

ON THE CONCEPT OF A RANDOM SEQUENCE¹

ALONZO CHURCH

Von Mises has based his frequency theory of probability on the notion of a *Kollektiv*,² that is, of an infinite sequence of trials of an event whose possible outcomes have each a definite probability but otherwise appear entirely at random. (Convenient illustrative examples are an infinite sequence of tosses of a coin, an infinite sequence of rolls of a die, and the like.)³

Abstractly the *Kollektiv* may be represented by an infinite sequence of points of an appropriate space, the *Merkmalraum*. Or if the number of possible outcomes of a trial is finite (and it may well be argued that this is always the case for any actual physical observation⁴), it is sufficient to employ an infinite sequence of natural numbers which are less than a fixed natural number. This infinite sequence—of points or of natural numbers—satisfies certain conditions which correspond to those appearing in the description of a *Kollektiv* as just given, and which we shall express by saying that it is a random sequence (*regellose Folge*).

For the present purpose it is largely sufficient to confine attention to the case that each trial has only two possible outcomes, as with the toss of a coin adjudged as falling heads or tails, or the roll of a die adjudged as showing or not showing an ace. The *Kollektiv* may then be represented abstractly by a random sequence of 0's and 1's: in the case of the coin, for instance, we may let 1 correspond to the fall of heads and 0 to tails.

The definition of a random sequence of 0's and 1's as given by von Mises may perhaps be put in the following form:

¹ Presented to the Society, April 8, 1939.

² Richard von Mises, *Grundlagen der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift, vol. 5 (1919), pp. 52–99; *Wahrscheinlichkeit, Statistik und Wahrheit*, Vienna, 1928; *Wahrscheinlichkeitsrechnung*, Leipzig and Vienna, 1931; and see especially the second edition of *Wahrscheinlichkeit, Statistik und Wahrheit*, Vienna, 1936, for its account of the objections which have been raised to von Mises's theory and the alternatives which have been proposed.

³ The introduction of an *infinite* sequence of trials (tosses of a coin, and so on) is, of course, an abstraction from the realities of the situation, made for the sake of the mathematical theory. It is an instance of the familiar device of employing the infinite as being, for certain purposes, a convenient and useful approximation to the large finite.

⁴ In cases where the number of possible outcomes of a trial is taken as infinite, either there is a further element of abstraction involved (the infinite again replacing the large finite), or else the problem considered has no direct physical application.

An infinite sequence a_1, a_2, \dots of 0's and 1's is a random sequence if the two following conditions are satisfied:

(1) If $f(r)$ is the number of 1's among the first r terms of a_1, a_2, \dots , then $f(r)/r$ approaches a limit p as r approaches infinity.

(2) If a_{n_1}, a_{n_2}, \dots is any infinite sub-sequence of a_1, a_2, \dots , formed by deleting some of the terms of the latter sequence according to a rule which makes the deletion or retention of a_n depend only on n and a_1, a_2, \dots, a_{n-1} , and if $g(r)$ is the number of 1's among the first r terms of a_{n_1}, a_{n_2}, \dots , then $g(r)/r$ approaches the same limit p as r approaches infinity.

The inclusion of condition (2) corresponds to the *Prinzip vom ausgeschlossenen Spielsystem* of von Mises.⁵ If a fixed number of wagers of "heads" are to be made, at fixed odds and in fixed amount, on the tosses of a coin, no advantage is gained in the long run if the player, instead of betting at random, follows some system, such as betting on every seventh toss, or (more plausibly) betting on the next toss after the appearance of four tails in succession, or (still more plausibly) making his n th bet after the appearance of $n+4$ tails in succession. This is accepted by von Mises as a sufficiently familiar and uncontroverted empirical generalization to be made fundamental to his theory in this way.

However, this definition, as given by von Mises or as rephrased above, while clear as to general intent, is too inexact in form to serve satisfactorily as the basis of a mathematical theory.

