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Inferring 3D Structure from Image Motion: The Constraint of Poinsot Motion*

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Abstract. Monocular observers perceive as three-dimensional (3D) many displays that depict three points rotating rigidly in space but rotating about an axis that is itself tumbling. No theory of structure from motion currently available can account for this ability. We propose a formal theory for this ability based on the constraint of Poinsot motion, i.e., rigid motion with constant angular momentum. In particular, we prove that three (or more) views of three (or more) points are sufficient to decide if the motion of the points conserves angular momentum and, if it does, to compute a unique 3D interpretation. Our proof relies on an upper semicontinuity theorem for finite morphisms of algebraic varieties. We discuss some psychophysical implications of the theory.

Key words. structure from motion, algebraic geometry, upper semicontinuity, observer theory, Poinsot motion, visual motion

1 Introduction

Monocular observers can perceive three-dimensional (3D) structures and motions in dynamic two-dimensional (2D) displays. This ability has generated a substantial body of literature, both theoretical [1]–[17] and experimental [18]–[25]. Yet it appears that no theory so far proposed can account for our perception of certain simple displays. These displays depict three points moving rigidly in space about an axis that is itself rotating in space. Such, for example, would be the motion of the three points were they attached to a precessing top. Detailed psychophysical studies of these displays remain to be done, but the verdict of casual observation is clear: one sees the points in three dimensions, rotating rigidly about a tumbling axis.

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A well-known theorem of Ullman and Fremlin [15] cannot explain this percept because the theorem requires three orthographic views of four noncoplanar points, whereas these displays have but three points. A theorem by Hoffman and Bennett [8] also cannot explain this percept because the theorem, although it needs but three orthographic views of three points, requires that the points rotate rigidly about a single fixed axis, whereas these displays exhibit tumbling motion. Other theoretical accounts fail on similar grounds: they require too many points or else require the points to move in ways less general than the motions actually depicted (and perceived) in these displays.

This circumstance led us to consider further constraints or assumptions that human vision might employ to interpret visual motion. A promising constraint, and the one we study here, is a constraint from classical mechanics: a freely moving rigid body, a body subject to no net torque, moves in such a manner that its angular-momentum vector remains constant [26], [27]. The behavior of such a rigid body, described by Poinsot [28] in 1834, is called *Poinsot motion.*

To keep our discussion reasonably self-contained we briefly review some relevant classical mechanics. We then use the constraint of Poinsot motion, together with an upper semi-continuity theorem from algebraic geometry, to prove a theorem about the inference of 3D structure from image motion. Finally we discuss some psychophysical implications.

For efficiency in locating numbered items, we number all theorems, lemmas, propositions, remarks, and displayed equations in a single sequence.

2 Conservation of Angular Momentum

In this section we briefly review the mechanics of a rigid body in motion. More details can be found in standard texts [26], [27].

Consider a rigid body made up of N points that have masses m_i and positions $\mathbf{r}_i$ with respect to an origin O. If the instantaneous angular velocity of the body is $\mathbf{w}$, then the instantaneous linear velocity $\mathbf{v}_i$ of each point mass is

$$\mathbf{v}_i = \mathbf{w} \times \mathbf{r}_i, \tag{1}$$

where $\times$ denotes the cross product of vectors. The angular momentum $\mathbf{L}$ of the body about O is then the sum of the angular momenta of the point masses:

$$\mathbf{L} = \sum_{i=1}^{N} m_i \mathbf{r}_i \times \mathbf{v}_i, \tag{2}$$

which by (1) can be written

$$\mathbf{L} = \sum_{i=1}^{N} m_i \mathbf{r}_i \times (\mathbf{w} \times \mathbf{r}_i) = \mathbf{I}\mathbf{w}, \tag{3}$$

where we view $\mathbf{I}$ as a symmetric rank-2 tensor (or a symmetric operator) that depends on the $\mathbf{r}_i$ and m_i. It is called the *inertia tensor* of the body. We can represent $\mathbf{I}$ as a matrix as follows. Recalling that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}),$$

where $\cdot$ denotes the dot product of vectors, we can rewrite (3) as

$$\mathbf{I}\mathbf{w} = \sum_{i=1}^{N} \left[m_i r_i^2 \mathbf{w} - m_i \mathbf{r}_i (\mathbf{r}_i \cdot \mathbf{w}) \right]. \tag{4}$$

Here r_i denotes the length of the vector $\mathbf{r}_i$. Thus $\mathbf{I}$, written as an operator, is

$$\mathbf{I} = \sum_{i=1}^{N} \left(m_i r_i^2 \mathbf{1} - m_i \mathbf{r}_i \mathbf{r}_i^* \right), \tag{5}$$

where $\mathbf{1}$ is the identity operator and $\mathbf{r}_i^*$ is the linear functional that takes dot product with $\mathbf{r}_i$. If we write $\mathbf{r}_i$ in terms of components as $\mathbf{r}_i = (x_i, y_i, z_i)$, then the corresponding matrix expression for $\mathbf{I}$ is

$$\mathbf{I} = \begin{pmatrix} \sum_{i=1}^N m_i(y_i^2 + z_i^2) & -\sum_{i=1}^N m_i x_i y_i & -\sum_{i=1}^N m_i x_i z_i \\ -\sum_{i=1}^N m_i x_i y_i & \sum_{i=1}^N m_i(x_i^2 + z_i^2) & -\sum_{i=1}^N m_i y_i z_i \\ -\sum_{i=1}^N m_i x_i z_i & -\sum_{i=1}^N m_i y_i z_i & \sum_{i=1}^N m_i(x_i^2 + y_i^2) \end{pmatrix}. \tag{6}$$

If we rotate our coordinate system by some rotation matrix A, then the components of the inertia tensor change by a similarity transformation:

$$\mathbf{I}' = A\mathbf{I}A^{\mathrm{T}}, \tag{7}$$

where A^{T} denotes the transpose of A.

PROPOSITION 8. If $N = 2$, then $\mathbf{r}_1$ and $\mathbf{r}_2$ are linearly independent if and only if the matrix $\mathbf{I}$ is nonsingular.

Proof. In view of (7), it suffices to prove the proposition after application of an arbitrary rotation A. Moreover, by multiplying by $1/m_1$, we may assume $m_1 = 1$. Thus we may take $\mathbf{r}_1 = (1, 0, 0)$ and $\mathbf{r}_2 = (a, b, 0)$ for some a, b. Let m denote the mass at $\mathbf{r}_2$. We then obtain

$$\mathbf{I} = \begin{pmatrix} mb^2 & -mab & 0 \\ -mab & ma^2 + 1 & 0 \\ 0 & 0 & m(a^2 + b^2) + 1 \end{pmatrix}.$$

Hence

$$\det \mathbf{I} = 0 \Leftrightarrow \det \begin{pmatrix} mb^2 & -mab \\ -mab & ma^2 + 1 \end{pmatrix} = 0,$$

which holds when $mb^2 = 0$, i.e., when $\mathbf{r}_2 = a\mathbf{r}_1$.

The eigenvectors of the inertia tensor are called the *principal axes of the body*. Any axis of symmetry of a body is a principal axis, and any plane of symmetry of a body is perpendicular to a principal axis. Each inertia tensor $\mathbf{I}$ has a uniquely associated inertia ellipsoid with equation

$$\mathbf{rIr} = 1. \tag{9}$$

The principal axes of the inertia tensor and of its associated elllipsoid are coincident. We discuss next the dynamical import of the principal axes.

According to classical mechanics, the behavior of the angular momentum is governed by the law

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}, \tag{10}$$

where $\mathbf{N}$ denotes the total torque about the point O. If $\mathbf{N}= 0$, i.e., if the body is subject to no net torque about the origin of the coordinate system, then it undergoes Poinsot motion: the angular velocity vector traces out a *polhode* on the inertia ellipsoid [26]–[32]. There are several cases. If the body has an axis of symmetry, then its general motion is easily described: the angular-velocity vector $\mathbf{w}$ precesses in a circle of fixed radius about the axis of symmetry. If $\mathbf{w}$ is parallel to the axis of symmetry, then the motion is rotation with constant angular speed about a fixed axis, viz., the symmetry axis. Indeed, a necessary condition for $\mathbf{w}$ to be constant is that it be directed along a principal axis of the inertia tensor. Such fixed-axis motion is stable if $\mathbf{w}$ lies along an axis of maximum or minimum moment of inertia, but it is unstable if $\mathbf{w}$ lies along the axis of intermediate moment of inertia. If the body is not symmetric about any axis, then the motion of $\mathbf{w}$ is more complex: the polhodes are fourth-order curves on the inertia ellipsoid (examples can be found in [29]–[32]).

We wish to investigate the constraint of constant angular momentum, a constraint that is formulated only for the case of continuous motion. In this case we can find, for any pair of times t_0 and $t + t_0$, an element of $SO(3, \mathbf{R})$ that rotates the object from its position at time t_0 to its position at time $t + t_0$. This element of $SO(3, \mathbf{R})$ can be represented by a 3×3 matrix. Its eigenvectors of eigenvalue unity and its trace give, respectively, a canonical axis and a canonical angle of rotation about this axis, which rotate the object from its position at time t_0 to its position at time $t + t_0$. In general, of course, an object does not undergo strictly fixed-axis motion during an interval t. In this case the canonical axis and angle represent a weighted time average of the instantaneous angular velocities of the body during the interval t. Any rotation, whether finite or infinitesimal, has a canonical axis and angle associated with it. If the angle of rotation is vanishingly small, the canonical axis and angle correspond (up to first order in t) to the actual angular velocity at that instant of time.

In this article we are interested in the discrete-time version of rigid motion with constant angular momentum. As above, we assume that our rigid body consists of N point masses, with $N \geq 3$. We are interested in the positions of these points in three dimensions at successive instants $\{t_j\}$ of time, instants separated by time intervals that are of equal length and that are small relative to the rate of motion of the body. We must formulate the constant-angular-momentum condition in this setting. For this purpose we first define (discrete-time) angular-velocity vectors for each interval, i.e., for each pair of successive positions of the body points. We use a 3D coordinate system in which one given point of the body remains fixed at the origin. This is a noninertial system and hence is dependent on the chosen point. The discrete-time angular momentum will be calculated with this point as the origin of our coordinate system. Such a choice is justified up to first order in t if the point is not undergoing large angular accelerations in the time intervals. The motivation for this choice of coordinates is to "mod out" the average motion of the body in our calculations. By "foveating" one of the N points of the body we are precisely eliminating the translation of this point. If we were to foveate the center of mass, then we would eliminate all translation and be left with pure rotation. However, our approach does not, in general, result in the center of mass being foveated; in fact, it is impossible, in general, to find the center of mass of three points over three views. In this coordinate system having one point fixed at the origin there is a rotation M_j of $\mathbf{R}^3$ (i.e., an element of SO(3, $\mathbf{R}$)) that carries the positions of the points of the body at time t_j to their positions at time t_{j+1}. If the degenerate case for which M_j is the identity is excluded, there will be a unique line l_j through the origin such that M_j is a rotation about l_j through some angle θ_j. Note that there are countably many choices for θ_j that differ by integer multiples of 2π. The vectors $\mathbf{w}_j$ on the line l_j satisfy $M_j\mathbf{w}_j = \mathbf{w}_j$. Our (discrete-time) angular-velocity vector for the time interval $[t_j, t_{j+1}]$ will be such a $\mathbf{w}_j$, subject to the additional condition that $|\mathbf{w}_j| = \sin\theta_j$, where θ_j is a choice of angle of the rotation. Note that for small angles θ_j the number $\sin\theta_j$ is very close to θ_j (where θ_j is measured in radians). The advantage of choosing $\sin\theta_j$ instead of θ_j itself for the length of $\mathbf{w}_j$ is that this choice can be expressed by an *algebraic* equation (see below). In addition, using $\sin\theta_j$ instead of θ_j reduces the ambiguity in θ, so that now $\mathbf{w}_j$ has but a twofold ambiguity, corresponding to choice of orientation. We will see below that this remaining ambiguity does not affect our result.

To summarize, our discrete-time angular-velocity vector $\mathbf{w}_j$ for the motion from time t_j to time t_{j+1} is specified by the equations

$$M_j\mathbf{w}_j = \mathbf{w}_j, \tag{11a}$$

$$|\mathbf{w}_j| = \sin\theta_j. \tag{11b}$$

At each time t_j we define the inertia tensor $\mathbf{I}_j$ by using (7), in which we substitute the coordinates of our N points at time t_j. (Here we are assuming that the inertia tensor does not change significantly over the time interval.) We then define the (discrete-time) angular momentum vector for the time interval $[t_j, t_{j+1}]$ to be $\mathbf{I}_j\mathbf{w}_j$. The constant (discrete-time) angular-momentum constraint may then be written

$$\mathbf{I}_j\mathbf{w}_j = \mathbf{I}_{j+1}\mathbf{w}_{j+1} \quad \forall j. \tag{12}$$

It should be stressed that this is only a discrete-time angular momentum; its construction was motivated by conservation of its continuous-time analog. However, the question of how one version relates to the other is as yet unresolved. The most we can conclude is that our discrete version of the angular momentum is in some sense the time average of the real angular momentum in each interval. Given such sparse data, this is the best approximation that we can make in the sense that other reasonable definitions of discrete-time angular momentum yield approximations to the continuous-time angular momentum that are no better than ours.

