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Kurt Gödel

COLLECTED WORKS

Volume III
Unpublished essays and lectures

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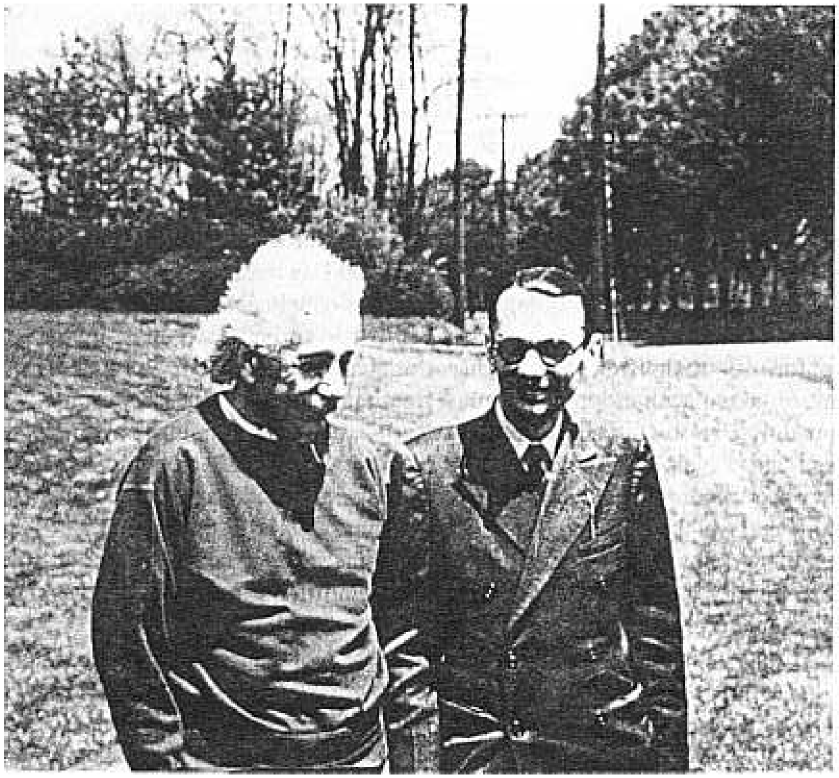
Prepared under the auspices of the Association for Symbolic Logic

New York Oxford
OXFORD UNIVERSITY PRESS
1995



Gödel and Einstein, 13 May 1947

Oskar Morgenstern

Introductory note to **1949b*

This item is the slightly edited text of a lecture that Gödel presented at the Institute for Advanced Study on 7 May 1949. It has been prepared from a handwritten manuscript (in English) in the Gödel *Nachlass*.

1. Introductory remarks

In his lecture, Gödel exhibits and discusses the properties of his new exact solution to Einstein's equation. The solution represents a possible universe, compatible with the constraints of relativity theory, in which aggregate matter (on a cosmological scale) is in a state of uniform, rigid rotation. Gödel's presentation here covers much the same ground as does the published account (*1949*) that appeared two months later. But the tone throughout is more relaxed and expansive, and there are a number of specific additions. They include the following:

(a) At the very beginning of the lecture, Gödel exhibits a Newtonian cosmological model in which, as in his own model, aggregate matter is in a state of uniform rigid rotation. In so doing, he partially anticipates later work by Heckmann and Schücking (*1955*) on Newtonian analogues of Gödel space-time. (See also *Lathrop and Teglas 1978.*)

(b) He discusses what it can mean to say that the universe in its entirety is rotating and notes the incompatibility of general relativity with (at least some versions of) "Mach's principle".

(c) He explains a generic connection in general relativity between rotation and temporal structure, and notes that it was this connection that first led him to look for "rotating solutions".

(d) Gödel gives an account of what he calls the "geometric meaning" of his solution that is perhaps more accessible, because less compressed, than the version in *1949*. (For an elegant redescription see *Chakrabarti, Geroch, and Liang 1983*.)