A plausible attempt to state the definition more exactly is the following (the numbers b_i serve as a convenient device to represent a function of a variable number of variables as a function of one variable):

An infinite sequence a_1, a_2, \dots of 0's and 1's is a random sequence if the two following conditions are satisfied:

(1) If $f(r)$ is the number of 1's among the first r terms of a_1, a_2, \dots , then $f(r)/r$ approaches a limit p as r approaches infinity.

(2) If ϕ is any function of positive integers, if $b_1 = 1$, $b_{n+1} = 2b_n + a_n$, $c_n = \phi(b_n)$, and the integers n such that $c_n = 1$ form in order of magnitude an infinite sequence n_1, n_2, \dots , and if $g(r)$ is the number of 1's among the first r terms of a_{n_1}, a_{n_2}, \dots , then $g(r)/r$ approaches the same limit p as r approaches infinity.

However it has been pointed out by various authors⁶ that the defi-

⁵ Loc. cit.

⁶ Erhard Tornier, *Wahrscheinlichkeitsrechnung und Zahlentheorie*, Journal für die reine und angewandte Mathematik, vol. 160 (1929), pp. 177-198; Hans Reichenbach, *Axiomatik der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift, vol. 34

dition in this form is self-contradictory, in the sense that it makes the class of random sequences associated⁷ with any probability p other than 0 or 1 an empty class. For the failure of (2) may always be shown by taking $\phi(x) = a_{\mu(x)}$, where $\mu(x)$ is the least positive integer m such that $2^m > x$: the sequence a_{n_1}, a_{n_2}, \dots will then consist of those and only those terms of a_1, a_2, \dots which are 1's. This means that the definition in this form does not satisfactorily represent the requirement that the deletion or retention of a_n shall not depend on a_n or on the sequence a_1, a_2, \dots as a whole, but only on n and a_1, a_2, \dots, a_{n-1} . Grave question is raised whether this requirement, made in vague terms by von Mises, can be satisfactorily represented in an exact definition at all.

This difficulty may be avoided by abandoning the attempt to define a random sequence and substituting some less restricted class of sequences, such as the admissible numbers of Copeland⁸ or the equivalent normal sequences of Reichenbach.⁹

These admissible numbers (to adopt Copeland's term) are closely related to the normal numbers of Borel¹⁰—indeed an admissible number associated with the probability $1/2$ is the same as a number *entièrement normal* to the base 2. The definition may be stated as follows: An infinite sequence a_1, a_2, \dots of 0's and 1's is an admissible number if it is associated with a probability p and if, for every positive integer m and every set of distinct positive integers r_1, r_2, \dots, r_k which are all less than or equal to m , the sequence whose n th term is the product $a_{nm+r_1} a_{nm+r_2} \dots a_{nm+r_k}$ is associated with the probability¹¹ p^k .

(1931–1932), pp. 568–619; E. Kamke, *Über neuere Begründungen der Wahrscheinlichkeitsrechnung*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 42 (1932–1933), pp. 14–27; Arthur H. Copeland, *Point set theory applied to the random selection of the digits of an admissible number*, American Journal of Mathematics, vol. 58 (1936), pp. 181–192.

⁷ An infinite sequence of 0's and 1's will be said to be associated with the probability p if $f(r)/r$ approaches p as r approaches infinity, where $f(r)$ is the number of 1's among the first r terms of the sequence.

⁸ Arthur H. Copeland, *Admissible numbers in the theory of probability*, American Journal of Mathematics, vol. 50 (1928), pp. 535–552. The infinite sequences of 0's and 1's are there taken as binary fractional expansions of real numbers between zero and one.

⁹ Loc. cit. (1931–1932). See also Hans Reichenbach, *Les fondements logiques du calcul des probabilités*, Annales de l'Institut Henri Poincaré, vol. 7 (1937), pp. 267–348.

¹⁰ Émile Borel, *Les probabilités dénombrables et leurs applications arithmétiques*, Rendiconti del Circolo Matematico di Palermo, vol. 27 (1909), pp. 247–271, reprinted as Note V to Borel's *Leçons sur la Théorie des Fonctions*, 2d edition, 1914, and 3d edition, 1928.