The motivation for our definition of discrete-time angular momentum is the following. Recall the definition of angular momentum in continuous time:

$$L(t) = \mathbf{I}(t)\omega(t),$$

where $\mathbf{I}(t)$ is the inertia tensor at time t and $\omega(t)$ is the instantaneous angular velocity at time t. We construct ω explicitly from the Lie derivative of the family of orthogonal rotations corresponding to the motion of the rigid object. Let $O(t)$ denote the matrix in SO(3, $\mathbf{R}$) representing rotation of a rigid object. (To each rigid object we can associate an orthogonal coordinate system fixed in the body. O is the rotation between this *body system* and the inertial system that we take to be the body system at time t_0.) Then $\omega(t_0)$ is the vector such that

$$O^{-1}(t_0)\left.\frac{dO}{dt}\right|_{t_0}\mathbf{r} = \omega(t_0)\times\mathbf{r}.$$

Here $O^{-1}(t_0)\,(dO/dt)\,|_{t_0}$ is in the Lie algebra so (3, $\mathbf{R}$). Now every rotation in SO(3, $\mathbf{R}$) is equivalent to a rotation about a fixed axis, i.e., every $A \neq I \in$ SO(3, $\mathbf{R}$) has a unique fixed direction $\hat{\mathbf{w}}$ such that $A\hat{\mathbf{w}} = \hat{\mathbf{w}}$, where the hats indicate normalization to a unit vector and I is the identity rotation. This is true for both finite and infinitesimal rotations. In the case of fixed-axis motion the matrices $O(t)$ are similar to

$$\begin{pmatrix} \cos\theta(t) & \sin\theta(t) & 0 \\ -\sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for the appropriate choice of coordinates. Notice that $d\theta(t)/dt = |\omega(t)|$. Intuitively, for sufficiently small intervals of time the body is undergoing *almost* fixed-axis motion, so that in the limit as $t \to t_0$ it is also true that $\hat{\mathbf{w}} \to \hat{\omega}(t_0)$. We can see this also from the definition of the Lie derivative at t_0. Let $\hat{\mathbf{w}}(t)$ be the fixed direction of $O(t)$ for t near t_0. Then

$$\left.\frac{dO}{dt}\right|_{t_0} = \lim_{t \to t_0} \frac{O(t) - I}{t - t_0}$$

and

$$\left.\frac{dO}{dt}\right|_{t_0} \omega(t_0) = \lim_{t \to t_0} \frac{O(t)\hat{\mathbf{w}}(t) - I\hat{\mathbf{w}}(t)}{t - t_0} = 0 \quad \forall t,$$

so that $\hat{\mathbf{w}}(t) \to \hat{\omega}(t_0)$. If we let

$$|\mathbf{w}(t)| = \frac{1}{t - t_0} \cos^{-1}\left(\frac{\operatorname{Tr} O(t) - 1}{2}\right)$$

and recall that the trace (denoted Tr) of a matrix is invariant under similarity transformations, then we obtain

$$\lim_{t \to t_0} |\mathbf{w}(t)| = |\omega(t_0)|.$$

3 Inferring 3D Structure

We are interested in the use of the Poinsot constraint for the purpose of inferring the depth (z) coordinates of moving points from their (orthographic) projections on the image (x, y) plane. Mathematically, this amounts to plugging in the x and y coordinates (in the system of equations consisting of (12) together with equations expressing rigidity of motion) and eliminating the $\mathbf{w}_j$'s, thereby obtaining values for the unknown z's. This can be done, with effort, for particular numerical values of the (x, y)'s. However, because of the complexity of the equations, it is very difficult to extract closed-form expressions for the z's in terms of the (x, y)'s. Our desired result, however, asserts that 1) when there are solutions, there are generically exactly two, and 2) for generic (x, y) data there are no solutions. How can we hope to obtain such results in the absence of some closed-form expression for the z's? The key is to exploit the algebraicity of the system of equations: for the first assertion we use results from algebraic geometry that state that under suitable conditions the number of solutions to a system of polynomial equations, equations depending on certain parameters, is an upper semicontinuous function of those parameters in a very strong sense. This technique enables us, in effect, to obtain the desired results simply by checking several test points. For the second assertion we show that the dimension of the solution set of our equations is suitably small. For more details about the algebro-geometric terminology and techniques the reader is referred to the appendix.

Here is our main result:

THEOREM 13. Suppose that three (or more) unit point masses move in space. Moreover suppose that three (or more) distinct images of the point masses are obtained, at equally spaced intervals of time, by using orthographic projection. Then the following two statements are true:

1. *Uniqueness.* If the point masses move rigidly in space and conserve discrete-time angular momentum, i.e., they satisfy equation (12) above, then the images are compatible, generically, with precisely two 3D interpretations in which the point masses move rigidly and conserve angular momentum with respect to one of the points. The two interpretations are mirror reflections of each other about the imaging plane.
2. *Measure-zero-distinguished premises.* For generic motions of the point masses (e.g., nonrigid and nonconservative motions) generically chosen images are compatible with *no* 3D interpretations in which the point masses move rigidly and conserve angular momentum with respect to one of the points. This implies that false targets have Lebesgue measure zero.

Proof. We take one of the three points to be the origin O of a Cartesian coordinate system whose z axis is taken to be orthogonal to the imaging plane. (This is the coordinate system in which we will make our calculations of angular momentum.) We let $\mathbf{r}_{ij} = (x_{ij}, y_{ij}, z_{ij})$, where $i = 1, 2$ and $j = 1, 2, 3$, denote the position vector of point mass i in frame j relative to O. (We use the term *frame* to denote the 3D situation at an instant of time; the term *view* denotes a 2D image.) The constraint that the point masses move rigidly over the three frames leads to six equations (studied previously in [8]):

$$\mathbf{r}_{11} \cdot \mathbf{r}_{11} = \mathbf{r}_{12} \cdot \mathbf{r}_{12} = \mathbf{r}_{13} \cdot \mathbf{r}_{13}, \tag{14}$$

$$\mathbf{r}_{21} \cdot \mathbf{r}_{21} = \mathbf{r}_{22} \cdot \mathbf{r}_{22} = \mathbf{r}_{23} \cdot \mathbf{r}_{23}, \tag{15}$$

$$\mathbf{r}_{11} \cdot \mathbf{r}_{21} = \mathbf{r}_{12} \cdot \mathbf{r}_{22} = \mathbf{r}_{13} \cdot \mathbf{r}_{23}. \tag{16}$$

Equations (14) and (15) state that the lengths of the position vectors $\mathbf{r}_{ij}$ remain constant over frames, whereas (16) states that the angle between the two position vectors in each frame remains constant. In these equations the components x_{ij} and y_{ij} are known from the image data and the six components z_{ij} must be solved for. Equations (14)–(16) have, generically, 64 solutions. Hence for these equations alone false targets have full measure. Thus the role of the angular-momentum constraint (12) is, first, to reduce the number of solutions from 64 to two and, second, to make the measure of false targets zero.

If the vectors $\mathbf{r}_{ij}$ satisfy the rigidity constraints (14)–(16), then the successive frames are related by rotations. Hence there are (discrete-time) angular-velocity vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ associated, respectively, to the rotation from frame 1 to frame 2 and to the rotation from frame 2 to frame 3 (see equation (11)). According to our conventions, conservation of angular momentum in discrete time is then expressed by the equation

$$\mathbf{I}_1\mathbf{w}_1 = \mathbf{I}_2\mathbf{w}_2, \tag{17}$$

where $\mathbf{I}_j$ denotes the inertia tensor in frame j (see equations (3) ff). We will call (14)–(17) the *Poinsot constraint*. A collection of unit point masses whose motion satisfies the Poinsot constraint is a collection that moves rigidly and whose motion conserves (discrete-time) angular momentum.

The strategy of our proof is to show that the set of 6-tuples of vectors $\{(\mathbf{r}_{ij}), i = 1, 2; j = 1, 2, 3\}$ representing, as above, a body undergoing Poinsot motion is a variety $E_{\mathbf{C}}$ in $\mathbf{C}^{18}$ whose projection onto the space $\mathbf{C}^{12}$ of image data $\{(x_{ij}, y_{ij})\}$ is a so-called *finite morphism.* We are ultimately interested in the set E of points in $E_{\mathbf{C}}$ with real coordinates. The proof of the first assertion of our theorem then hinges on the application of the upper semicontinuity theorem for finite morphisms (see appendix, Theorem A5) to the finite morphism $E_{\mathbf{C}} \to \mathbf{C}^{12}$ and on its interpretation for E. The second assertion of the theorem is proved by showing that the dimension of E is suitably small. One immediate problem in handling $E_{\mathbf{C}}$ is that, of the equations (14)–(17) that define Poinsot motion, one of them – equation (17) – involves variables other than (x_{ij}, y_{ij}, z_{ij}). In fact, it involves $\mathbf{w}_1$ and $\mathbf{w}_2$. We are thus led to construct an appropriate space in whch $\mathbf{w}_1$ and $\mathbf{w}_2$ are well-defined functions, in order to obtain a solution variety for (14)–(17) in this space, and then to project this variety back into (x_{ij}, y_{ij}, z_{ij})-space, thereby obtaining $E_{\mathbf{C}}$. To ensure the finiteness of the morphism $E_{\mathbf{C}} \to \mathbf{C}^{12}$ we will need to keep good control over the various algebraic aspects of this construction.

The following proof is organized into 10 steps, labeled A through J. Each step begins with a less technical discussion of what is to be accomplished in that step and then proceeds with the technical details. For those interested in following the proof in detail, this organization should help to see its logical structure. For those not interested in following the proof in detail, the less technical discussions at the start of each step should give the general idea and intuitive meaning of the proof.

Step A. Our first task is to analyze (14)–(16). In terms of components (14)–(16) may be written

$$f_1(z_{ij}) = z_{11}^2 - z_{12}^2 + c_1 = 0, \tag{18}$$

$$f_2(z_{ij}) = z_{11}^2 - z_{13}^2 + c_2 = 0, \tag{19}$$

$$f_3(z_{ij}) = z_{21}^2 - z_{22}^2 + c_3 = 0, \tag{20}$$

$$f_4(z_{ij}) = z_{21}^2 - z_{23}^2 + c_4 = 0, \tag{21}$$

$$f_5(z_{ij}) = z_{11}z_{21} - z_{12}z_{22} + c_5 = 0, \tag{22}$$

$$f_6(z_{ij}) = z_{11}z_{21} - z_{13}z_{23} + c_6 = 0, \tag{23}$$

where

$$c_1 = x_{11}^2 + y_{11}^2 - x_{12}^2 - y_{12}^2, \tag{24}$$

$$c_2 = x_{11}^2 + y_{11}^2 - x_{13}^2 - y_{13}^2, \tag{25}$$

$$c_3 = x_{21}^2 + y_{21}^2 - x_{22}^2 - y_{22}^2, \tag{26}$$

$$c_4 = x_{21}^2 + y_{21}^2 - x_{23}^2 - y_{23}^2, \tag{27}$$

$$\begin{array}{ccccl} R_{\mathbf{C}} & \subset & X_{\mathbf{C}} & = & \mathbf{C}^{18} \quad \{(x_{ij}, y_{ij}, z_{ij})\} \\ & \pi\downarrow & & & \downarrow\pi \\ & & Y_{\mathbf{C}} & = & \mathbf{C}^{12} \quad \{(x_{ij}, y_{ij})\}, \qquad i = 1,2; j = 1,2,3 \end{array}$$

Fig. 1. Structural setting for $R_{\mathbf{C}}$, which is defined by equations (18)–(23).

$$c_5 = x_{11}x_{21} + y_{11}y_{21} - x_{12}x_{22} - y_{12}y_{22}, \quad (28)$$
$$c_6 = x_{11}x_{21} + y_{11}y_{21} - x_{13}x_{23} - y_{13}y_{23}, \quad (29)$$

Equations (18)–(23) can be regarded as defining an affine variety $R_{\mathbf{C}}$ (for rigidity) in a complex affine space $X_{\mathbf{C}} = \{(x_{ij}, y_{ij}, z_{ij}) \mid i = 1, 2; j = 1, 2, 3\} = \mathbf{C}^{18}$. Since we are given the (x_{ij}, y_{ij}) from the images, we can view the (x_{ij}, y_{ij}) as parameters in these equations. The space of all possible parameters is a complex affine space $Y_{\mathbf{C}} = \{(x_{ij}, y_{ij}) \mid i = 1, 2; j = 1, 2, 3\} = \mathbf{C}^{12}$. $X_{\mathbf{C}}$ and $Y_{\mathbf{C}}$ are related by a morphism $\pi : X_{\mathbf{C}} \to Y_{\mathbf{C}}$ given by $(x_{ij}, y_{ij}, z_{ij}) \mapsto (x_{ij}, y_{ij})$. For each parameter point $y \in Y_{\mathbf{C}}$ the set $\pi^{-1}(\{y\})$ is a six-dimensional (6D) complex affine space with coordinates z_{ij}. Figure 1 displays the various spaces and maps.

Step B. We now want to show that for generic choices of the x_{ij} and y_{ij}, i.e., for generic choices of the constants c_i, that (a) equations (18)–(23) have only finitely many solutions for the unknown z_{ij}'s and (b) these equations have no additional solutions at infinity when we view the 6D space of possible z_{ij}'s as being embedded in a 6D projective space. In the terminology of algebraic geometry, this means that if we restrict the map $\pi : R_{\mathbf{C}} \to Y_{\mathbf{C}}$ (mentioned in the previous paragraph) away from a nongeneric measure-zero subset of $Y_{\mathbf{C}}$, then the result is a so-called finite morphism. (A technical definition of finite morphism is given in the appendix.) The next few paragraphs are devoted entirely to proving that π is a finite morphism. The reader not interested in the details of this proof can now skip to step C.

For each choice of parameters $y = \{(x_{ij}, y_{ij})\} \in Y_{\mathbf{C}}$ we can view the complex affine space $\pi^{-1}(\{y\}) = \mathbf{C}^6$ as an affine open subset of complex projective space $\mathbf{P}^6(\mathbf{C})$. In this sense $\mathbf{P}^6(\mathbf{C}) = \mathbf{C}^6 \cup \mathbf{P}^5(\mathbf{C})$, where we call the projective space $\mathbf{P}^5(\mathbf{C})$ the *points at infinity* relative to our original affine space $\mathbf{C}^6$. Algebraically, this is expressed in coordinates as follows: As a system of homogeneous coordinates on $\mathbf{P}^6(\mathbf{C})$ we take $\{\{Z_{ij}\}, T\}$ and let

$$z_{ij} = \frac{Z_{ij}}{T}. \quad (30)$$

The space $\mathbf{P}^5(\mathbf{C})$ at infinity is then the locus $T = 0$ in $\mathbf{P}^6(\mathbf{C})$. Its homogeneous coordinates are the $\{Z_{ij}\}$. (For more details a good first reference is Fulton [35].) Let $\overline{R_{\mathbf{C}}}$ denote the closure of $R_{\mathbf{C}}$ in $\mathbf{P}^6(\mathbf{C}) \times Y_{\mathbf{C}}$. $\overline{R_{\mathbf{C}}}$ is defined by a collection of homogeneous polynomials in $\{\{Z_{ij}\}, T\}$ that may be obtained from the polynomials $f_1, \ldots, f_6$ in $\{z_{ij}\}$ that define $R_{\mathbf{C}}$ in $\mathbf{C}^6 \times Y_{\mathbf{C}}$. To do this we first take the ideal I generated by $f_1, \ldots, f_6$ in the polynomial ring $\mathbf{C}[x_{ij}, y_{ij}, z_{ij}]$. ($I$ is the set of all polynomials that can be written in the form $g_1 f_1 + \cdots + g_n f_n$ for some polynomials $g_1, \ldots, g_n$ in x_{ij}, y_{ij}, z_{ij}.) Now, for each f in I, view f as a polynomial in the $\{z_{ij}\}$ whose coefficients are polynomials in $\{(x_{ij}, y_{ij})\}$. Let $\deg_z f$ denote the degree of f with respect to the z variables only. Then, if $d = \deg_z f$, by using (30) we see that $F = T^d f$ is a homogeneous polynomial of degree d in $\{\{Z_{ij}\}, T\}$. The projective variety $\overline{R_{\mathbf{C}}}$ is the one defined by *all* such F's (i.e., by the F's that come by means of this procedure from *all* the f's in I).

