DAVID B. MALAMENT†

IN DEFENSE OF DOGMA: WHY THERE CANNOT BE A RELATIVISTIC QUANTUM MECHANICS OF (LOCALIZABLE) PARTICLES

> ... *although it is not a theorem*, it is widely believed that it is impossible to reconcile quantum mechanics and relativity, except in the context of a quantum field theory. A quantum field theory is a theory in which the fundamental ingredients are fields rather than particles; the particles are little bundles of energy in the field. (Weinberg 1987, 78–79; italics added)

IN SOME QUARTERS, AT LEAST, IT COUNTS as the "received view" that there cannot be a relativistic, quantum mechanical theory of (localizable) particles. In the attempt to reconcile quantum mechanics with relativity theory, that is, one is driven to a field theory; all talk about "particles" has to be understood, at least in principle, as talk about the properties of, and interactions among, quantized fields. I want to suggest, today, that it *is* possible to capture this thesis in a convincing "no–go theorem". Indeed, it seems to me that various technical results on the "non–localizability" of particles in (so–called) relativistic quantum mechanics, going back some thirty years, are best understood as versions of such a theorem.[1]

I am well aware that not everyone agrees with this interpretation. Gordon Fleming, who has been centrally involved in this work himself, has an altogether different view of its significance (if I understand him correctly). I shall welcome any comments Gordon may have in the discussion period. But it is not my purpose to have a debate, or try to force agreement. My remarks are going to be largely expository in character,

†I wish to thank Rob Clifton, William Demopoulos, Gordon Fleming, Michael Friedman, Geoffrey Hellman, Simon Saunders, and Abner Shimony for helpful comments.

[1]See Fleming (1965a,b), Hegerfeldt (1974, 1985, 1989), Hegerfeldt and Ruijsenaars (1980), Jancewicz (1977), Perez and Wilde (1977), Ruijsenaars (1981), Schlieder (1971), and Skagerstam (1976).

R. Clifton (ed.), Perspectives on Quantum Reality, 1–10.

and are intended primarily for persons who are not already familiar with the technical results I have in mind.

The particular theorem I am going to discuss is just a bit different in surface formulation than ones I have seen in the literature. The difference is unimportant when it comes to proving the theorem. Standard arguments carry over intact. But it may serve to clarify just how high is the cost of trying to hold on to a particle theory. I want to use the theorem to argue that in attempting to do so (i.e., hold on to a particle theory), one commits oneself to the view that the *act* of performing a particle detection experiment here can statistically influence the *outcome* of such an experiment there, where "here" and "there" are spacelike related. Of course, this sort of "act–outcome" correlation is to be distinguished from the sort that involves only the outcomes of the two experiments.

I have always taken for granted that relativity theory rules out "act–outcome" correlations across spacelike intervals. For that reason, it seems to me that the result *does* bear its intended weight as a "no–go theorem"; it *does* show that there is no acceptable middle ground between ordinary, non–relativistic quantum (particle) mechanics and relativistic quantum field theory. But I have no inclination to argue if someone wants to claim, to the contrary, that the existence of such correlations *is* compatible with "relativity theory" (properly understood). I would like to avoid recent controversy concerning the nature of "causal connection" and "signaling", and concerning just which sorts of "non–locality" are and are not excluded by the theory.[2] What seems to me most important is that there is an empirical issue at stake here — whether Mother Nature *does* allow for act–outcome correlations across spacelike intervals. The point on which I want to stand is this: to whatever extent we have evidence that She does *not* allow such correlations, we have evidence that quantum mechanical phenomena must ultimately be given a field-theoretic interpretation.

Let me now turn to the theorem itself. One has a choice in discussions of particle localization. One can consider either localization in regions of spacetime or in regions of "space", as determined relative to some free–falling observer. I am going to opt for the latter because doing so sharpens a comparison of the classical and relativistic contexts.

Let M be Minkowski spacetime, and let $\mathcal{S}$ be a family of parallel spacelike hyperplanes that cover M. (We will be working throughout

[2]See Maudlin (1994), and references therein.

with this one family. "Hyperplane dependence", such as Gordon Fleming discusses, will play no role here whatsoever.) Let us take a *spatial set* to be any bounded open set within some particular hyperplane in $\mathcal{S}$. (As will become clear, nothing would be lost if we restricted attention to bounded open sets that are, in any desired sense, sufficiently "large". The problem we will be discussing is not that of trying to localize particles in small regions of space, but rather in any bounded regions whatsoever, no matter how large.)