(e) He makes several remarks that help one visualize the configuration of light-cones in Gödel space-time and visualize its time-like and null geodesics. (As represented in a standard space-time diagram, the latter are helices of bounded radius that cyclically intersect the [vertical] worldlines of the major mass points. Thus free particles and light rays exhibit a kind of boomerang effect. See *Kundt 1956*, *Chandrasekhar and Wright 1961*, and *Lathrop and Teglas 1978*.)

(f) He asserts in passing the existence of yet other solutions to Einstein's equation representing possible universes in which (i) aggregate matter is rotating, but no closed timelike curves are present; (ii) aggregate matter is expanding as well as rotating. (Gödel sketches an existence proof in *1952*, but he does not exhibit any solutions explicitly.)

Gödel's remarks (c) concerning how he first came to look for "rotating solutions" are of particular biographical interest. He makes clear that his technical investigations were driven, at least initially, by antecedent philosophical interests. In what follows, I'll make a number of non-technical explanatory remarks about the generic connection between rotation and temporal structure that is at issue here, and then add a few details to Gödel's account (d) of the "geometrical meaning" of his solution.

2. Cosmic rotation and "objective" time

Gödel came to look for solutions to Einstein's equation representing rotating universes in the course of trying to bolster an argument that relativity theory supports a particular conception of time, one that he identified with Kant and certain species of "idealist philosophy". It is that argument, in its fully bolstered form, that one finds in *1949a*.

What links Kant and relativistic physics, according to Gödel, is a common denial of the "objective existence of ... time in the Newtonian sense". In the case of relativistic physics, of course, denial is supposed to follow from the relativity of simultaneity (and of temporal relations more generally). But Gödel himself raises a possible difficulty with this position.

To be sure, relativity theory teaches us that it does not make sense to speak of simultaneity until one relativizes consideration to particular individuals. (It would be more appropriate to speak of relativization to time-like curves or worldlines. It is not essential that they be "traversed" by conscious agents, or any objects for that matter.) But in the context of cosmology, at least, one is given at the outset a distinguished class of worldlines, namely, those of the major mass points of the universe (stars or galaxies). And so it is natural to understand all attributions of temporal structure as relativized to that class. This is in fact done all the time. It is done, for example, by cosmologists when they talk about the "first three minutes" of the universe. In this context, therefore, one does recover an "objective", in the sense of uniquely distinguished, temporal structure.

Gödel's search for "rotating solutions" was prompted by a desire to counter this objection. It turns out that in relativity theory it is a

highly contingent matter whether one can talk about simultaneity in any natural sense *even after relativizing consideration to a particular family of worldlines.* Gödel apparently understood this early on, and undertook to find a cosmological model, unlike all others previously discovered, in which the worldlines of the major mass points would not support a natural notion of relative simultaneity. But this was tantamount to looking for a rotating universe, because, and this is the crucial point, *in a cosmological model, non-rotation of the major mass points is precisely the necessary and sufficient condition for there to exist a natural notion of simultaneity relative to their worldlines.* This is the generic connection between rotation and temporal structure that Gödel discusses in his lecture.

What does one mean by a "natural" notion of relative simultaneity? Consider a congruence of worldlines, and a spatial slice (i.e., a three-dimensional space-like hypersurface) that intersects the congruence. Standardly, at least, one construes the latter as a "simultaneity slice" relative to the former if and only if the slice is everywhere orthogonal to the individual worldlines of the congruence (with respect to the background space-time metric). This identification of relative simultaneity with orthogonality is taken for granted in standard presentations of "special relativity" and in discussions of cosmology. Moreover, one can easily show that the identification is forced if one requires of any candidate notion of relative simultaneity that it respect certain weak symmetry conditions.

In any case, this is the identification Gödel has in mind when he speaks of "a very natural definition of simultaneity independent of coordinates". And the italicized assertion above can be captured, more precisely, this way: in a cosmological model, the congruence of worldlines of the major mass points is twist-free (i.e., has everywhere vanishing rotation) if and only if the congruence admits an orthogonal foliation (i.e., if there exists a one-parameter family of slices everywhere orthogonal to the worldlines).

