected ideal point at (1, 0), let $f(R)$ be the normal distribution with a mean of 0 and an SD of 0.5 (in Fig. 4 we assumed the SD is 1) The incumbent now maximizes his probability of victory by choosing the position at (0.5, 0.25), which lies closer to the expected ideal point of the voter about whom there is least uncertainty. As we would expect, the incumbent's chances of winning, compared to the previous case, decline (to 33% from 45%).

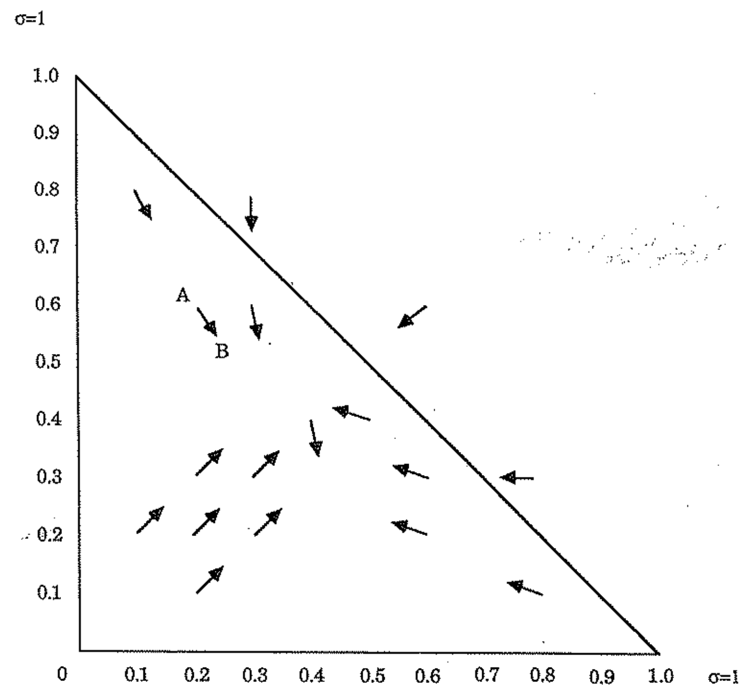

Fig. 4. Optimal directions for the challenger.

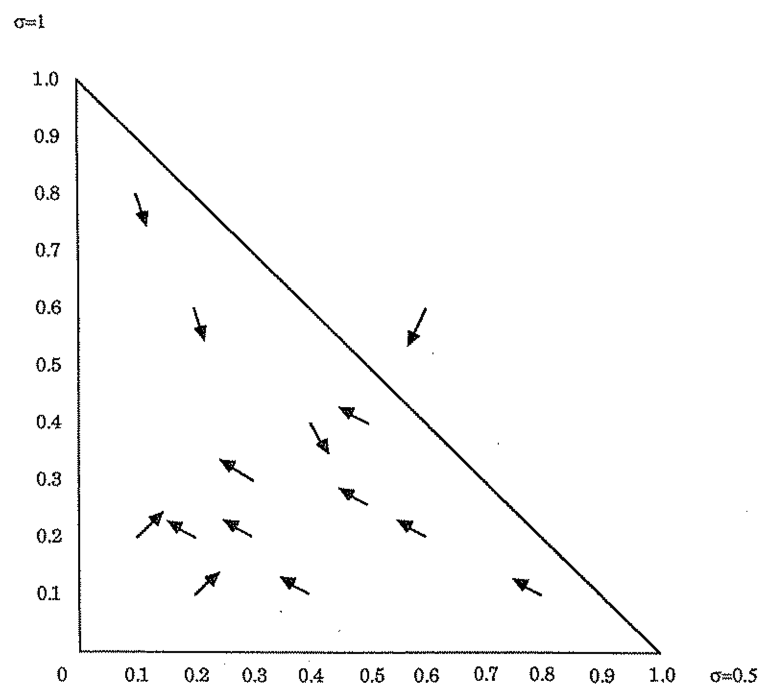

Fig. 5. Optimal directions, with a slight change of variance.