We continue to use the notation π for the projection map $\mathbf{P}^6(\mathbf{C}) \times Y_{\mathbf{C}} \to Y_{\mathbf{C}}$. Define $R_y = R_{\mathbf{C}} \cap \pi^{-1}(\{y\})$, $\overline{R_y} = \overline{R_{\mathbf{C}}} \cap \pi^{-1}(\{y\})$. R_y and $\overline{R_y}$ have concrete descriptions as follows: A point $y \in Y_{\mathbf{C}}$ corresponds to particular numerical values (in $\mathbf{C}$) for $\{(x_{ij}, y_{ij})\}$. Thus, given any

$$
\begin{array}{ccccccc}
 & & & & X_{\mathbf{C}} & & \\
 & & & & \| & & \\
R_{\mathbf{C}} & \subset & \mathbf{C}^6 \times \mathbf{C}^{12} & = & \mathbf{C}^6 \times Y_{\mathbf{C}} & & \{x_{ij}, y_{ij}, z_{ij}\} \\
\cap & & \cap & & & & \\
\overline{R_{\mathbf{C}}} & \subset & \mathbf{P}^6(\mathbf{C}) \times \mathbf{C}^{12} & = & \mathbf{P}^6(\mathbf{C}) \times Y_{\mathbf{C}} & & \{x_{ij}, y_{ij}, Z_{ij}, T\} \\
 & & \downarrow \pi & & & & \\
 & & \mathbf{C}^{12} & = & Y_{\mathbf{C}} & & \{x_{ij}, y_{ij}\}
\end{array}
$$

Fig. 2. Structural setting for $R_{\mathbf{C}}$ and $\overline{R_{\mathbf{C}}}$.

polynomial f in the variables $\{(x_{ij}, y_{ij}, z_{ij})\}_{ij}$, for any $y \in Y$ we can evaluate $\{(x_{ij}, y_{ij})\}$ at y and obtain a polynomial in the z_{ij} only (with coefficients in $\mathbf{C}$). Denote this polynomial by f_y. With this notation R_y is the affine variety in $\mathbf{C}^6$ defined by $f_{1y}, \ldots, f_{6y}$. Similarly, $\overline{R_y}$ is the projective variety in $\mathbf{P}^6(\mathbf{C})$ defined by the F_y. Figure 2 displays the various spaces and maps.

We will now show that $R_y = \overline{R_y}$ (for y outside of a measure-zero subset of $Y_{\mathbf{C}}$), i.e., we will show that the system of homogeneous polynomials F described above has no solutions at infinity. To compute solutions at infinity we set $T = 0$ in each F and obtain a homogeneous polynomial $\widetilde{F}$ in $\{Z_{ij}\}$ only. From the way in which F is obtained from f it is clear that $\widetilde{F}$ is nothing other than the part of f that has the highest degree in the z_{ij}, except that we replace z_{ij} by Z_{ij}. To carry out our computation here we need to note that the ideal I generated by $f_1, \ldots, f_6$ contains, in particular, the polynomials

$$\begin{aligned} f_7 &= z_{21}^2 f_1 + z_{12}^2 f_3 - (z_{21}z_{11} + z_{12}z_{22}) f_5 \\ &= c_3 z_{12}^2 - c_5 z_{11} z_{21} + c_1 z_{21}^2 - c_5 z_{12} z_{22} \\ &= 0, \end{aligned} \tag{31}$$

$$\begin{aligned} f_8 &= z_{21}^2 f_2 + z_{13}^2 f_4 - (z_{11}z_{21} + z_{13}z_{23}) f_6 \\ &= c_4 z_{13}^2 - c_6 z_{11} z_{21} + c_2 z_{21}^2 - c_6 z_{13} z_{23} \\ &= 0. \end{aligned} \tag{32}$$

Associated to these $f_1, \ldots, f_8$ are the homogeneous polynomials $F_1, \ldots, F_8$:

$$F_1 = Z_{11}^2 - Z_{12}^2 + c_1 T^2 = 0, \tag{33}$$
$$F_2 = Z_{11}^2 - Z_{13}^2 + c_2 T^2 = 0, \tag{34}$$
$$F_3 = Z_{21}^2 - Z_{22}^2 + c_3 T^2 = 0, \tag{35}$$
$$F_4 = Z_{21}^2 - Z_{23}^2 + c_4 T^2 = 0, \tag{36}$$
$$F_5 = Z_{11} Z_{21} - Z_{12} Z_{22} + c_5 T^2 = 0, \tag{37}$$
$$F_6 = Z_{11} Z_{21} - Z_{13} Z_{23} + c_6 T^2 = 0, \tag{38}$$
$$\begin{aligned} F_7 &= c_3 Z_{12}^2 - c_5 Z_{11} Z_{21} + c_1 Z_{21}^2 - c_5 Z_{12} Z_{22} \\ &= 0, \end{aligned} \tag{39}$$
$$\begin{aligned} F_8 &= c_4 Z_{13}^2 - c_6 Z_{11} Z_{21} + c_2 Z_{21}^2 - c_6 Z_{13} Z_{23} \\ &= 0. \end{aligned} \tag{40}$$

Setting $T = 0$ in $F_1, \ldots, F_8$, we obtain

$$\widetilde{F}_1 = Z_{11}^2 - Z_{12}^2 = 0, \tag{41}$$
$$\widetilde{F}_2 = Z_{11}^2 - Z_{13}^2 = 0, \tag{42}$$
$$\widetilde{F}_3 = Z_{21}^2 - Z_{22}^2 = 0, \tag{43}$$
$$\widetilde{F}_4 = Z_{21}^2 - Z_{23}^2 = 0, \tag{44}$$
$$\widetilde{F}_5 = Z_{11} Z_{21} - Z_{12} Z_{22} = 0, \tag{45}$$
$$\widetilde{F}_6 = Z_{11} Z_{21} - Z_{13} Z_{23} = 0, \tag{46}$$
$$\begin{aligned} \widetilde{F}_7 &= c_3 Z_{12}^2 - c_5 Z_{11} Z_{21} + c_1 Z_{21}^2 - c_5 Z_{12} Z_{22} \\ &= 0, \end{aligned} \tag{47}$$
$$\begin{aligned} \widetilde{F}_8 &= c_4 Z_{13}^2 - c_6 Z_{11} Z_{21} + c_2 Z_{21}^2 - c_6 Z_{13} Z_{23} \\ &= 0. \end{aligned} \tag{48}$$

The solutions to $\tilde{F}_1, \ldots, \tilde{F}_8$ in $\mathbf{P}^5(\mathbf{C})$, with homogeneous coordinates $\{Z_{ij}\}$, are the points in $\overline{R_{\mathbf{C}}}$ at infinity. Observe that although f_7 and f_8 are in the ideal generated by $f_1, \ldots, f_6$ it is *not* the case that $\tilde{F}_7$ and $\tilde{F}_8$ are contained in the ideal generated by $\tilde{F}_1, \ldots, \tilde{F}_6$. For instance, any assignment of values to the Z_{ij} such that $Z_{11} = Z_{12} = Z_{13}$ and $Z_{21} = Z_{22} = Z_{23}$ are solutions to $\tilde{F}_1, \ldots, \tilde{F}_6$ but are not solutions to $\tilde{F}_7$ and $\tilde{F}_8$ (for generic values of $c_1, \ldots, c_6$).

Equations (41)–(46) have at most the solutions $Z_{11} = \pm Z_{12} = \pm Z_{13}$, $Z_{21} = \pm Z_{22} = \pm Z_{23}$. For any choices of these signs we can express all the Z_{ij} in terms of, say, Z_{11} and Z_{21}. $\tilde{F}_7$ and $\tilde{F}_8$ then become homogeneous quadratic polynomials in Z_{11} and Z_{21}, say G_7, G_8, whose coefficients are expressions in $c_1, \ldots, c_6$. Thus for generic choices of $c_1, \ldots, c_6$, i.e., for (x_{ij}, y_{ij}) outside of some proper closed subvariety $\mathcal{D}_1 \subset Y_{\mathbf{C}}$, equations G_7 and G_8 will be independent, and hence $Z_{11} = Z_{21} = 0$ will be the unique solution. (In fact, if $G_7 = AZ_{11}^2 + BZ_{11}Z_{21} + CZ_{21}^2$, $G_8 = DZ_{11}^2 + EZ_{11}Z_{21} + FZ_{21}^2$, where A, B, C, D, E, F are polynomials in $c_1, \ldots, c_6$, then $\mathcal{D}_1$ is the variety in $\mathbf{C}^{12}$ where the three determinants

$$\begin{vmatrix} A & B \\ D & E \end{vmatrix}, \quad \begin{vmatrix} A & C \\ D & F \end{vmatrix}, \quad \begin{vmatrix} B & C \\ E & F \end{vmatrix}$$

all vanish.) It follows that the unique solution to the system (41)–(48) is $Z_{ij} = 0$ for all i and j, but this does not correspond to a point in $\mathbf{P}^5(\mathbf{C})$ (the points in $\mathbf{P}^5(\mathbf{C})$ correspond to *lines* through the origin in z_{ij}-space). We have thus shown that there are no solutions at infinity for $\overline{R_y}$ for $y \notin \mathcal{D}_1$, i.e., that $R_y = \overline{R_y}, y \notin \mathcal{D}_1$. Let $Y_{\mathbf{C},1} = Y_{\mathbf{C}} - \mathcal{D}_1$, and let $R_{\mathbf{C}}$ now denote the variety in $X_{\mathbf{C},1} = Y_{\mathbf{C},1} \times \mathbf{C}^6$ defined by (18)–(23). We have shown that $R_{\mathbf{C}}$ contains no points at infinity. Thus while $R_{\mathbf{C}}$ is *a priori* a closed subvariety of $Y_{\mathbf{C},1} \times \mathbf{C}^6$, it is, moreover, a subvariety of $Y_{\mathbf{C},1} \times \mathbf{P}^6(\mathbf{C})$, i.e., it is the projective variety defined by $\tilde{F}_1, \ldots, \tilde{F}_8$. We have shown, then, that this $R_{\mathbf{C}}$ is actually a *closed* subvariety of $Y_{\mathbf{C},1} \times \mathbf{P}^6(\mathbf{C})$. Hence the projection map $R_{\mathbf{C}} \xrightarrow{\pi} Y_{\mathbf{C},1}$ is a projective morphism. In fact, $R_{\mathbf{C}} \xrightarrow{\pi} Y_{\mathbf{C},1}$ is a *finite* morphism. Namely, by Fact A4 in the appendix it suffices to show that for $y \in Y_{\mathbf{C},1}$ the set R_y consists of finitely many points. But if it

$$\begin{array}{ccccc} R_{\mathbf{C}} & \subset & \mathbf{C}^6 \times Y_{\mathbf{C},1} & = & X_{\mathbf{C},1} \\ {\scriptstyle \pi|_{R_{\mathbf{C}}}\ \text{finite}} \downarrow & & \downarrow {\scriptstyle \pi} & & \\ Y_{\mathbf{C},1} & = & Y_{\mathbf{C},1} & = & Y_{\mathbf{C}} - \mathcal{D}_1 \end{array}$$

Fig. 3. Map π, which when restricted to $R_{\mathbf{C}}$, is a finite morphism.

contained infinitely many points it would have a positive dimensional component, which would then intersect the $\mathbf{P}^5$ at infinity, i.e., we would then have $R_y \neq \overline{R_y}$, a contradiction.

We summarize as follows: There is a proper subvariety $\mathcal{D}_1$ of $Y_{\mathbf{C}} = \mathbf{C}^{12}$, so that if we denote $Y_{\mathbf{C},1} = Y_{\mathbf{C}} - \mathcal{D}_1$ and we let $R_{\mathbf{C}}$ denote the subvariety of $X_{\mathbf{C},1} = Y_{\mathbf{C},1} \times \mathbf{C}^6$ defined by $f_1 = \cdots = f_6 = 0$ (equations (18)–(23)), then the projection $\pi : R_{\mathbf{C}} \to Y_{\mathbf{C},1}$ is a finite morphism (as illustrated in figure 3).

Step C. At this point we have established that $\pi : R_{\mathbf{C}} \to Y_{\mathbf{C},1}$ is a finite morphism, where $Y_{\mathbf{C},1}$ is obtained from $Y_{\mathbf{C}}$ by deleting a measure-zero subset of nongeneric image data. To get uniqueness of interpretations and to assure that the measure of false targets is zero we now need to impose the constraint of conservation of angular momentum, viz., $\mathbf{I}_1\mathbf{w}_1 = \mathbf{I}_2\mathbf{w}_2$. To construct $\mathbf{w}_1$ and $\mathbf{w}_2$ we must first construct discrete-time rotation matrices O_1 and O_2; O_1 takes the vector $\mathbf{r}_{i1}$ to the vector $\mathbf{r}_{i2}$ and O_2 takes $\mathbf{r}_{i2}$ to $\mathbf{r}_{i3}$. Then $\mathbf{w}_1$ will be an eigenvector of O_1 whose length encodes the amount of rotation from frame 1 to frame 2 and $\mathbf{w}_2$ will be an eigenvector of O_2 whose length encodes the amount of rotation from frame 2 to frame 3.