¹¹ Copeland imposes the further conditions $p \neq 0$, $p \neq 1$.

The admissible numbers have properties which are sufficient to form a basis for a large part of the theory of probability, and they have the important advantage that their existence, for any assigned probability p , can be proved.¹² Their use for this purpose, however, is open to certain objections from the point of view of completeness of the theory, as has been forcibly urged by von Mises,¹³ and it is therefore desirable to consider further the question of finding a satisfactory form for the definition of a random sequence.

The purpose of the present note is to call attention to the following possibility in this connection.

It may be held that the representation of a *Spielsystem* by an arbitrary function ϕ is too broad. To a player who would beat the wheel at roulette a system is unusable which corresponds to a mathematical function known to exist but not given by explicit definition; and even the explicit definition is of no use unless it provides a means of calculating the particular values of the function. As a less frivolous example, the scientist concerned with making predictions or probable predictions of some phenomenon must employ an effectively calculable function: if the law of the phenomenon is not approximable by such a function, prediction is impossible. Thus a *Spielsystem* should be represented mathematically, not as a function, or even as a definition of a function, but as an effective algorithm for the calculation of the values of a function.

Now a formal definition of effective calculability, for functions of positive integers, has been proposed by the author,¹⁴ and the adequacy of this definition to represent the empirical notion of an effective calculation finds strong support in a recent result of Turing.¹⁵

¹² Copeland, loc. cit. (1928). See also Borel, loc. cit. (1909).

¹³ Mathematische Annalen, vol. 108 (1933), pp. 771–772; *Wahrscheinlichkeit, Statistik und Wahrheit*, 2d edition, pp. 116–117.

¹⁴ Alonzo Church, *An unsolvable problem of elementary number theory*, American Journal of Mathematics, vol. 58 (1936), pp. 345–363. A function of positive integers is defined to be effectively calculable if it has either of the two equivalent properties of λ -definability (in the sense of Church-Kleene) or general recursiveness (in the sense of Herbrand-Gödel). Cf. S. C. Kleene, *λ -definability and recursiveness*, Duke Mathematical Journal, vol. 2 (1936), pp. 340–353, and *General recursive functions of natural numbers*, Mathematische Annalen, vol. 112 (1936), pp. 727–742.

¹⁵ A. M. Turing, *On computable numbers, with an application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society, (2), vol. 42 (1936–1937), pp. 230–265; *A correction*, ibid., vol. 43 (1937), pp. 544–546; and *Computability and λ -definability*, Journal of Symbolic Logic, vol. 2 (1937), pp. 153–163. Turing proves the equivalence of λ -definability and general recursiveness to a notion of computability whose definition, briefly stated, is as follows: A function ϕ is computable if it is possible to make a computing machine, with a finite number of parts of finite size, which will calculate $\phi(n)$ for any assigned n , printing intermediate calculations

It is therefore suggested that this definition of effective calculability be employed in order to define a random sequence as follows:

An infinite sequence a_1, a_2, \dots of 0's and 1's is a random sequence if the two following conditions are satisfied:

(1) If $f(r)$ is the number of 1's among the first r terms of a_1, a_2, \dots , then $f(r)/r$ approaches a limit p as r approaches infinity.

(2) If ϕ is any effectively calculable function of positive integers, if $b_1 = 1, b_{n+1} = 2b_n + a_n, c_n = \phi(b_n)$, and the integers n such that $c_n = 1$ form in order of magnitude an infinite sequence n_1, n_2, \dots , and if $g(r)$ is the number of 1's among the first r terms of a_{n_1}, a_{n_2}, \dots , then $g(r)/r$ approaches the same limit p as r approaches infinity.