Any candidate for a relativistic, quantum mechanical representation of a single (localizable) particle, presumably, will include the following elements[3]:

(a) a Hilbert space $\mathcal{H}$ (the rays of which represent the pure states of the particle);

(b) an assignment to each spatial set Δ of a projection operator P_Δ on $\mathcal{H}$;

(c) a strongly continuous, unitary representation $\mathbf{a} \mapsto U(\mathbf{a})$ in $\mathcal{H}$ of the translation group in M.

We think of P_Δ as representing the "proposition" or "eventuality" that the particle in question would be found in Δ if a particular detection experiment were performed. We will be interested in four constraints on the structure

$$(\mathcal{H}, \Delta \mapsto P_\Delta, \mathbf{a} \mapsto U(\mathbf{a})).$$

(1) *Translation Covariance Condition:* For all vectors $\mathbf{a}$ in M, and all spatial sets Δ,

$$P_{\Delta + \mathbf{a}} = U(\mathbf{a})\ P_\Delta\ U(-\mathbf{a})$$

(where $\Delta + \mathbf{a}$ is the set that results from translating Δ by the vector $\mathbf{a}$).

[3]These elements will also be included in a classical representation. The difference is this. In the relativistic context, the theory determines a representation $\Delta \mapsto P_\Delta$ for every family of parallel spacelike hyperplanes; we are simply *choosing* one with which to work. In the classical context, there is only one family to consider. ((c), as formulated, makes reference to Minkowski spacetime, but only to its structure as an affine space. So it too is neutral as between the classical and relativistic contexts.)

(2) *Energy Condition:* For all future directed, unit timelike vectors $\mathbf{a}$ in M, if $H(\mathbf{a})$ is the unique self-adjoint ("Hamiltonian") operator satisfying

$$U(t\,\mathbf{a}) = e^{-i\,t\,H(\mathbf{a})},$$

the spectrum of $H(\mathbf{a})$ is bounded below, i.e., there exists a real number $k(\mathbf{a})$ such that $(\varphi, H(\mathbf{a})\,\varphi) \geq k(\mathbf{a})$ for all unit vectors φ in the domain of $H(\mathbf{a})$.

(3) *Localizability Condition:* If Δ_1 and Δ_2 are disjoint spatial sets in a single (common) hyperplane,

$$P_{\Delta_1} P_{\Delta_2} = P_{\Delta_2} P_{\Delta_1} = \mathbf{0}$$

(where $\mathbf{0}$ is the zero operator on $\mathcal{H}$).

(4) *Locality Condition:* If Δ_1 and Δ_2 are spatial sets (not necessarily in the same hyperplane) that are spacelike related

$$P_{\Delta_1} P_{\Delta_2} = P_{\Delta_2} P_{\Delta_1}.$$

I shall comment on these in turn. The "covariance condition" is quite weak. It is formulated solely in terms of the translation group, rather than the full Poincaré group, and is entirely neutral as between a classical and a relativistic framework. The condition bears the following interpretation. Let Δ and $\mathbf{a}$ be given. P_Δ represents the "proposition" that the particle would be found in Δ if a particular detection experiment were performed. We can imagine conducting that experiment, not at its original site, but rather at another that is displaced from the first by the vector $\mathbf{a}$. We understand $P_{\Delta + \mathbf{a}}$ to represent the "proposition" that if this particular *displaced* detection experiment were performed, the particle would be found in $\Delta + \mathbf{a}$.

The second condition asserts that the energy of the particle, as determined by any free-falling observer, has a finite ground state. If it failed, the particle could serve as an infinite energy source (the likes of which we just do not seem to find in nature). Think about it this way. We could first tap the particle to run all the lights in Canada for a week. To be sure, in the process of doing so, we would lower its energy state. Then we could run all the lights for a second week, and lower the energy state of the particle still further. And so on. If the particle had no finite ground state, this process could continue forever. There would never come a stage at which we had extracted all available energy.

The "localizability condition" captures the requirement that the particle cannot be detected in two disjoint spatial sets *at a given time* (as determined relative to our background free falling observer).[4] Notice that the condition does not rule out the possibility that the particle can travel with arbitrarily large finite speed; it only rules out "infinite speed". This condition too is entirely neutral as between the classical and relativistic frameworks.