We now explicitly construct the matrix O_1. The construction for O_2 is analogous. In what follows we will use an overbar to denote normalization to a unit vector. We first note that if the vectors $\mathbf{r}_{1j}$, $\mathbf{r}_{2j}$ are linearly independent, the following three unit vectors define orthonormal coordinates in frame j : $\overline{\mathbf{r}_{1j}}$, $\overline{\mathbf{r}_{1j} \times \mathbf{r}_{2j}}$, and $\overline{(\mathbf{r}_{1j} \times \mathbf{r}_{2j}) \times \mathbf{r}_{1j}}$. The rotation of these orthonormal coordinates (and therefore of the points) from frame 1 to frame 2 is then given by the matrix

$$O_1 = \begin{pmatrix} \overline{\mathbf{r}_{12}} \\ \overline{\mathbf{r}_{12} \times \mathbf{r}_{22}} \\ \overline{(\mathbf{r}_{12} \times \mathbf{r}_{22}) \times \mathbf{r}_{12}} \end{pmatrix}^{\mathbf{T}} \begin{pmatrix} \overline{\mathbf{r}_{11}} \\ \overline{\mathbf{r}_{11} \times \mathbf{r}_{21}} \\ \overline{(\mathbf{r}_{11} \times \mathbf{r}_{21}) \times \mathbf{r}_{11}} \end{pmatrix}. \tag{49}$$

The normalizations to unit vectors used in the definition of O_1 involve square roots, which are not polynomial functions. Since our method of proof requires that we use polynomial equations exclusively, we must rework the definition of O_1 to make it polynomial. The mathematical details of this reworking are contained in the next few paragraphs. The reader not interested in the details of this reworking can skip to Step D.

To make our rotation matrices algebraic it will now be convenient to introduce variables corresponding to the lengths of the vectors $\mathbf{r}_{1j}$ and $\mathbf{r}_{1j} \times \mathbf{r}_{2j}$ so that we can represent these lengths in polynomial expressions. To do this we introduce variables l_j, n_j, where $j = 1, 2, 3$, satisfying the equations

$$n_j^2 = \mathbf{r}_{1j} \cdot \mathbf{r}_{1j}, \tag{50}$$

$$l_j^2 = (\mathbf{r}_{1j} \times \mathbf{r}_{2j}) \cdot (\mathbf{r}_{1j} \times \mathbf{r}_{2j}). \tag{51}$$

In terms of components these equations may be written

$$n_j^2 - x_{1j}^2 - y_{1j}^2 - z_{1j}^2 = 0, \tag{52}$$

and

$$l_j^2 - (x_{1j}y_{2j} - x_{2j}y_{1j})^2 - (x_{2j}z_{1j} - x_{1j}z_{2j})^2 - (y_{1j}z_{2j} - y_{2j}z_{1j})^2 = 0, \tag{53}$$

where $j = 1, 2, 3$. Now (52) and (53), together with (18)–(23), can be regarded as defining a variety $\widetilde{\mathrm{R}}_{\mathbf{C}}$ in $X'_{\mathbf{C}} = X_{\mathbf{C},1} \times \{(n_j, l_j), j = 1, 2, 3\} = X_{\mathbf{C},1} \times \mathbf{C}^6 = Y_{\mathbf{C},1} \times \mathbf{C}^{12}$. The projection q from $X'_{\mathbf{C}}$ to $X_{\mathbf{C},1}$ (which forgets the n_j and l_j) induces a *finite* morphism $\widetilde{\mathrm{R}}_{\mathbf{C}} \to R_{\mathbf{C}}$; in fact, according to (52) and (53), n_j and l_j satisfy *monic* polynomials whose coefficients are functions on $R_{\mathbf{C}}$.

Using the variables l_j, n_j (where $j = 1, 2, 3$) introduced above, we can rewrite the matrix O_j as

$$O_1 = \begin{pmatrix} \mathbf{r}_{12}/n_2 \\ \mathbf{r}_{12} \times \mathbf{r}_{22}/l_2 \\ (\mathbf{r}_{12} \times \mathbf{r}_{22}) \times \mathbf{r}_{12}/(n_2 l_2) \end{pmatrix}^{\mathbf{T}} \begin{pmatrix} \mathbf{r}_{11}/n_1 \\ \mathbf{r}_{11} \times \mathbf{r}_{21}/l_1 \\ (\mathbf{r}_{11} \times \mathbf{r}_{21}) \times \mathbf{r}_{11}/(n_1 l_1) \end{pmatrix}, \tag{54a}$$

$$O_2 = \begin{pmatrix} \mathbf{r}_{13}/n_3 \\ \mathbf{r}_{13} \times \mathbf{r}_{23}/l_3 \\ (\mathbf{r}_{13} \times \mathbf{r}_{23}) \times \mathbf{r}_{13}/(n_3 l_3) \end{pmatrix}^{\mathbf{T}} \begin{pmatrix} \mathbf{r}_{12}/n_2 \\ \mathbf{r}_{12} \times \mathbf{r}_{22}/l_2 \\ (\mathbf{r}_{12} \times \mathbf{r}_{22}) \times \mathbf{r}_{12}/(n_2 l_2) \end{pmatrix}. \tag{54b}$$

Note that for a given set of $\mathbf{r}_{ij}$'s there are many matrices O_1, O_2 corresponding to different choices of sign for the l_j's and n_j's; each point of the variety $\widetilde{\mathrm{R}}_{\mathbf{C}}$ corresponds to one such choice, i.e., for each point of $\widetilde{\mathrm{R}}_{\mathbf{C}}$ there is precisely one matrix O_1 and one matrix O_2. These matrices have the property that $O_j\mathbf{r}_{ij} = \pm\mathbf{r}_{i,j+1}$ $(i = 1, 2)$. Therefore to ensure that the O_j have the desired meaning we will impose the equations

$$O_j\mathbf{r}_{ij} = \mathbf{r}_{i,j+1}, \quad j = 1, 2, \quad i = 1, 2, \tag{55}$$

$$\operatorname{Det} O_1 = \operatorname{Det} O_2 = 1. \tag{56}$$

Thus the $\mathbf{r}_{ij}$ are related by the rigid rotations O_j in SO(3, **C**). These equations define a subvariety $R'_{\mathbf{C}}$ of $\widetilde{\mathrm{R}}_{\mathbf{C}}$. Since $\widetilde{\mathrm{R}}_{\mathbf{C}} \to R_{\mathbf{C}}$ is finite, $R'_{\mathbf{C}} \to R_{\mathbf{C}}$ is finite and is therefore projective by Fact A3 in the appendix.

For these matrices to make sense, i.e., in order that none of the l_j's or n_j's be zero, we must restrict our attention to those arrays $\{\mathbf{r}_{ij}\}$ for which $\mathbf{r}_{1j}$ and $\mathbf{r}_{2j}$ are linearly independent for $j = 1, 2, 3$. A sufficient condition for this is that the projections of $\mathbf{r}_{1j}$ and $\mathbf{r}_{2j}$ into the image plane be independent. Thus let $\mathcal{D}_2 \subset Y_{\mathbf{C}} = \mathbf{C}^{12}$ denote the set of those $\{(x_{ij}, y_{ij})\}_{ij}$ in which (x_{1j}, y_{1j}) and (x_{2j}, y_{2j}) are dependent for at least one value of j, $1 \le j \le 3$. $\mathcal{D}_2$ is a variety in $\mathbf{C}^{12}$ defined by the vanishing of the appropriate determinants. We will let $Y_{\mathbf{C},2} = Y_{\mathbf{C}} - (\mathcal{D}_1 \cup \mathcal{D}_2)$, $X_{\mathbf{C},2} = Y_{\mathbf{C},2} \times \mathbf{C}^6$, and $X'_{\mathbf{C},2} = X_{\mathbf{C},2} \times \mathbf{C}^6$. We will now use the symbol $R_{\mathbf{C}}$ to denote the variety in $X_{\mathbf{C},2}$ defined by (18)–(23); we will let $R'_{\mathbf{C}}$ denote the variety in $X'_{\mathbf{C},2}$ defined by (51) and (52). The point is that with this new notation the projections $R'_{\mathbf{C}} \to R_{\mathbf{C}}$ and $R_{\mathbf{C}} \to Y_{\mathbf{C},2}$ are finite morphisms and at each point $\mathbf{r}' = \{(x_{ij}, y_{ij}, z_{ij}, n_j, l_j)\}$ of $R'_{\mathbf{C}}$ the matrices $O_1(\mathbf{r}')$ and $O_2(\mathbf{r}')$ defined in (54) and (55) make sense (see figure 4). Indeed, if at a point $\mathbf{r}' \in R'_{\mathbf{C}}$

$$
\begin{array}{ccccccc}
R'_{\mathbf{C}} & \subset & X'_{\mathbf{C},2} & = & Y_{\mathbf{C},2} \times \mathbf{C}^6 \times \mathbf{C}^6 & & \{x_{ij}, y_{ij}, z_{ij}, l_j, n_j\} \\
{\scriptstyle q|_{R'_{\mathbf{C}}}\ \text{finite}} \downarrow & & \downarrow {\scriptstyle q} & & & & \\
R_{\mathbf{C}} & \subset & X_{\mathbf{C},2} & = & Y_{\mathbf{C},2} \times \mathbf{C}^6 & & \{x_{ij}, y_{ij}, z_{ij}\} \\
{\scriptstyle \pi|_{R'_{\mathbf{C}}}\ \text{finite}} \downarrow & & \downarrow {\scriptstyle \pi} & & & & \\
Y_{\mathbf{C},2} & = & Y_{\mathbf{C},2} & = & Y_{\mathbf{C}} - (\mathcal{D}_1 \cup \mathcal{D}_2) & & \{x_{ij}, y_{ij}\}
\end{array}
$$

Fig. 4. $R_{\mathbf{C}}$ which is defined by (18)–(23), and $R'_{\mathbf{C}}$ which is defined additionally by (49), (50), and (56). To each point $\mathbf{r}'$ in $R'_{\mathbf{C}}$ are associated the rotation matrices O_1 and O_2.

the coordinates $\{(x_{ij}, y_{ij}, z_{ij}, n_j, l_j)\}$ are all real numbers, then $O_1(\mathbf{r}')$ and $O_2(\mathbf{r}')$ are rotation matrices in SO(3, R); $O_1(\mathbf{r}')$ sends $\mathbf{r}_{i1}$ to $\mathbf{r}_{i2}$ ($i = 1, 2$) and $O_2(\mathbf{r}')$ sends $\mathbf{r}_{i2}$ to $\mathbf{r}_{i3}$ ($i = 1, 2$).

Step D. We now consider vector variables $\mathbf{w}_1$ and $\mathbf{w}_2$, representing possible discrete-time angular velocities (see section 2), where each $\mathbf{w}_j$ varies on a copy of $\mathbf{C}^3$. (For the reader following the technical details we form the variety $R'_{\mathbf{C}} \times \mathbf{C}^3 \times \mathbf{C}^3$, on which we have coordinates x_{ij}, y_{ij}, z_{ij}, n_j, l_j, $\mathbf{w}_1$, $\mathbf{w}_2$. It is on this variety that we can formulate our conservation equations.)

First of all, according to equations (11), which define the $\mathbf{w}_j$, we must impose the eigenvector conditions

$$O_1\mathbf{w}_1 = \mathbf{w}_1, \qquad (57)$$

$$O_2\mathbf{w}_2 = \mathbf{w}_2. \qquad (58)$$

Secondly, we impose the conservation equation

$$\mathbf{I}_1\mathbf{w}_1 = \mathbf{I}_2\mathbf{w}_2. \qquad (59)$$

Finally, we must impose the length conditions (11) for $\mathbf{w}_1$ and $\mathbf{w}_2$. We use the fact that if O is a matrix that expresses a rotation through an angle θ about some axis, then

$$\cos\theta = \frac{\mathrm{Tr}(O) - 1}{2}. \qquad (60)$$

It follows that

$$\sin^2\theta = 1 - \left(\frac{\mathrm{Tr}(O) - 1}{2}\right)^2. \qquad (61)$$

Hence or length condition (11) on $\mathbf{w}_j$ implies the equations

$$\mathbf{w}_j \cdot \mathbf{w}_j = 1 - \left(\frac{\mathrm{Tr}(O_j) - 1}{2}\right)^2. \qquad (62)$$

In the following few paragraphs we will show that the number of *complex* solutions to these constraint equations is generically not more than two. The reader not interested in the mathematical details can skip to step E.

We are interested in the subvariety $E''_{\mathbf{C}}$ of $R'_{\mathbf{C}} \times \mathbf{C}^3 \times \mathbf{C}^3$ defined by (57)–(59) *and* (62) (see figure 5). We can complete $E''_{\mathbf{C}}$ to a projective variety $\overline{E''_{\mathbf{C}}}$ over $R'_{\mathbf{C}}$: $\overline{E''_{\mathbf{C}}}$ is the projective completion of $E''_{\mathbf{C}}$ in the $\mathbf{w}$-coordinates, i.e., $E''_{\mathbf{C}}$ is a subvariety of $R'_{\mathbf{C}} \times \mathbf{C}^6$ and $\overline{E''_{\mathbf{C}}}$ is the closure of $E''_{\mathbf{C}}$ in $R'_{\mathbf{C}} \times \mathbf{P}^6(\mathbf{C})$. Now let $E'_{\mathbf{C}}$ be the image of $\overline{E''_{\mathbf{C}}}$ in $R'_{\mathbf{C}}$ by means of p, and let $E_{\mathbf{C}}$ be the image of $E'_{\mathbf{C}}$ in $R_{\mathbf{C}}$ by means of the morphism $q \circ p$, where $p : R'_{\mathbf{C}} \times \mathbf{P}^6(\mathbf{C}) \to R'_{\mathbf{C}}$ and $q : R'_{\mathbf{C}} \to R_{\mathbf{C}}$ are the projections. Both p and q are projective morphisms, as we have seen. Therefore $q \circ p$ is projective (Fact A1 in the appendix). Since a projective morphism is closed (Fact A2 in the appendix), it follows that $E_{\mathbf{C}}$ is a closed subvariety of $R_{\mathbf{C}}$. Since $R_{\mathbf{C}}$ is finite over $Y_{\mathbf{C},2}$ and $E_{\mathbf{C}} \subset R_{\mathbf{C}}$ is closed, it follows that $\pi|_{E_{\mathbf{C}}} : E_{\mathbf{C}} \to Y_{\mathbf{C},1}$ is finite.