The existence of random sequences in this sense is an immediate consequence of a result of Doob,¹⁶ or alternatively of a theorem of Wald,¹⁷ if use is made of the fact that the set of effectively calculable functions can be represented as a subset of an effectively enumerable set and is therefore itself (noneffectively) enumerable. From Doob's theorem, taken in conjunction with Borel's result¹⁸ that the infinite sequences of 0's and 1's associated with the probability 1/2 (regarded as binary fractional expansions of real numbers between zero and one) form a set of measure one, it follows that the random sequences associated with the probability 1/2 (similarly regarded) form a set of measure one; and from the existence of random sequences associated with the probability 1/2 the existence of random sequences associated with other probabilities is readily derived. From Wald's Theorem I it follows as a corollary that the set of random sequences associated with a fixed probability has the power of the continuum.

That every random sequence is an admissible number is also easily demonstrated. On the other hand, the set of random sequences is more restricted than the set of admissible numbers; this follows, for example, from the existence of admissible numbers a_1, a_2, \dots such that

and the final result on a tape with which the machine must be supplied (no upper limit is placed on the time or on the length of tape required for a particular calculation). Actually, Turing imposes several further conditions on the computing machine, but these are more or less clearly nonessential.

¹⁶ J. L. Doob, *Note on probability*, *Annals of Mathematics*, (2), vol. 37 (1936), pp. 363–367. The author is indebted to A. H. Copeland for calling his attention to the significance of Doob's theorem in this connection, as well as to the matter of effective constructibility of admissible numbers (footnote 19), and for other suggestions.

¹⁷ Abraham Wald, *Sur la notion de collectif dans le calcul des probabilités*, *Comptes Rendus des Séances de l'Académie des Sciences*, vol. 202 (1936), pp. 180–183, and *Die Widerspruchsfreiheit des Kollektivbegriffes der Wahrscheinlichkeitsrechnung*, *Ergebnisse eines mathematischen Kolloquiums*, vol. 8 (1937), pp. 38–72.

¹⁸ *Loc. cit.* (1909).

a_n is an effectively calculable function of n , a property¹⁹ which clearly cannot be possessed by any random sequence.

Thus an existence proof for random sequences is necessarily non-constructive.

Use of the above proposed definition of a random sequence as fundamental to the theory of probability is consequently open to the objection that by its means such otherwise apparently combinatorial matters as elementary questions of probability in connection with the tossing of a coin are made to depend on the powerful (and dubious) non-constructive methods of analysis. It is clear, however, that any definition of a random sequence more stringent than this one would have the same disadvantage, and on the other hand that no definition in any respect less stringent could be regarded as even approximately representing von Mises's intention or as being free from such objections as those brought by him against the use of admissible numbers or normal sequences.

Nevertheless it would seem to be of interest to investigate criteria of randomness of intermediate strength, in particular the definition of a random sequence which results if the condition that ϕ be effectively calculable is replaced by the condition that ϕ be primitive recursive. Since the primitive recursive functions are effectively enumerable, sequences satisfying this criterion can be effectively constructed in accordance with Wald's Theorem V.²⁰

PRINCETON UNIVERSITY

¹⁹ The example of von Mises, *Mathematische Annalen*, vol. 108 (1933), p. 769, is readily specialized so as to have this property. A very neat effective construction of an admissible number is implied in a paper of D. G. Champernowne, *The construction of decimals normal in the scale of ten*, *Journal of the London Mathematical Society*, vol. 8 (1933), pp. 254-260.

²⁰ *Loc. cit.* (1937). Wald relies on the common notion of effectiveness and has no exact definition. His proof is entirely applicable here. Wald also remarks on a criterion of randomness—in general more stringent than that proposed in the present paper—which consists, in effect, in replacing the condition that ϕ be effectively calculable by the condition that ϕ be definable within a fixed system of symbolic logic L . There are, however, several objections to this criterion. It is unavoidably relative to the choice of the particular system L and thus has an element of arbitrariness which is artificial. If used within the system L , it requires the presence in L of the semantical relation of *denotation* (known to be problematical on account of the Richard paradox). If it is used outside of L , it becomes necessary to say more exactly what is meant by "definable in L ," and the questions of consistency and completeness of L are likely to be raised in a peculiarly uncomfortable way.