The "locality condition", in contrast, does impose the stamp of relativity. It is here, by the way, that I have altered slightly the formulation that one standardly finds in the literature. Let Δ_1 and Δ_2 be spatial sets that are spacelike related, but do not necessarily belong to the same hyperplane. Usually it is required that the corresponding projection operators P_{Δ_1} and P_{Δ_2} be orthogonal, i.e., that they satisfy the condition formulated in (3). The intended interpretation is clear. Since relativity theory rules out the possibility that the particle travels at superluminal speed, it should be impossible to detect the particle in two places, spacelike related to one another. I find this stronger condition entirely plausible as a constraint on a candidate for a relativistic theory of a single particle. But *it is just not needed*. It suffices to require that the two operators P_{Δ_1} and P_{Δ_2} commute. This weaker condition carries the following interpretation. Perhaps it *is* possible for the particle to be detected in both places. Still, the probability that it be detected in Δ_2 must be statistically independent of whether a detection experiment is performed at Δ_1, and *vice versa*.[5]

---

[4]It is this condition, in the present context, that captures what is essential to a "particle theory". In contrast to a particle, a "field" is spread out throughout all of space and so can, in a sense, be found in two (disjoint) places at one time.

[5]This interpretation is supported by the following simple argument that goes back, at least, to Lüders (1951). It takes for granted that conditional probabilities in quantum mechanics can be computed using the so-called "Lüders rule". (In effect, we are restricting attention to a particular class of idealized, non-disturbing measurements.)

The case of interest is that in which Δ_1 and Δ_2 are spatial sets that do not belong to the same hyperplane. Let us assume that Δ_1's hyperplane is earlier than Δ_2's, and let us assume that the particle starts out in a state represented by the density operator W. If no detection experiment is performed at Δ_1, the probability that the particle will be detected at Δ_2 is given by $tr(WP_{\Delta_2})$. But the probability of the later event, conditional on the assumption that the detection experiment *is* performed at Δ_1 (but not conditional on the outcome), is given, instead, by $tr(W' P_{\Delta_2})$ where

$$W' = P_{\Delta_1} W P_{\Delta_1} + (I - P_{\Delta_1}) W (I - P_{\Delta_1}).$$

W′ is a mixture of the two outcome states that could result from the first detection experiment ("particle detected at Δ_1" and "particle not detected at Δ_1") weighted by their respective probabilities. So the condition in which we are interested, the one

The theorem can now be formulated as follows.

Proposition: If the structure $(\mathcal{H}, \mathbf{a} \mapsto U(\mathbf{a}), \Delta \mapsto P_\Delta)$ satisfies conditions (1)–(4), $P_\Delta = \mathbf{0}$ for all spatial sets Δ.

We can think about it this way. Any candidate relativistic particle theory satisfying the four conditions must predict that, no matter what the state of the particle, the probability of finding it in any spatial set is 0. The conclusion is unacceptable. So the proposition has the force of a "no–go theorem" to the extent that one considers (1) through (4) reasonable constraints.

II

The proposition is an elementary consequence of the following non–elementary technical lemma of Borchers (1967). The use I will make of it is almost exactly the same as in Jancewicz (1977). So far as I know, the first person to recognize the relevance of the lemma to questions of "non–localizability" was Schlieder (1971).

---

that captures the requirement that the act of performing the first experiment cannot statistically influence the outcome of the second, is:

$$\text{For all density operators } W, \ \ tr(WP_{\Delta_2}) = tr(W' P_{\Delta_2}). \qquad (*)$$

But, as we verify below, (*) is strictly equivalent to the condition that P_{Δ_1} and P_{Δ_2} commute. (Since the latter condition is symmetric in Δ_1 and Δ_2, it follows that (*) is also equivalent to its symmetric counterpart in which the roles of Δ_1 and Δ_2 are interchanged.)

If P_{Δ_1} and P_{Δ_2} commute, (*) follows immediately from basic properties of the trace operator. For any density operator W,

$$\begin{aligned} tr(W' P_{\Delta_2}) &= tr(P_{\Delta_1} W P_{\Delta_1} P_{\Delta_2}) + tr[(I - P_{\Delta_1}) W (I - P_{\Delta_1}) P_{\Delta_2}] \\ &= tr(W P_{\Delta_1} P_{\Delta_2} P_{\Delta_1}) + tr[W (I - P_{\Delta_1}) P_{\Delta_2} (I - P_{\Delta_1})] \\ &= tr(W P_{\Delta_1} P_{\Delta_2}) + tr[W (I - P_{\Delta_1}) P_{\Delta_2}] = tr(W P_{\Delta_2}). \end{aligned}$$