Now since $\pi : E_{\mathbf{C}} \to Y_{\mathbf{C},2}$ is a finite morphism, by Theorem 13 we find the following:

RESULT 63. $T_{\mathbf{C}} = \{y \in Y_{\mathbf{C},2} \mid \pi^{-1}(\{y\}) \cap E_{\mathbf{C}}$

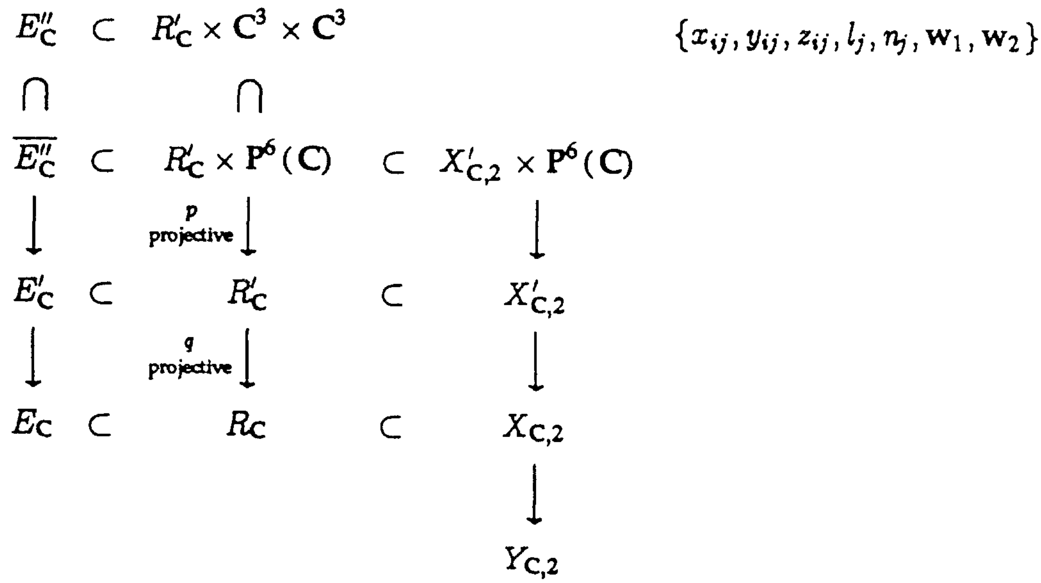

Fig. 5. E''_C which is defined by (18)–(23), (49), (50), (56)–(59), and (62). $\overline{E''_C}$ is the projective completion of E''_C in the $\mathbf{w}_j$ variables.

contains more than two points} is a *closed* subvariety of $Y_{C,2}$.

Note that the fact that T_C is a closed subvariety of $Y_{C,2}$ means that T_C has measure zero in $Y_{C,2}$.

Step E. Up to this point all of our results have concerned complex solutions to our constraint equations. We are, of course, ultimately interested in the real solutions. In the following paragraphs we will define a set E of real vectors $\mathbf{r}_{ij}$ that satisfy our constraint equations. This will be essential in finding the number of real solutions to our equations. Recall that the map $\pi : \mathbf{R}^{18} \to \mathbf{R}^{12}$ takes $\{x_{ij}, y_{ij}, z_{ij}\}$ to $\{x_{ij}, y_{ij}\}$. As a matter of notation let $S = \pi(E)$. Intuitively, S represents the set of all *displays* that are consistent with a Poinsot motion interpretation. The next few paragraphs will be devoted to showing (a) for generic $s \in S$, the set $\pi^{-1}(S) \cap E$ contains exactly two points (which correspond to 3D interpretations that are mutual reflections in the x, y plane) and (b) the set $\pi^{-1}(S) \cap E$ has Lebesgue measure zero in $X(= \mathbf{R}^{18})$. The reader not interested in the mathematical details can now skip to Step F.

Result 63 is all that we need to extract from the complex geometry of our equations. We now consider the underlying real geometry. Let $\overline{E''}$, E'', E', E denote, respectively, the subsets of $\overline{E''_C}$, E''_C, E'_C, E_C of points with real coordinates. As illustrated in figure 6, we let

$$\begin{aligned} Y &= Y_{C,2} \cap \mathbf{R}^{12} \quad (= \mathbf{R}^{12} - (\mathcal{D}_1 \cup \mathcal{D}_2) \cap \mathbf{R}^{12}),\\ X &= X_{C,2} \cap \mathbf{R}^{18} \quad (= Y \times \mathbf{R}^6),\\ X' &= X'_C \cap \mathbf{R}^{24},\\ R &= R_C \cap X,\\ R' &= R'_C \cap X',\\ S &= \pi(E) \subset Y. \end{aligned}$$

We note that *all* the polynomial equations defining the variety E''_C have *real* coefficients and that the maps $E''_C \to E'_C \to E_C \to \mathbf{C}^{12}$ are induced by projections and hence are defined over $\mathbf{R}$. Thus E'', E', and E are $\mathbf{R}$-varieties, S is a semialgebraic set (see the appendix), and $E'' \to E \to S \subset \mathbf{R}^{12}$ are $\mathbf{R}$-morphisms. (Note that E may equally well be defined as the image of E'' in X_2.)

$$
\begin{array}{lllll}
E'' & \subset & & X' \times \mathbf{R}^6 \subset \mathbf{R}^{24} \times \mathbf{R}^6 & \{x_{ij}, y_{ij}, z_{ij}, n_j, l_j, \mathbf{w}_j\} \\
\cap & & & \quad\cap & \\
\overline{E''} & \subset & & X' \times \mathbf{P}^6(\mathbf{R}) \subset \mathbf{R}^{24} \times \mathbf{P}^6(\mathbf{R}) & \\
\downarrow p & & & \downarrow p & \\
E' & \subset R' \subset & & X' = X \times \mathbf{R}^6 \subset \mathbf{R}^{24} & \{x_{ij}, y_{ij}, z_{ij}, n_j, l_j\} \\
\downarrow q & & & \downarrow q & \\
E & \subset R \subset & & X = Y \times \mathbf{R}^6 \subset \mathbf{R}^{18} & \{x_{ij}, y_{ij}, z_{ij}\} \\
\downarrow \pi & & & \downarrow \pi & \\
S & \subset & & Y = Y_{\mathbf{C},2} \cap \mathbf{R}^{12} \subset \mathbf{R}^{12} & \{x_{ij}, y_{ij}\} \\
 & & & \quad = \mathbf{R}^{12} - (\mathcal{D}_1 \cup \mathcal{D}_2) \cap \mathbf{R}^{12} &
\end{array}
$$

Fig. 6. Relationships between the real spaces involved in the proof of Theorem 13.

Since $E'_{\mathbf{C}} = p(\overline{E''_{\mathbf{C}}})$ and $E_{\mathbf{C}} = q(E'_{\mathbf{C}})$ by definition, we have $E' \subset p(\overline{E''})$ and $E \subset q(E')$. We will show that $\overline{E''} = E''$. Moreover, we will show that $E' = p(E'')$ and $E = q(E')$ (the corresponding statement is false in general for complexified maps of real varieties). These results mean that the points of E represent 3D motions for which the Poinsot constraint holds in the ordinary sense and not in some virtual sense for infinite or complex-valued angular velocities as an artifact of our equations. Precisely, we show the following:

CLAIM 64

(a) $\overline{E''} = E''$.
(b) $E = q(E')$.
(c) $E' = p(E'')$.

Proof. Let $\mathbf{r}' \in R'$, so that $O_1(\mathbf{r}')$ and $O_2(\mathbf{r}')$ are in $SO(3, \mathbf{R})$. Moreover, neither O_1 nor O_2 is the identity matrix since otherwise c_1, c_3 and c_5 (or c_2, c_4, and c_6) would be 0, so that $\mathbf{r}'$ would project to $\mathcal{D}_1$, the possibility of which we have excluded. Now any nontrivial matrix in $SO(3, \mathbf{R})$ has a *unique* (up to scalar multiple) nonzero eigenvector of eigenvalue 1, which is a real vector. Now we have noted that the inertia operator $\mathbf{I}_2$, say, is nonsingular provided that $\mathbf{r}_{12}$, $\mathbf{r}_{22}$ are linearly independent (see Proposition 8). Therefore (since we are working over the complement of $\mathcal{D}_2$) equation (59) may be written

$$\mathbf{w}_2 = \mathbf{I}_2^{-1}\mathbf{I}_1\mathbf{w}_1.$$

Thus the variation of $(\mathbf{w}_1, \mathbf{w}_2)$ is restricted by equations (57)–(59) to a one-dimensional (1D) vector space: If $(a, b, c) = \mathbf{v}$ is a (real) eigenvector for O_1 of eigenvalue 1, we can take this space to be the 1D subspace of $\mathbf{C}^3 \times \mathbf{C}^3$ consisting of those points of the form $(t\mathbf{v}, t\mathbf{I}_2^{-1}\mathbf{I}_1\mathbf{v})$, $t \in \mathbf{C}$. Equation (62) for $j = 1$ is then $t^2(a^2 + b^2 + c^2) = k_1$, where

$$k_1 = 1 - \left(\frac{\mathrm{Tr}(O_1(\mathbf{r}')) - 1}{2}\right)^2$$

is a real nonnegative constant that depends on $\mathbf{r}'$. Hence, since $a^2 + b^2 + c^2 > 0$, we have

$$\begin{array}{ccc} E & \subset & X \\ \pi \downarrow & & \downarrow \pi \\ S & \subset & Y \end{array}$$

Fig. 7. Spaces and maps involved in the statement of Theorem 13.

$t \in \mathbf{R}$, with $0 \leq t < \infty$. Thus equations (57)–(59) and (62) have no simultaneous solutions at infinity (in the $\mathbf{w}_1$ and $\mathbf{w}_2$ variables), i.e., every point of $\overline{E''_{\mathbf{C}}}$ lying over a real point $\mathbf{r}'$ in $E'_{\mathbf{C}}$ is in fact in $E''_{\mathbf{C}}$ and is real. This proves parts (a) and (b) of Claim 64. Also if $\mathbf{r} \in R$ has real Coordinates $\{(x_{ij}, y_{ij}, z_{ij})\}$, then if $\mathbf{r}' \in R'$ projects to $\mathbf{r}$, it follows that $\mathbf{r}'$ also has real coordinates $\{(x_{ij}, y_{ij}, z_{ij}, n_j, l_j)\}$ since the n_j and l_j are the squared lengths of real vectors. This proves part (c) of Claim 64.

We summarize: E is the set of 6-tuples $\{(x_{ij}, y_{ij}, z_{ij})\}_{i=1,2; j=1,2,3}$ of vectors in $\mathbf{R}^3$ that are nondegenerate in the sense that they do not give image data in $\mathcal{D}_1 \cup \mathcal{D}_2$ and that represent a body exhibiting Poinsot motion. S consists of those nondegenerate real image data $\{(x_{ij}, y_{ij})\}$ for which there exists real $\{z_{ij}\}$, so that the array $\{(x_{ij}, y_{ij}, z_{ij})\}$ is in E, i.e., S consists of those nondegenerate real image data that can be interpreted as arising from a Poinsot motion in three dimensions. Assertions 1 and 2 of our main Theorem 13 may now be stated in terms of E, S, X, Y (see figure 7):

1. For generic $s \in S$, the set $\pi^{-1}(S) \cap E$ contains exactly two points (which correspond to 3D interpretations that are mutual reflections in the x, y plane).
2. The set $\pi^{-1}(S) \cap E$ has Lebesgue measure zero in X.

Step F. The next few paragraphs will be devoted to finding the irreducible components of E. Intuitively, a set W is irreducible only if it cannot be decomposed into distinct subcomponents, each component being the zero set of a distinct set of polynomials. For example, in $\mathbf{R}^2$ the solution set to the equation $xy = 0$ is not irreducible since it has two distinct components, viz., the x axis (i.e., $y = 0$) and the y axis (i.e., $x = 0$). The reason we want to find the irreducible components of E is that these irreducible components will be used later in our proof in conjunction with an *upper semicontinuity theorem* from algebraic geometry. The reader not interested in the mathematical details can now skip to Step G.

Let η be the map that associates to any array in E the vectors in its first two frames, i.e,

$$\eta((\mathbf{r}_{11}, \mathbf{r}_{21}), (\mathbf{r}_{12}, \mathbf{r}_{22}), (\mathbf{r}_{13}, \mathbf{r}_{23})) = ((\mathbf{r}_{11}, \mathbf{r}_{21}), (\mathbf{r}_{12}, \mathbf{r}_{22})).$$

Thus $\eta : E \to (\mathbf{R}^3)^4 = \mathbf{R}^{12}$. (Note that this $\mathbf{R}^{12}$ is different from the $\mathbf{R}^{12}$ containing Y, which consists of the projections into the x, y plane of the vectors in all three frames.)

Let F denote the set of all $((\mathbf{r}_{11}, \mathbf{r}_{21}), (\mathbf{r}_{12}, \mathbf{r}_{22}))$ in $\mathbf{R}^{12}$ such that

Property 65. For some $O_1 \in \mathrm{SO}(3, \mathbf{R})$, $O_1(\mathbf{r}_{i1}) = \mathbf{r}_{i2}$, $i = 1, 2$.

Property 66. The projections into the x, y plane of $\mathbf{r}_{11}$ and $\mathbf{r}_{21}$ are linearly independent.