The assertion is a special case of Frobenius' theorem. (See *Wald 1984*, page 434.) One can make it at least intuitively plausible by considering an analogue. Think about an ordinary rope. In its natural twisted state, the rope cannot be sliced in such a way that the slice is orthogonal to all fibers. But if the rope is first untwisted, such a slicing is possible. Thus orthogonal sliceability is equivalent to fiber untwistedness. The assertion above merely extends this intuitive equivalence to the four-dimensional "space-time ropes" (i.e., congruences of worldlines) determined by the major mass points of the universe.

By finding the first cosmological model in relativity theory in which the major mass points are in a state of rotation, Gödel found the first one in which there is no natural, distinguished notion of temporal structure, not even one determined by relativization to the worldlines of the major mass points. He could make no claim that our universe, in fact, has this property. But he could hold that out as a possibility; our universe just might exhibit some very small net rotation. And even if it does not, he could claim that, at least so far as relativity theory is concerned, it is an entirely accidental matter that it does not.

3. The "geometrical meaning" of Gödel space-time

Gödel's account of the "geometrical meaning" of his solution is not so terse as that in *1949*, but it does still leave out many details. Here we add just a few, and make explicit the connection between Gödel's abstract characterization of his metric and his two representations involving particular coordinate systems.

First, consider the coordinate-free characterization. Quite generally, a relativistic space-time may be taken to be a structure (M, g_{ik}), where M is a connected, smooth, four-dimensional manifold, and g_{ik} is a smooth semi-Riemannian metric on M of signature $(+,-,-,-)$. In Gödel space-time M is $\mathbb{R}^4$, and (M, g_{ik}) can be decomposed as a metric product of $\mathbb{R}$ (with its usual positive definite metric) and a structure $(\mathbb{R}^3, h_{ik})$, where h_{ik} has signature $(+,-,-)$. Any discussion of the "geometrical meaning" of Gödel's solution concerns the latter. The metric h_{ik} is of form

$$h_{ik} = h'_{ik} + t_i t_k$$

where

1. h'_{ik} is a complete metric on $\mathbb{R}^3$ of signature $(+,-,-)$ and constant positive curvature; and
2. $t^i = h'^{ik} t_k$ is a unit time-like Killing field with respect to h'_{ik}.

(In Gödel's presentation of his metric, he carries a free scale parameter a. The corresponding constant curvature of h'_{ik} comes out to be $(1/4a^2)$.)

The transition from h'_{ik} to h_{ik} is what Gödel describes as "stretching the metric in the ratio $\sqrt{2}$ in a direction of Clifford parallels". The "Clifford parallels" here are the integral curves of the Killing field t^i. Notice that (i) h'_{ik} and h_{ik} agree in their determinations of whether vectors are orthogonal to t^i; (ii) they agree in the squared lengths they assign to vectors orthogonal to t^i; and (iii) whereas t^i has length 1 with respect to h'_{ik}, it has length $\sqrt{2}$ with respect to h_{ik}.

We can easily recover this abstract characterization of the Gödel metric starting from either of the two coordinate-dependent expressions he exhibits in the lecture. Consider the first:

$$a^2[dx_0^2 - dx_1^2 + (1/2)e^{2x_1}dx_2^2 - dx_3^2 + 2e^{x_1}dx_0dx_2].$$

Here the coordinates $x_0, \ldots, x_3$ range over all of $\mathbb{R}$. We arrive at the structure $(\mathbb{R}^3, h_{ik})$ simply by dropping the term $-dx_3^2$ and restricting the reduced metric to any hyperplane of constant x_3 value. This reduced metric can be cast in the form