4. THE PARETO OPTIMALITY OF THE INCUMBENT'S LOCATION

The previous sections dealt with the challenger's decision. We now turn to the choice made by the first-mover, to ask whether he will choose a Pareto-optimal position. In this model the concept of Pareto optimality is not straightforward. A candidate does not know for certain the preferences of voters, and therefore we cannot speak of a candidate who believes that all voters prefer one position over another. We can, however, modify the definition of Pareto optimality to ask the following question. Will the incumbent, who must announce his position before the challenger does, choose point I′ over point I if he expects that with probability >0.5 each voter would prefer point I′ over point I? Note that the conventional definition of Pareto optimality is identical to ours if candidates have perfect information about voter's preferences. Similarly, if we suppose that voters have circular indifference curves, and that the only uncertainty concerns the location of their ideal points, our question is whether the incumbent will choose a Pareto-optimal point relative to the candidates' beliefs about the voters' expected ideal points.

Consider an incumbent who is initially at point I. Let the challenger maximize his own chances of winning by choosing point C, at an infinitesimal distance from I. Now suppose instead that the incumbent chooses a Pareto-superior point, I′, at an infinitesimal distance from point I and, in response, the challenger now chooses point C′. Let the incumbent's probability of winning when he has position I, and the challenger has position C, be $w(\mathrm{I}, \mathrm{C})$. By assumption $w(\mathrm{I}', \mathrm{C}) > w(\mathrm{I}, \mathrm{C})$, but we have yet to prove that $w(\mathrm{I}', \mathrm{C}') > w(\mathrm{I}, \mathrm{C})$.

If the probability distributions of the parameters describing the voters' preferences are differentiable, then so is the function $w(\mathrm{I}, \mathrm{C})$. We posit that the challenger chooses his position optimally, which implies that

$$\frac{\partial w(i, c)}{\partial c} = 0 \tag{4}$$

and, therefore

$$\begin{aligned} \frac{\mathrm{d}w(i, c)}{\mathrm{d}i} &= \frac{\partial w(i, c)}{\partial i} + \frac{\partial w(i, c)}{\partial c}\left(\frac{\mathrm{d}c}{\mathrm{d}i}\right) \\ &= \frac{\partial w(i, c)}{\partial i}. \end{aligned} \tag{5}$$

Recall that, in general, I and C are vectors in multidimensional space. The expressions $\partial w/\partial c$ and $\partial w/\partial i$ should therefore be interpreted as gradient partial derivatives; $\mathrm{d}w/\mathrm{d}i$ is a gradient total derivative and $\mathrm{d}c/\mathrm{d}i$ is a Jacobian matrix. Because the r.h.s. side of equation (5) is a positive for movements toward Pareto-superior points $w(\mathrm{I}', \mathrm{C}') > w(\mathrm{I}, \mathrm{C})$, and the incumbent improves his chances of victory by moving to a Pareto-superior point.

We note several implications of this solution:

1. The incumbent maximizes his chances of victory by choosing a position that is Pareto optimal. The proof is by contradiction—if the incumbent chooses a position that is not Pareto optimal, then we have just seen that he can improve his chances of victory by moving away from that position. Moreover, since associated with each position is a probability that the incumbent wins, there must exist one or more points which maximize this probability. Any such point, however, cannot be a point that is not Pareto optimal, and therefore must be a point that is Pareto optimal. The result thus obtained is identical to that which holds with probabilistic voting.
2. The result is stronger than that found in much of the literature. We do not merely claim that the incumbent maximizes his chance of victory by choosing a Pareto-optimal point. We state that for any position whatsoever, a movement towards a Pareto-superior point in the neighborhood of the initial point will increase the incumbent's probability of victory.
3. Note however, that we cannot make the statement that, regardless of what the incumbent does, the challenger should choose a Pareto-optimal position. If the incumbent happens to choose a position that is not Pareto optimal, then the challenger does best by adopting a position that is near the incumbent's and which therefore need not be Pareto optimal either. In contrast, under probabilistic voting where voters are uncertain about the positions of the candidates the challenger should choose a Pareto-optimal position regardless of what the incumbent does (see Coughlin and Palfrey 1985; Ordeshook 1971).

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Our conclusions held under the assumption that voters have perfect information about the candidates, but that candidates do not have perfect information about the voters. Undoubtedly, real elections see both types of uncertainty. We suspect, though cannot prove, that the characteristics of an equilibrium will lie somewhere between those obtained in our model and those obtained elsewhere.

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