We *claim* that $\eta(E) = F$. It is clear that $\eta(E) \subset F$ since Property 66 holds for any array E because we have excluded the degenerate locus $\mathcal{D}_2$. Thus the content of the claim is that given $((\mathbf{r}_{11}, \mathbf{r}_{21}), (\mathbf{r}_{12}, \mathbf{r}_{22})) \in F$, there exists a pair of vectors $(\mathbf{r}_{13}, \mathbf{r}_{23})$ and a rotation $O_2 \in \mathrm{SO}(3, \mathbf{R})$ such that $O_2(\mathbf{r}_{i2}) = \mathbf{r}_{i3}$ (where $i = 1, 2$) and the (discrete-time) angular momentum of the motion $(\mathbf{r}_{11}, \mathbf{r}_{21}) \mapsto (\mathbf{r}_{12}, \mathbf{r}_{22})$ is the same as that of the motion $(\mathbf{r}_{12}, \mathbf{r}_{22}) \mapsto (\mathbf{r}_{13}, \mathbf{r}_{23})$. This means that $\mathbf{w}_2 = \mathbf{I}_2^{-1}\mathbf{I}_1\mathbf{w}_1$. Now $\mathbf{I}_j$ is determined by $(\mathbf{r}_{1j}, \mathbf{r}_{2j})$, $j = 1, 2$, so that $\mathbf{I}_1$, $\mathbf{I}_2$ are determined by the point of F. Moreover, O_1 is uniquely determined by the point of F since the vectors in the first frame are linearly independent and a rotation of $\mathbf{R}^3$ is uniquely determined by its effect on two linearly independent vectors. Finally, $\mathbf{w}_1$ is determined by O_1 up to a factor of ± 1 by the conditions that $\mathbf{w}_1$ is parallel to the axis of rotation of O_1 and that $|\mathbf{w}_1|^2 = \sin^2\theta$, where θ

is the angle of the rotation expressed by O_1. Choose one of the two possible values of $\mathbf{w}_1$, and let $\mathbf{w}_2 = \mathbf{I}_2^{-1}\mathbf{I}_1\mathbf{w}_1$. Now any vector $\mathbf{w}$ is a discrete-time angular velocity vector for some rotation O. In fact, O is the rotation about an axis parallel to $\mathbf{w}$ through an angle θ such that $\sin^2\theta = |\mathbf{w}|^2$. Notice that this does not uniquely determine θ. However, by choosing one such θ we obtain an O as desired. In our case choose an O_2 for $\mathbf{w}_2$ in this manner. We can then let $(\mathbf{r}_{13}, \mathbf{r}_{23}) = O_2(\mathbf{r}_{12}, \mathbf{r}_{22})$, and then it is clear that $((\mathbf{r}_{11}, \mathbf{r}_{21}), (\mathbf{r}_{12}, \mathbf{r}_{22}), (\mathbf{r}_{13}, \mathbf{r}_{23}))$ is in E.

F is an *irreducible variety* in $\mathbf{R}^3$. F is isomorphic to the product variety $V \times SO(3, \mathbf{R})$, where V is the set of points $(\mathbf{r}_{11}, \mathbf{r}_{21}) \in \mathbf{R}^3 \times \mathbf{R}^3$, which are linearly independent. In fact, we have already noted that in view of the independence, O_1 is uniquely determined by the vectors in question. V is the complement of the variety W in $\mathbf{R}^3 \times \mathbf{R}^3$ (defined by the 2×2 minors of the matrix)

$$\begin{pmatrix} \mathbf{r}_{11} \\ \mathbf{r}_{21} \end{pmatrix}.$$

Since the codimension of W in $\mathbf{R}^3 \times \mathbf{R}^3$ is at least 2, $V = \mathbf{R}^3 \times \mathbf{R}^3 - W$ is connected. Moreover, it is nonsingular (since it is an open subset of $\mathbf{R}^3 \times \mathbf{R}^3$). Hence V is irreducible (appendix, Fact A8). Moreover $SO(3, \mathbf{R})$ is irreducible since it is a connected (algebraic) group (appendix, Fact A9). Hence the product $V \times SO(3, \mathbf{R})$ is irreducible (appendix, Fact A10), so that F is irreducible.

We now look more closely at the map $\eta : E \to F$. In particular, we want to study the set $\eta^{-1}(P)$ for a point $P = ((\mathbf{r}_{11}, \mathbf{r}_{21}), (\mathbf{r}_{12}, \mathbf{r}_{22}))$ in F. As we noted above, P determines $\mathbf{I}_1$ and $\mathbf{I}_2$ uniquely and it determines O_1 uniquely, whence it determines $\mathbf{w}_1$ up to a factor of ± 1. Hence $\mathbf{w}_2$ is determined up to ± 1 by the relation $\mathbf{w}_2 = \mathbf{I}_2^{-1}\mathbf{I}_1\mathbf{w}_1$. Each point in $\eta^{-1}(P)$ is then of the form

$$((\mathbf{r}_{11}, \mathbf{r}_{21}), (\mathbf{r}_{12}, \mathbf{r}_{22}), (O_2(\mathbf{r}_{12}), O_2(\mathbf{r}_{22}))),$$

where $O_2 \in SO(3, \mathbf{R})$ is a rotation with axis $\mathbf{w}_2$, through an angle θ such that $|\mathbf{w}_2|^2 = \sin^2\theta$. Thus the possible choices for O_2 correspond to choices of θ, $0 \le \theta < 2\pi$, such that $\sin^2\theta$ is a given fixed number. In general there are four such angles: If θ is one of them (say, $0 < \theta < \pi/2$), then the others are $\pi - \theta$, $\pi + \theta$, $2\pi - \theta$. We summarize.

RESULT 67. The map $\eta : E \to \eta(E)$ is generically *four to one*: If $P = ((\mathbf{r}_{11}, \mathbf{r}_{21}), (\mathbf{r}_{12}, \mathbf{r}_{22}))$ is in $\eta(E)$, then

$$\eta^{-1}(P) = \left\{((\mathbf{r}_{11}, \mathbf{r}_{21}), (\mathbf{r}_{12}, \mathbf{r}_{22}), (O_2^{(l)}(\mathbf{r}_{12}), O_2^{(l)}(\mathbf{r}_{22})))\right\}_{l=1,\dots,4},$$

where $O_2^{(l)}$, $l = 1, \dots, 4$, in $SO(3, \mathbf{R})$ represent rotations about the same axis through four different angles: θ, $\pi - \theta$, $\pi + \theta$, $2\pi - \theta$. It follows that E is generically a four-sheeted cover of $\eta(E)$.

We now ask, Is E irreducible? We know that E has an irreducible image by the algebraic map η and that E is generically a four-sheeted cover of $\eta(E)$. Thus each irreducible component must be a union of sheets, which may be more precisely stated as follows:

RESULT 68. Let $E_1, \dots, E_n$ denote the irreducible components of E. Then $n \le 4$, and $\dim E_k = \dim \eta(E) = 9$ for each k. Let $P \in \eta(E)$, with $\eta^{-1}(P) = \{e_1, \dots, e_4\}$. Then each irreducible component E_k contains at least one of $e_1, \dots, e_4$.

Step G. At this point we have established that the real solutions E to our constraint equations have at most four irreducible components. We now consider each of the irreducible components of E, which we have denoted by E_k. We denote their images under the map π by $\pi(E_k) = S_k$. Since the E_k are irreducible, so also are the S_k. Intuitively, S_k consists of image data $\{x_{ij}, y_{ij}\}$ that are compatible with real solutions to our constraint equations, i.e., that give rise to solutions $\{x_{ij}, y_{ij}, z_{ij}\}$ contained in E_k. We will now develop more detailed information about the irreducible sets S_k. The reader not interested in the mathematical details can now skip to Step H.

Now we turn our attention to the map $\pi : E \to S$, which is surjective by definition of S. Let $S_k = \pi(E_k)$, $k = 1, \dots, n$. We know that the

S_k are irreducible semialgebraic sets (since they are algebraic images of the irreducible varieties E_k; see appendix, Fact, A7) and that $S = \cup_k S_k$. It follows from Result 68 that

RESULT 69. If $\{e_1, \ldots, e_4\}$ are as in Result 68, then each S_k contains at least one of the $\pi(e_1), \ldots, \pi(e_4)$.

Since each irreducible component of S is one of the S_k and since all the S_k have the same dimension (of 9), it follows that "generic on S_k for *all* k" implies "generic on S." Therefore to prove assertion 1 of Theorem 13 it suffices to prove the following:

Assertion 70. For each k, for generic $s \in S_k$, the set $\pi^{-1}(\{s\}) \cap E$ contains exactly two points, which correspond to configurations that are reflections of each other in the x, y plane.

Now for any $e \in E$ the point e' in X, which represents the reflection of e, is also in E; this is true because the equations for the Poinsot constraint are invariant under the transformation $z_{ij} \mapsto -z_{ij}$. Since by the definition of S the set $\pi^{-1}(\{s\}) \cap E$ contains at least one point, it follows that $\pi^{-1}(\{s\}) \cap E$ contains at least two points for *all* $s \in S$. Hence to show Assertion 70 it suffices to show that if $T_k = \{s \in S_k \,|\, \pi^{-1}(\{s\}) \cap E$ has more than two points$\}$, then $\dim T_k < \dim S_k$. Now $\pi^{-1}(\{s\}) \cap E \subset \pi^{-1}(\{s\}) \cap E_{\mathbf{C}}$. Hence $T_k \subset S_k \cap T_{\mathbf{C}}$, where $T_{\mathbf{C}}$ is as in Result 63. Therefore to prove Assertion 70 it suffices to prove the following:

Assertion 71. For each $k = 1, \ldots, n$ it is the case that $\dim(T_{\mathbf{C}} \cap S_k) < \dim S_k$ (where $T_{\mathbf{C}}$ is as in Result 63.

Step H. We have just finished a careful examination of the sets S_k. Recall that each set S_k consists of image data $\{x_{ij}, y_{ij}\}$ that are compatible with real solutions to our constraint equations in E_k. We will now show that for generic image data in each S_k the number of real solutions to our constraint equations is precisely two. Here is where we explicitly use the upper semicontinuity theorem. When this theorem is used, it suffices to find one point in each S_k for which there are precisely two real solutions to our constraint equations and no complex solutions. Finding such points constitutes a rigorous proof that for generic image data in each S_k there are precisely two real solutions. We now produce a point on each S_k for which there are precisely two real solutions. The reader not interested in the mathematical details can now skip to Step I.

Now $T_{\mathbf{C}} \cap S_k$ is a closed subvariety of the irreducible semialgebraic set S_k in the sense that it is the locus of points of S_k defined by the vanishing of certain polynomials. In fact, it is the subvariety for which the real and imaginary parts of the complex polynomials defining $T_{\mathbf{C}}$ vanish separately. It is a fact (appendix, Fact A11) that a proper subvariety of an *irreducible* variety has positive codimension. Therefore it remains only to show that $T_{\mathbf{C}} \cap S_k$ is a *proper* subvariety of S_k. To do this we need only produce one point $s_k \in S_k$ (for each k) such that $\pi^{-1}(s_k) \cap E_{\mathbf{C}}$ contains exactly two points, both of which are in E, i.e., have real coordinates. For this purpose, in view of Result 69 we will choose a concrete point $P \in \eta(E)$ such that $\eta^{-1}(P) = \{e_1, \ldots, e_4\}$, we will let $s_k = \pi(e_k)$, and we will then simply check $\pi^{-1}(s_k) \cap E$ for each of these s_k. (If there are fewer than four components we will thereby have done some unnecessary checking, but this is, of course, harmless.)

For our point P we choose

$$P = (((7.00000, 2.00000, 3.00000), (5.00000, 1.00000, 9.00000)), ((6.39369, 1.42001, 4.37085), (3.01394, 1.30948, 9.80823))), \quad (72)$$

(where the numbers are truncated decimals derived from double-precision computations). We then compute that the corresponding e_k's are

$$e_1 = (P, ((5.43181, 0.678917, 5.6599), (0.988604, 1.71018, 10.1537))), \quad (73)$$

$$e_2 = (P, ((-4.16471, 5.60007, 3.64615), (-1.00359, 10.2764, 0.623041))); \quad (74)$$

$$e_3 = (P, ((-2.71738, 7.32657, 0.968083),$$

$$(2.94818, 9.90028, -0.54107))), \quad (75)$$

$$e_4 = (P, ((6.87914, 2.40542, 2.98183), (4.94038, 1.33405, 8.9896))), \quad (76)$$

with the associated s_k's:

$$s_1 = (((7.00000, 2.00000), (5.00000, 1.00000)), ((6.39369, 1.42001), (3.01394, 1.30948)), ((5.43181, 0.678917), (0.988604, 1.71018))), \quad (77)$$

$$s_2 = (((7.00000, 2.00000), (5.00000, 1.00000)), ((6.39369, 1.42001), (3.01394, 1.30948)), ((-4.16471, 5.60007), (-1.00359, 10.2764))), \quad (78)$$

$$s_3 = (((7.00000, 2.00000), (5.00000, 1.00000)), ((6.39369, 1.42001), (3.01394, 1.30948)), ((-2.71738, 7.32657), (2.94818, 9.90028))), \quad (79)$$

$$s_4 = (((7.00000, 2.00000), (5.00000, 1.00000)), ((6.39369, 1.42001), (3.01394, 1.30948)), ((6.87914, 2.40542), (4.94038, 1.33405))). \quad (80)$$

In practice, the easiest way to compute $\pi^{-1}(\{s\}) \cap E$ is first to compute $\pi^{-1}(\{s\}) \cap R_C$. In fact, (see [8]) for each point of Y the equations (14)–(16) defining R_C have 64 solutions, which may be computed explicitly. Beginning with one of our s_k's, then, we can check systematically which of these 64 explicitly computed points in $\pi^{-1}(\{s\}) \cap R$ satisfy the Poinsot constraint (embodied by the equations (57)–(59), (62)). We know *a priori* that e_k and its reflection in the x, y plane, viz., e'_k, both satisfy the constraint; the question is whether there are any other points that satisfy it. Having carried out these computations (using *Mathematica*), we find that for each $k = 1, \ldots, 4$ the set $\pi^{-1}(\{s\}) \cap E = \{e_k, e'_k\}$.

Step I. We have now found that for generic image data in each of the four sets S_k the number of real solutions to our constraint equations is precisely two. We did this by examining concrete test cases, finding all the real solutions to our equations, and then invoking the upper semicontinuity theorem. But we must now be concerned with one issue: the *multiplicity* of these real solutions. Recall that if $f(z) = (z - z_0)^n g(z)$ is a polynomial with a zero at z_0 and with $g(z_0) \neq 0$, then the multiplicity of the solution z_0 is n. For instance, the polynomial $y = x^2$ can be rewritten as $y(x) = (x - 0)^2$, so that the solution 0 has multiplicity two. We must now establish that each of the real solutions to our equations that were found in our concrete test cases have multiplicity one. The reason we must do this is that solutions with multiplicity n can break up into n distinct solutions if the parameters to the equations change (which they do along each S_k). In the case of the parabola $y = x^2$, for instance, the solution at 0 becomes two solutions near 0 if the parabola is translated very slightly down the y axis. To determine the multiplicity of our real solutions, we will now apply a Jacobian test to each solution. The reader not interested in the mathematical details can skip to Step J.