$$a^2[(1/2)dx_0^2 + e^{x_1}dx_0dx_2 - dx_1^2] + (a^2/2)[dx_0 + e^{x_1}dx_2]^2.$$

Let h'_{ik} be the metric determined by the first term

$$a^2[(1/2)dx_0^2 + e^{x_1}dx_0dx_2 - dx_1^2],$$

and let t_i be the field $(a/\sqrt{2})(dx_0 + e^{x_1}dx_2)$ determined by the second. Then the inverse h'^{ik} is given by

$$\left(\frac{1}{a^2}\right)[-2e^{-2x_1}\left(\frac{\partial}{\partial x_2}\right)^2 + 4e^{-x_1}\left(\frac{\partial}{\partial x_0}\right)\left(\frac{\partial}{\partial x_2}\right) \quad \left(\frac{\partial}{\partial x_1}\right)^2],$$

and $t^i = h'^{ik}t_k$ comes out to be, simply, $(\sqrt{2}/a)(\partial/\partial x_0)$. Clearly t^i has unit length with respect to h'_{ik} and is a Killing field with respect to that metric (since x_0 does not appear in any of the coefficients of h'_{ik}).

It remains to show that $(\mathbb{R}^3, h'_{ik})$ is a complete manifold with constant curvature $(1/4a^2)$. To do so we define a map

$$(x_0, x_1, x_2) \mapsto (u_0, u_1, u_2, u_3)$$

from $\mathbb{R}^3$ into $\mathbb{R}^4$ by setting

$$\begin{aligned}
u_0 &= 2a[\cos(x_0/2\sqrt{2})\cosh(x_1/2) - (1/2\sqrt{2})x_2\, e^{x_1/2}\sin(x_0/2\sqrt{2})]\\
u_1 &= 2a[\sin(x_0/2\sqrt{2})\cosh(x_1/2) + (1/2\sqrt{2})x_2\, e^{x_1/2}\cos(x_0/2\sqrt{2})]\\
u_2 &= 2a[-\sin(x_0/2\sqrt{2})\sinh(x_1/2) + (1/2\sqrt{2})x_2\, e^{x_1/2}\cos(x_0/2\sqrt{2})]\\
u_3 &= 2a[\cos(x_0/2\sqrt{2})\sinh(x_1/2) + (1/2\sqrt{2})x_2\, e^{x_1/2}\sin(x_0/2\sqrt{2})].
\end{aligned}$$

A straightforward computation determines that

$$\begin{aligned}
u_0^2 + u_1^2 - u_2^2 - u_3^2 &= 4a^2\\
du_0^2 + du_1^2 - du_2^2 - du_3^2 &= a^2[(1/2)dx_0^2 + e^{x_1}dx_0dx_2 - dx_1^2]
\end{aligned}$$

at every point in the image of the map. Moreover, one can verify that the map is a diffeomorphism if one restricts x_0, say, to the interval $[-\pi, \pi)$. Indeed, under that restriction one can explicitly solve for the x coordinates in terms of the u coordinates; e.g.,

$$x_0 = 2\sqrt{2}\,\arccos[(u_0 + u_3)/((u_0 + u_3)^2 + (u_2 - u_1)^2)^{1/2}].$$

Thus we see that $(\mathbb{R}^3, h'_{ik})$ is an isometric covering space of the manifold

$$H = \{(u_0, \quad , u_3) \in \mathbb{R}^4 \mid u_0^2 + u_1^2 - u_2^2 - u_3^2 = 4a^2\}$$

with respect to the metric induced on H by the background flat metric on $\mathbb{R}^4$ of signature $(+,+,-,-)$. It is a standard result that the latter is a complete manifold of constant curvatûre $(1/4a^2)$. (See, for example, *O'Neill 1983*, page 113.)