We now check that, for each k, both e_k and e'_k have multiplicity one as points on the fiber of E_C over s_k, i.e., that they are nonsingular points of the fiber. Since E_C is a subvariety of R_C, it suffices to show that they have multiplicity one in the fiber of R_C over s_k. To show this we may apply the Jacobian criterion for nonsingularity to the fiber of R_C over s_k: It suffices to show that the Jacobian matrix

$$\left(\frac{\partial f_l}{\partial z_m}\right), \quad l, m = 1, \ldots, 6,$$

has nonzero determinant when evaluated at e_k or e'_k (here the six functions f are as given in (18)–(23) and the six z_m's are what we have elsewhere called the six z_{ij}'s). This determinant is easy to compute and is, in fact, far from zero in each case. One might wonder whether our evaluation of the Jacobian determinant is subject to rounding error due to the use of floating point arithmetic. It is. However, this is not a problem, since (a) we want only to ascertain that this determinant is not zero and (b) the values of the determinant we obtained were far from zero, so that they could not be due simply to rounding error. This concludes the proof of the assertion 1 of Theorem 13.

Step J. The final step in our proof is to show that the (Lebesgue) measure of false targets is zero. This follows from the fact that the image data S that are compatible with Poinsot interpretations have measure zero in the set of all possible image data. The argument is presented more precisely in the following paragraph.

We now prove the second assertion of Theorem 13. We have seen that $\dim(S) = 9$, so that S has Lebesgue measure zero in $\mathbf{R}^{12}$. Since only points of S have interpretations that satisfy the Poinsot constraint, we are done. Now to see further that this implies that false targets have measure zero, observe that since S has measure zero in $\mathbf{R}^{12}$, it follows that $\pi^{-1}(S)$ has Lebesgue measure zero in $\mathbf{R}^{18}$ and *a fortiori* $\pi^{-1}(S)-E$ has Lebesgue measure zero in $\mathbf{R}^{18}$. But $\pi^{-1}(S)-E$ is precisely the set of false targets: It is the set of those 3D arrays that are not in E, i.e., that do not represent Poinsot motions but that produce image data in S, i.e., image data that have a Poinsot interpretation. This concludes the proof of assertion 2 of Theorem 13.

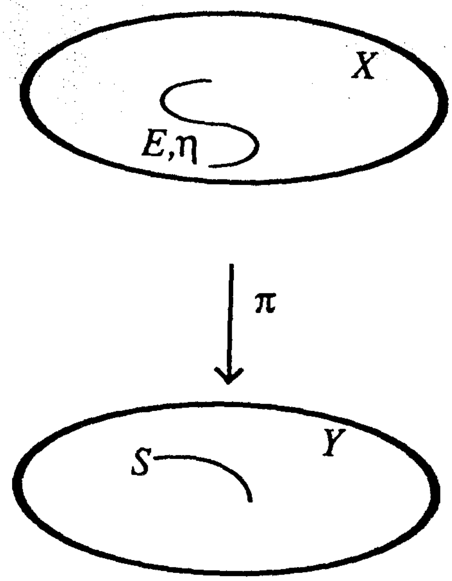

Fig. 8. Observer structure of the Poinsot motion inference.

4 Formulation as an Observer

Theorem 13 licenses a class of inferences: The premises for these inferences are certain dynamical images; the conclusions are certain 3D structures in motion. The abstract form of these inferences can be described as follows (see figure 8). The set of possible premises is the set $Y = \mathbf{R}^{12}$ of all possible three views of two vectors (we can now restrict our consideration to real numbers and ingore the complex numbers that arose in the course of the proof). The set of possible conclusions is the set $X = \mathbf{R}^{18}$ of all possible 3D interpretations for elements of Y. Those interpretations satisfying the Poinsot constraint form a nine-dimensional subset E of X. The conclusions X and premises Y are related by a function π given by $(x_{ij}, y_{ij}, z_{ij}) \mapsto (x_{ij}, y_{ij})$, and for each premise $y \in Y$ the set $\pi^{-1}(\{y\})$ is the set of all 3D conclusions compatible with the premise y. Those premises y that have at least one compatible conclusion that satisfies the Poinsot constraint form a subset S of Y. Clearly, $S = \pi(E)$. Moreover, S has Lebesgue measure zero in Y. Thus for most $y \in Y$ none of the compatible conclusions satisfies the Poinsot constraint, and hence the probability of false targets for this inference is zero. For premises $s \in S$ the number of compatible conclusions that satisfy the Poinsot constraint is, generically, two. Therefore the conclusion associated to such an s is best thought of as a probability measure, say, η_s, supported on these two conclusions. The weight given to a particular conclusion by this measure can be thought of as the frequency with which that interpretation is perceived, given that one is viewing the display s.

Thus the inference of structure from motion examined here is specified by a six-tuple (X, Y, E, S, π, η). This six-tuple precisely satisfies the definition of *observer* given in observer theory [36], [37]. According to the *observer thesis* [36], [37] every perceptual capacity, whether instantiated in neurons or in silicon, can be described as an instance of a single formal structure, viz., the observer.

DEFINITION 81. An *observer* is a six-tuple (X, Y, E, S, π, η) where

1. X and Y are measurable spaces. E is an event of X. S is an event of Y. Points of X and Y are measurable.
2. π is a mesurable map from X onto Y such that $\pi(E) = S$.
3. η is a Markovian kernel that associates to each point s of S a probability measure on E that gives the set $\pi^{-1}(s) \cap E$ a probability of one.

The present theory of structure from Poinsot motion is a specific example in support of the observer thesis.

5 Implications and Comments

The analysis presented here is a departure from previous analyses in an interesting respect: Whereas previous analyses of the inference of 3D structure from image motion have relied exclusively on kinematical and geometrical constraints, such as rigid motion or fixed-axis motion, the present analysis introduces a dynamical constraint—Poinsot motion. The dynamical nature of this constraint is evident in its use of the inertia tensor, which incorporates the masses of the moving points, and in its assumption that there is no net force acting on the system of points.

The present analysis is, of course, but a first step in this direction. We have assumed, for instance, that all the visible points have equal masses and that these masses alone determine the appropriate inertia tensor. This assumption will not, in general, be valid. If this assumption is nevertheless used by human vision, we should be able to concoct displays that are systematically misperceived by subjects in ways predicted by the foregoing analysis. However, it might be theoretically possible to infer, from the motion of the visible points, more detailed information about the *true* inertia tensor of the body to which the points are attached. If so, it would be of some interest to ask psychophysically whether human vision can, from displays of structure from motion, infer such information. Indeed, pilot experiments carried out in our laboratory suggest that human subjects can infer dynamical properties of moving points from 2D displays. In these experiments subjects were shown displays of three points undergoing Poinsot motion. In each display one point had a high mass, one had an intermediate mass, and one had a low mass. The ratio of these masses was an independent variable of the experiment; the ratios were 16:1:1/16 or 4:1:1/4 or 2:1:1/2 or 1:1:1. The subject's task was to view the Poinsot motion display for roughly 30 s (exactly 900 distinct frames) and then to choose which one of the three points was of intermediate mass. The displays were controlled so that subjects could not use a simple strategy based on only the relative 2D velocities of the points to perform the task. In particular, the point of lightest mass was not always the point with the fastest average 2D speed. The pilot data suggest that subjects can determine well above chance which of the three points has intermediate mass. This result indicates that human vision might well use dynamical constraints for the interpretation of motion. It also suggests that further theoretical analyses should be pursued, along the lines of the analysis presented here but relaxing the assumption that all the points have equal mass. (Some other psychophysical studies also suggest that subjects can infer information about the relative masses of colliding disks just from their 2D motions [38]–[41]. Such experiments are, of course, quite different from the one just proposed, but their positive results can be taken as encouraging: Perhaps relative masses can be inferred as well from displays of structure from motion.)

Human vision might make assumptions about the general form of the inertia tensor. For example, it would be convenient to assume that the body has an axis of symmetry, so that the inertia tensor has a twofold degeneracy (two of the eigenvalues are equal). In this case one can show that Poinsot motion of the body has constant *magnitude* of angular velocity. Therefore one could pursue an analysis based on constraint equations (14)–(16) and, instead of equation (17), use the following equation, which states that the magnitude of the angular velocity is constant:

$$\begin{aligned}\overline{\mathbf{r}_{12}} \cdot \overline{\mathbf{r}_{11}} &+ (\overline{\mathbf{r}_{12} \times \mathbf{r}_{22}}) \cdot (\overline{\mathbf{r}_{11} \times \mathbf{r}_{21}}) \\ &+ [\overline{(\mathbf{r}_{12} \times \mathbf{r}_{22}) \times \mathbf{r}_{12}}] \cdot [\overline{(\mathbf{r}_{11} \times \mathbf{r}_{21}) \times \mathbf{r}_{11}}] \\ &= \overline{\mathbf{r}_{13}} \cdot \overline{\mathbf{r}_{12}} + (\overline{\mathbf{r}_{13} \times \mathbf{r}_{23}}) \cdot (\overline{\mathbf{r}_{12} \times \mathbf{r}_{22}}) \\ &\quad + [\overline{(\mathbf{r}_{13} \times \mathbf{r}_{23}) \times \mathbf{r}_{13}}] \cdot [\overline{(\mathbf{r}_{12} \times \mathbf{r}_{22}) \times \mathbf{r}_{12}}] \end{aligned} \tag{81}$$

A number of empirical studies suggest that axes of symmetry (local and global) are important in the visual perception of motion [3], [4], [42], [43] and in mental rotations of mental images [44]–[47].

One might just drop equations (14)–(16) altogether, i.e., drop the assumption of rigidity, and see what can be inferred about 3D structure and motion by using the above equation alone or by using the more general equation (17). There are many directions to go in pursuing dynamical, as opposed to kinematical, constraints in the perception of structure from motion.

One interesting consequence of pursuing dynamical constraints is that one automatically gets 3D interpretations in which the motion is smooth. If one just uses the kinematical constraint of rigid motion, then an object can undergo arbitrary accelerations and jerks from frame to frame and still satisfy the rigidity constraint. The same is true for a fixed-axis motion constraint or a planar motion constraint. However, the human visual perception of 3D structure is greatly impaired for displays involving such jerks and accelerations, even when care is taken to avoid any problems due to failure of point correspondence [15] from frame to frame. Human vision seems to prefer smooth interpretations of the motion; dynamical constraints such as Poinsot motion may provide just the right notion of smoothness.

Appendix: Some Results from Algebraic Geometry

We now briefly review some basic terminology and facts from algebraic geometry that are used in the proof of Theorem 13. We work first with the complex numbers **C**, even though our ultimate interest is in solutions to equations over the real numbers **R**. For any positive integer n, $\mathbf{C}^n$ denotes the set of ordered n-tuples of complex numbers; we call it n-dimensional complex *affine space*. The usual coordinates on this space are called *affine coordinates*. $\mathbf{P}^n(\mathbf{C})$ denotes n-dimensional complex *projective space*. By definition, the points of $\mathbf{P}^n(\mathbf{C})$ are the *lines* (1D complex linear subspaces) through the origin in $\mathbf{C}^{n+1}$. The ordinary coordinates on this $\mathbf{C}^{n+1}$ are called *homogeneous coordinates* for $\mathbf{P}^n(\mathbf{C})$. Thus the homogeneous coordinates of a point in $\mathbf{P}^n(\mathbf{C})$ are specified only up to scalar multiplication. We note that the origin in $\mathbf{C}^{n+1}$ does not, by itself, correspond to any point of $\mathbf{P}^n(\mathbf{C})$.

We are interested in solutions of polynomial equations on affine and projective spaces. Given a collection of polynomials in the affine coordinates of $\mathbf{C}^n$, the locus of points in $\mathbf{C}^n$ where these polynomials vanish is called the *affine (algebraic) variety* determined by the polynomials. Similarly, given a collection of homogeneous polynomials in the coordinates of $\mathbf{C}^{n+1}$ (a polynomial is homogeneous if all its monomials have the same total degree), there is a well-defined set of lines through the origin on which these polynomials vanish. The corresponding set of points in $\mathbf{P}^n(\mathbf{C})$ is called the *projective variety* determined by the polynomials.

Let V be a variety, affine or projective. In any case it can be shown that V is covered by open sets each of which is an affine variety. Now every affine variety U can, by definition, be represented as a set of points in some affine space $\mathbf{C}^n$, as we have described above. In this sense, given any polynomial function on $\mathbf{C}^n$ we can restrict it to U. The functions on U obtained in this manner will be called *polynomial functions on* U. Now if V is an arbitrary variety and f is a function defined locally on V, it is called a polynomial function on V if it is a polynomial function on each affine open set U of V contained in its domain of definition.

If W is a variety, a subset $W' \subset W$ is called a *closed subvariety* of W if there exist, locally, polynomial functions $f_1, \ldots, f_n$ on W such that $W' = \{w \in W \mid f_1(w) = \cdots = f_n(w) = 0\}$. A variety W is called *irreducible* if whenever W' and W'' are closed subvarieties of W such that $W = W' \cup W''$, then $W = W'$ or $W = W''$.

Let V, W be any varieties. V and W may be affine, projective, or suitable open subsets of affine or projective varieties. A mapping

$\varphi : V \to W$ is called a *morphism* if for any polynomial function g on W, $g \circ \varphi$ is a polynomial function on V. A morphism φ is called *projective* if V is representable as a closed subvariety of $W \times \mathbf{P}^n(\mathbf{C})$ (for some n) in such a way that φ is induced by the projection of $W \times \mathbf{P}^n(\mathbf{C})$ onto W. We can then think of V as a family of projective varieties $\{V_w\}$ in $\mathbf{P}^n(\mathbf{C})$ parametrized by the points of W, where V_w is $\varphi^{-1}(w) \subset V$. A morphism φ is called a *finite morphism* if the polynomial functions on V are locally obtained from the polynomial functions on W by adjoining finitely many new functions, each of which satisfies a monic polynomial whose coefficients are polynomial functions on W. This will be true, for example, if the new functions are pth roots (for some p) of polynomial functions on W. A morphism φ is called *quasi finite* if $\varphi^{-1}(w)$ is a finite set for all w in W. Finite morphisms are quasi finite, but the converse is false in general.

Our proof will use the following:

FACT A1. The composition of projective morphisms is projective [33, II, Exer. 4.9].

FACT A2. The image of a projective morphism $\varphi : V \to W$ is a closed subvariety of W [33, II, Thm. 4.9].

FACT A3. A finite morphism is projective. (This follows directly from the definition of finite morphism given above.)

FACT A4. A quasi-finite projective morphism is finite [33, III, Cor. 11.5].

One of our main tools is the following:

THEOREM A5. Suppose $\varphi : V \to W$ is a finite morphism. Then for any integer n the set

$$\{w \in W \mid \varphi^{-1}(w) \text{ has at least } n \text{ points}\}$$

is a closed subvariety of W.