We can execute very much the same computational argument starting from Gödel's second representation of his metric (in cylindrical coordinates):

$$4a^2[dt^2 - dr^2 - dy^2 + (sh^4 r - sh^2 r)d\varphi^2 + 2\sqrt{2}\ sh^2 r\ d\varphi\ dt].$$

If we drop the term $-dy^2$ and regroup the other terms, we arrive at the expression

$$4a^2[(1/2)dt^2 - dr^2 - \sinh^2 r\ d\varphi^2 + \sqrt{2}\sinh^2 r\ d\varphi\ dt] \\ + 2a^2[dt + \sqrt{2}\sinh^2 r\ d\varphi]^2$$

for h_{ik}. If we use the two terms to define, respectively, the metric h'_{ik} and the field t_i, then again a simple computation establishes that $t^i = h'^{ik} t_k$ is a unit time-like Killing field with respect to h'_{ik}. (It comes out to be $(1/a\sqrt{2})(\partial/\partial t)$.) To map $(\mathbb{R}^3, h'_{ik})$ isometrically onto H (with its induced metric) we set

$$\begin{aligned} u_0 &= 2a\ \cos(t/\sqrt{2})\cosh r \\ u_1 &= 2a\ \sin(t/\sqrt{2})\cosh r \\ u_2 &= 2a\sinh r\ \sin(\varphi - (t/\sqrt{2})) \\ u_3 &= 2a\sinh r\ \cos(\varphi - (t/\sqrt{2})). \end{aligned}$$

To gain further insight into the two displayed maps of $(\mathbb{R}^3, h'_{ik})$ onto H, we can recast them, making use of the fact that H has a natural Lie group structure.

Consider the four-dimensional (associative, distributive) algebra of what Gödel calls the "hyperbolic quaternions". Its elements can be construed as vectors

$$\varphi = w_0 + w_1 j_1 + w_2 j_2 + w_3 j_3$$

with real coefficients $w_0, \quad , w_3$. Multiplication is defined by the requirement that 1 serve as an identity element, and by the table

$$\begin{aligned} j_1^2 = -1 \quad j_2^2 &= j_3^2 = 1 \\ j_1 \cdot j_2 = -j_2 \cdot j_1 &= j_3 \\ j_2 \cdot j_3 = -j_3 \cdot j_2 &= -j_1 \\ j_3 \cdot j_1 = -j_1 \cdot j_3 &= j_2. \end{aligned}$$

If we define the conjugate and norm of φ by setting

$$\begin{aligned} \overline{\varphi} &= w_0 - w_1 j_1 - w_2 j_2 - w_3 j_3 \\ \text{norm}(\varphi) &= \varphi \cdot \overline{\varphi} = w_0^2 + w_1^2 - w_2^2 - w_3^2, \end{aligned}$$

then it follows easily that for all φ and ψ, $\overline{\varphi \cdot \psi} = \overline{\psi} \cdot \overline{\varphi}$ and hence

$$\text{norm}(\varphi \cdot \psi) = \text{norm}(\varphi)\ \text{norm}(\psi).$$

To simplify notation now, let us identify the hyperbolic quaternion $w_0 + w_1 j_1 + w_2 j_2 + w_3 j_3$ with the corresponding element $(w_0, \ldots, w_3)$ in $\mathbb{R}^4$. Then H is identified with the set of hyperbolic quaternions of norm $4a^2$, and it acquires a natural Lie group structure: given two elements u and u' in H, we simply take their product to be $(1/4a^2)u \cdot u'$. The norm-product condition above guarantees that the operation is well-defined. The element u has $\overline{u}$ for an inverse.

Notice that for all real numbers x_0, x_1, x_2, the quadruples

$$\begin{aligned} &(\cos x_0, \sin x_0, 0, 0) \\ &(\cosh x_1, 0, 0, \sinh x_1) \\ &(1, x_2, x_2, 0) \end{aligned}$$

all have norm one. So their dot product has norm 1. Straightforward multiplication establishes that the associated map

$$(x_0, x_1, x_2) \mapsto 2a(\cos x_0, \sin x_0, 0, 0) \cdot (\cosh x_1, 0, 0, \sinh x_1) \cdot (1, x_2, x_2, 0)$$

is essentially just the first of the two maps from $(\mathbb{R}^3, h'_{ik})$ onto H displayed above. This is where it "comes from". [To match Gödel's coefficients exactly one has to take the dot product of

$$\begin{aligned} &(\cos(x_0/2\sqrt{2}),\ \sin(x_0/2\sqrt{2}), 0, 0) \\ &(\cosh(x_1/2), 0, 0,\ \sinh(x_1/2)) \\ &(1, x_2/2, x_2/2, 0).] \end{aligned}$$

Notice also that for every fixed element w of norm 1, the map from H to H defined by

$u \mapsto$

is an isometry. (Maps of this form are norm preserving. Hence they preserve the flat metric on $\mathbb{R}^4$ and the induced metric on H.) Moreover, elements of the first type above—$(\cos x_0, \sin x_0, 0, 0)$—form a one-parameter group. So the family of maps $\{\varphi_{x_0}\}_{x_0 \in \mathbb{R}}$ defined by

$$u \mapsto (\cos x_0, \sin x_0, 0, 0) \cdot u$$

$$(t, r, \varphi) \mapsto 2a(\cos(t/\sqrt{2}), \sin(t/\sqrt{2}), 0, 0)\cdot$$
$$(\cosh r, 0, \sinh r \, \sin\varphi, \sinh r \, \cos\varphi)$$

of H.

Consider again the abstract characterization of Gödel's metric. Why should one be interested in a metric of this sort in the first place? As Gödel explains, the answer is that one knows "in advance" that it will provide a solution to Einstein's equation for the case in which the energy-momentum source is a cosmic dust field (or a perfect fluid, if one does not want to allow a non-zero "cosmological constant"). One can show, quite generally, that if h'_{ik} is a three-dimensional metric of constant curvature α (not necessarily positive), t^i is a unit time-like Killing field with respect to h'_{ik}, and h_{ik} is of form

$$h_{ik} = h'_{ik} \quad \beta t_i t_k,$$

then the Ricci tensors corresponding to h'_{ik} and h_{ik} have forms

$$R'_{ik} \quad 2\alpha h'_{ik}$$
$$R_{ik} \quad 2(1-\beta)\alpha h'_{ik} + (2\beta^2\alpha + 6\beta\alpha)t_i t_k.$$

So if one takes $\beta = 1$, the latter reduces to $R_{ik} = 8\alpha t_i t_k$, or to $R_{ik} = 4\alpha t_i t_k$ if one rescales t^i to be of unit length with respect to h_{ik}. Therefore

(working with the rescaled t^i)

$$R_{ik} \quad (1/2)h_{ik}R = 4\alpha t_i t_k - 2\alpha h_{ik}.$$

To cast the right hand side in the desired form

$$8\pi\rho\kappa t_i t_k + \lambda h_{ik}$$

one need only set $\rho = \alpha/2\pi\kappa$ and $\lambda = -2\alpha$. Clearly, the construction works only if the initial constant curvature α is positive (since the gravitational constant κ and the mass density ρ are positive), though the magnitude of α is not constrained.

David B. Malament[a]

[a]This note was written while I was a Fellow at the Center for Advanced Study in the Behavioral Sciences. I am grateful for the financial support provided by the National Science Foundation (#BN587-00864) and the University of Chicago. I am also grateful to Howard Stein for helpful comments on an earlier version.

Lecture on rotating universes
(**1949b*)

A few years ago, in a note in *Nature*, Gamow [*1946*] suggested that the whole universe might be in a state of uniform rotation and that this rotation might explain the observed rotation of the galactic systems. Indeed, if the primordial matter out of which the galaxies were formed by condensation was in a state of rotation, the galaxies themselves will possess a much faster rotation. For in consequence of the law of conservation of angular momentum, their angular velocity will increase as the square of the ratio of contraction. Therefore they will exhibit a rotation even in the coordinate system in which the primordial matter was at rest, that is to say, a rotation in the frame of reference defined by the totality of galaxies. In exactly the same way, the rotation of the galaxies in its turn can be used to explain the rotation of the fixed stars and planetary systems and therewith essentially all rotations occurring in astronomy. Of course, there arises at once the objection that by this theory the axes of rotation of all astro-