Sketch of Proof. This theorem may be stated equivalently as follows: Let $N : W \to \mathbf{Z}$ be the function defined by $N(w)$ = number of points in V_w; then N is upper semicontinuous for the Zariski topology of W. It follows directly from the definitions that if $\varphi : V \to W$ is a finite morphism, then $\varphi_* \mathcal{O}_V$ is a *coherent* $\mathcal{O}_W$ module. By [33, II, Exer. 5.8] the function $w \mapsto \mathrm{rk}_w \varphi_* \mathcal{O}_V$ is then Zariski upper semicontinuous on W, where $\mathrm{rk}_w \varphi_* \mathcal{O}_V$ denotes the rank at w of the $\mathcal{O}_W$-module $\varphi_* \mathcal{O}_V$. Finally, we conclude with the fact that if $\varphi : V \to W$ is a finite morphism, $\mathrm{rk}_w \varphi_* \mathcal{O}_V$ is equal to the number of points in $\varphi^{-1}(w)$ (counted with multiplicities).

We will also need some facts about real algebraic varieties. A real affine variety V is a subset of $\mathbf{R}^n$ defined by the vanishing of a collection of polynomials in n variables with real coefficients; V is also called an *algebraic set.* The notion of a *polynomial function on* V is defined just as in the complex case, except that now we consider only real polynomials. Similarly, we define a morphism $\varphi : V \to W$ of real varieties to be a map that associates polynomial functions on W to polynomial functions on V. We note that if U is any complex variety in $\mathbf{C}^n$, the set of real points of U (i.e., $U \cap \mathbf{R}^n$) is a real variety; in fact, it is defined in $\mathbf{R}^n$ by the real and imaginary parts separately of the complex polynomials defining U in $\mathbf{C}^n$. $\mathbf{R}^n$ itself is a variety defined by the polynomial that is identically 0. A *semialgebraic* set in $\mathbf{R}^n$ is a subset defined by a collection of polynomial inequalities and equalities, i.e., by a collection of relations of the form

$$f_i = 0, \quad g_j \geq 0, \quad h_k > 0,$$

where the $\{f_i\}$, $\{g_j\}$, and $\{h_k\}$ are finite sets of polynomials.

FACT A6. If $\varphi : \mathbf{R}^{n_1} \to \mathbf{R}^{n_2}$ is a morphism and $V \subset \mathbf{R}^{n_1}$ is a variety, then $\varphi(V)$ is a semialgebraic set in $\mathbf{R}^{n_2}$. (This is the famous theorem of Tarski–Seidenberg; see, e.g., [48, section 2.2.8] or [49].

If $A \subset \mathbf{R}^n$ is a set, the *Zariski closure of* A, denoted $Z(A)$, is the smallest algebraic set containing A; it exists because the intersection of any collection (finite or infinite), of algebraic sets is again an algebraic set. $Z(A)$ is the variety defined by all those polynomals that vanish on A. An algebraic set V is *irreducible* if for any algebraic sets W_1 and W_2, $V = W_1 \cup W_2 \Longrightarrow V = W_1$ or $V = W_2$. A semialgebraic set S in $\mathbf{R}^n$ is irreducible if $Z(S)$ is irreducible; this is so if and only if the polynomials that vanish on S form a *prime ideal* in the ring $\mathbf{R}[x_1, \ldots, x_n]$ of polynomial functions on $\mathbf{R}^n$[48, section 2.8.3]. It follows from this and Fact A6 that the following is true:

FACT A7. If $\varphi : \mathbf{R}^n \to \mathbf{R}^m$ is a projection morphism and A is an irreducible semialgebraic set of $\mathbf{R}^n$, then $\varphi(A)$ is an irreducible semialgebraic set.

Let V be an algebraic set in $\mathbf{R}^n$, and let $x \in V$. V is *nonsingular of dimension d at x* if there is a neighborhood U of x in $\mathbf{R}^n$ and if there are $n-d$ polynomials $f_1, \ldots, f_{n-d}$ such that $V \cap U = \{u \in U \mid f_1(u) = \cdots = f_{n-d}(u) = 0\}$ and the gradients $\nabla f_i(x)$, $i = 1, \ldots, n-d$, are linearly independent. A variety is nonsingular if it is nonsingular at every point. We need the following facts:

FACT A8. A nonsingular, connected variety is irreducible. (This follows from [50, section 2.2.6.]).

From this we get

FACT A9. A connected algebraic group is irreducible.

We will also need the following:

FACT A10. The product of irreducible varieties is irreducible [48, section 2.8.3.].

FACT A11. Suppose S is an irreducible semialgebraic set in $\mathbf{R}^n$ and W an algebraic set in $\mathbf{R}^n$. Suppose $W \cap S$ is properly contained in S. Then $\dim(W \cap S) < \dim S$.

To prove this let $V = Z(S)$. Then $W \cap S \underset{\neq}{\subset} V$. But then $\dim(W \cap V) < \dim V$ by [50, section 2.2.9] (which asserts that Fact A11 holds for the special case that S is an algebraic set). Now $Z(W \cap S) \subset W \cap V$, so that $\dim Z(W \cap S) \leq \dim W \cap V < \dim V$, i.e., $\dim Z(W \cap S) < \dim Z(S)$. We conclude using the fact that $\dim(S) = \dim Z(S)$ for any semialgebraic set S [48, section 2.8.2].

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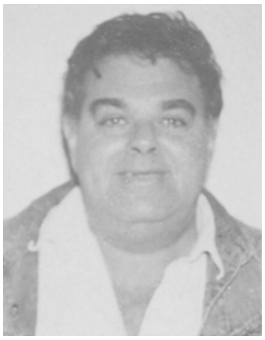
References

1. J. Aloimonos and A. Bandyopadhyay, "Perception of structure from motion: Lower bound results," Tech. Report 158, Department of Computer Science, University of Rochester, Rochester, NY, 1985.
2. B.M. Bennett and D.D. Hoffman, "The computation of structure from fixed-axis motion: Nonrigid structures," *Biol. Cybernet.*, vol. 51, 1985, pp. 293–300.
3. E.H. Carlton and R.N. Shepard, "Psychologically simple motions as geodesic paths: I. Asymmetric objects," *J. Math. Psychol.*, vol. 34, 1990, pp. 127–188.
4. E.H. Carlton and R.N. Shepard, "Psychologically simple motions as geodesic paths: II. Symmetric objects," *J. Math. Psych.*, vol. 34, 1990. pp. 189–228.
5. O.D. Faugeras and S. Maybank, "Motion from point matches: Multiplicity of solutions," *Internat. J. Comput. Vis.*, vol. 4, 1990, pp. 225–246.
6. N. Grzywacz and E. Hildreth, "Incremental rigidity scheme for recovering structure from motion: Position-based versus velocity-based formulations," *J. Opt. Soc. Am. A*, vol. 4, 1987, pp. 503–518.
7. D.D. Hoffman and B.M. Bennett, "Inferring the relative three-dimensional positions of two moving points," *J. Opt. Soc. Am. A*, vol. 2, 1985, pp. 350–353.
8. D.D. Hoffman and B.M. Bennett, "The computation of structure from fixed-axis motion: Rigid structures," *Biol. Cybernet.*, vol. 54, 1986, pp. 71–83.
9. D.D. Hoffman and B.E. Flinchbaugh, "The interpretation of biological motion," *Biol. Cybernet.*, vol. 42, 1982, pp. 197–204.
10. T. Huang and C. Lee, "Motion and structure from orthographic projections," *IEEE Trans. Patt. Anal. Mach. Intell.*, vol. 11, 1989, pp. 536–540.
11. J. Koenderink and A. van Doorn, "Depth and shape from differential perspective in the presence of bending deformations," *J. Opt. Soc. Am. A*, vol. 3, 1986, pp. 242–249.
12. J. Koenderink and A. van Doorn, "Invariant properties

of the motion parallax field due to the movement of rigid bodies relative to an observer," *Opt. Acta*, vol. 22, 1975, pp. 773–791.
13. E. Kruppa, "Zur Ermittlung eines Objektes aus zwei Perspektiven mit innerer Orientierung," *Akad. Wiss. Wien Math. Naturwiss. Kl. Sitzungsberichte*, vol. 122, 1913, pp. 1939–1948.
14. H.C. Longuet-Higgins, "A computer algorithm for reconstructing a scene from two perspective projections," *Nature*, vol. 293, 1981, pp. 133-135.
15. S. Ullman, *The Interpretation of Visual Motion*, MIT Press, Cambridge, MA, 1979.
16. A. Waxman and K. Wohn, "Contour evolution, neighborhood deformation, and image flow: Textured surfaces in motion," in *Image Understanding 1985–1986*, W. Richards and S. Ullman, eds., Ablex, Norwood, NJ, 1987, pp. 72–98.
17. J.A. Webb and J.K. Aggarwal, "Structure from motion of rigid and jointed objects," *Artif. Intell.*, vol. 19, 1982, pp. 107–130.
18. M.L. Braunstein, D.D. Hoffman, and F.E. Pollick, "Discriminating rigid from nonrigid motion: Minimum points and views," *Percept. Psychophys.*, vol. 47, 1990, pp. 205–214.
19. M. Braunstein, D. Hoffman, L. Shapiro, G. Andersen, and B. Bennett, "Minimum points and views for the recovery of three-dimensional structure," *J. Exp. Psychol. Hum. Percept.*, vol. 13, 1987, pp. 335–343.
20. J.J. Gibson and E.J. Gibson, "Continuous perspective transformations and the perception of rigid motion," *J. Exp. Psychol.*, vol. 54, 1957, pp. 129–138.
21. B. Green, "Figure coherence in the kinetic depth effect," *J. Exp. Psychol.*, vol. 62, 1961, pp. 272–282.
22. J.S. Lappin, J.F. Donner, and B. Kottas, "Minimal conditions for the visual detection of structure and motion in three dimensions," *Science*, vol. 209, 1980, pp. 717–719.
23. V.S. Ramachandran, S. Cobb, and D. Rogers-Ramachandran, "Perception of 3-D structure from motion: The role of velocity gradients and segmentation boundaries," *Percept. Psychophys.*, vol. 44, 1988, pp. 390–393.
24. J.T. Todd, R.A. Akerstrom, F.D. Reichel, and W. Hayes, "Apparent rotation in three-dimensional space: Effects of temporal, spatial, and structural factors," *Percept. Psychophys.*, vol. 43, 1988, pp. 179–188.
25. H. Wallach and D. O'Connell, "The kinetic depth effect," *J. Exp. Psychol.*, vol. 45, 1953, pp. 205–217.
26. H. Goldstein, *Classical Mechanics*, Addison-Wesley, Reading, MA, 1980.
27. K. Symon, *Mechanics*, Addison-Wesley, Reading, MA, 1971.
28. L. Poinsot, "Theorie nouvelle de la rotation des corps," *J. Liouville*, vol. 16, 1851.
29. A. Gray, *A Treatise on Gyrostatics and Rotational Motion*, Dover, New York, 1959.
30. W.D. Macmillan, *Theoretical Mechanics: Dynamics of Rigid Bodies*, Dover, New York, 1960.
31. E.J. Routh, *Dynamics of a System of Rigid Bodies: Advanced Part*, Macmillan, London, 1892.
32. A.G. Webster, *The Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies*, Stechert-Hafner, New York, 1920.
33. R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.
34. A. Grothendieck, "Etude cohomologique des faisceaux cohérents," *Publ. Math. IHES*, vol. 11, 1961; vol. 17, 1963.
35. W. Fulton, *Algebraic Curves: An Introduction to Algebraic Geometry*, Addison-Wesley, New York, 1989.
36. B.M. Bennett, D.D. Hoffman, and C. Prakash, *Observer Mechanics*, Academic Press, New York, 1989.
37. B.M. Benett, D.D. Hoffman, and C. Prakash, "Unity of perception," *Cognition*, vol. 38, 1991, pp. 295–334.
38. D.L. Gilden and D.R. Proffitt, "Understanding collision dynamics," *J. Exp. Psychol. Hum. Percept.*, vol. 15, 1989, pp. 372–383.
39. M.K. Kaiser and D.R. Proffitt, "The development of sensitivity to causally relevant dynamic information," *Child Dev.*, vol. 55, 1984, pp. 1614–1624.
40. M.K. Kaiser and D.R. Proffitt, "Observers' sensitivity to dynamic anomalies in collisions," *Percept. Psychophys.*, vol. 42, 1987, pp. 275–280.
41. J.T. Todd and W.H. Warren, "Visual perception of relative mass in dynamic events," *Perception*, vol. 11, 1982, pp. 325–335.
42. M. McBeath and R.N. Shepard, "Apparent motion between shapes differing in location and orientation: A window technique for estimating path curvature," *Percept. Psychophys.*, vol. 46, 1989, 333–337.
43. D.R. Proffitt, D. Gilden, M.K. Kaiser, and S. Whelan, "The effect of configural orientation on perceived trajectory in apparent motion," *Percept. Psychophys.*, vol. 45, 1988, pp. 465–474.
44. M.A. Just and P.A. Carpenter, "Cognitive coordinate systems: Accounts of mental rotation and individual differences in spatial abilities," *Psychol. Rev.*, vol. 92, 1985, pp. 137–192.
45. L.M. Parsons, "Imagined spatial transformations of one's hand and feet," *Cog. Psychol.*, vol. 19, 1987, pp. 178–241.
46. L.M. Parsons, "Imagined spatial transformations of one's body," *J. Exp. Psychol. Gen.*, vol. 116, 1987, pp. 172–191.
47. R.N. Shepard, *Mental Images and Their Transformations*, MIT Press, Cambridge, MA, 1982.
48. J. Bochnak and M. Coste, M. Roy, *Géométrie algébrique réele*, Springer-Verlag, New York, 1987.
49. A. Seidenberg, "A new decision method for elementary algebra," *Ann. Math.*, vol. 60, 1954, pp. 365–374.
50. S. Akbulut and H. King, *Topology of Real Algebraic Sets*, in press.



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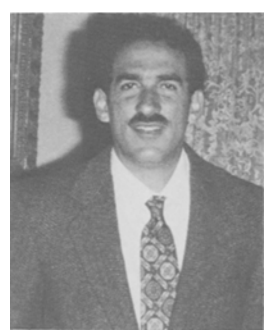

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