Conversely, assume that (*) holds. Let ϕ be any unit vector in $\mathcal{H}$, and let W be the projection operator $P_{[\phi]}$ whose range is the one–dimensional subspace spanned by ϕ. Then we have

$$\begin{aligned} (\phi, P_{\Delta_2}\phi) = tr(WP_{\Delta_2}) = tr(W' P_{\Delta_2}) &= tr(W P_{\Delta_1} P_{\Delta_2} P_{\Delta_1}) + tr[W (I - P_{\Delta_1}) P_{\Delta_2} (I - P_{\Delta_1})] \\ &= (\phi, P_{\Delta_1} P_{\Delta_2} P_{\Delta_1} \phi) + (\phi, (I - P_{\Delta_1}) P_{\Delta_2} (I - P_{\Delta_1}) \phi). \end{aligned}$$

So if

$$A = P_{\Delta_2} - P_{\Delta_1} P_{\Delta_2} P_{\Delta_1} - (I - P_{\Delta_1}) P_{\Delta_2} (I - P_{\Delta_1}),$$

$(\phi, A\phi) = 0$ for all unit vectors ϕ. Since A is self–adjoint, this is only possible if $A = \mathbf{0}$. Thus,

$$P_{\Delta_2} = P_{\Delta_1} P_{\Delta_2} P_{\Delta_1} + (I - P_{\Delta_1}) P_{\Delta_2} (I - P_{\Delta_1}).$$

Multiplying both sides of the equation by P_{Δ_1}, first on the left, and then on the right, we have

$$P_{\Delta_1} P_{\Delta_2} = P_{\Delta_1} P_{\Delta_2} P_{\Delta_1} = P_{\Delta_2} P_{\Delta_1}.$$

Lemma: Let $V(t) = e^{-itH}$ be a strongly continuous, one-parameter group of unitary operators on a Hilbert space whose generator H has a spectrum bounded from below. Let P_1 and P_2 be two projection operators such that:

(i) $P_1 P_2 = \mathbf{0}$, and

(ii) there is an $\varepsilon > 0$ such that for all t, if $|t| < \varepsilon$,

$$[P_1, V(t) P_2 V(-t)] = \mathbf{0}.$$

Then $P_1 V(t) P_2 V(-t) = \mathbf{0}$ for *all* t (and hence $V(t) P_2 V(-t) P_1 = \mathbf{0}$ for *all* t).

The proof of the proposition (from the lemma) is as follows.

Assume that conditions (1) – (4) hold, and let Δ be any spatial set. We show that $P_\Delta = \mathbf{0}$. We can certainly find a vector $\mathbf{a}$, tangent to the hyperplane of Δ, such that (see figure 1)

(a) Δ and $\Delta + \mathbf{a}$ are disjoint, and

(b) for all future directed, unit timelike vectors $\mathbf{a}_1$, and all sufficiently small t (in absolute value), Δ and $\Delta + \mathbf{a} + t\,\mathbf{a}_1$ are spacelike related.

It follows from (a) and the localizability condition that

(a′) $P_\Delta P_{\Delta+\mathbf{a}} = P_{\Delta+\mathbf{a}} P_\Delta = \mathbf{0}$

It follows from (b), the translation covariance condition, and the locality condition that:

(b′) For all future directed, unit timelike vectors $\mathbf{a}_1$, and all sufficiently small t,

$$[P_\Delta, U(t\mathbf{a}_1) P_{\Delta+\mathbf{a}} U(-t\mathbf{a}_1)] = [P_\Delta, P_{\Delta+\mathbf{a}+t\mathbf{a}_1}] = \mathbf{0}$$

Now we invoke the lemma (taking $V(t) = U(t\mathbf{a}_1)$, $P_1 = P_\Delta$, and $P_2 = P_{\Delta+\mathbf{a}}$) - it is applicable by the energy condition - and conclude that:

(c) For all future directed, unit timelike vectors $\mathbf{a}_1$, and *all* t,

$$P_\Delta \; U(t\mathbf{a}_1)P_{\Delta+\mathbf{a}}\,U(-t\mathbf{a}_1) = U(t\mathbf{a}_1)P_{\Delta+\mathbf{a}}\,U(-t\mathbf{a}_1)\,P_\Delta = \mathbf{0}.$$

and therefore

$$P_\Delta \; P_{\Delta+\mathbf{a}+t\mathbf{a}_1} = P_{\Delta+\mathbf{a}+t\mathbf{a}_1}\,P_\Delta = \mathbf{0}.$$

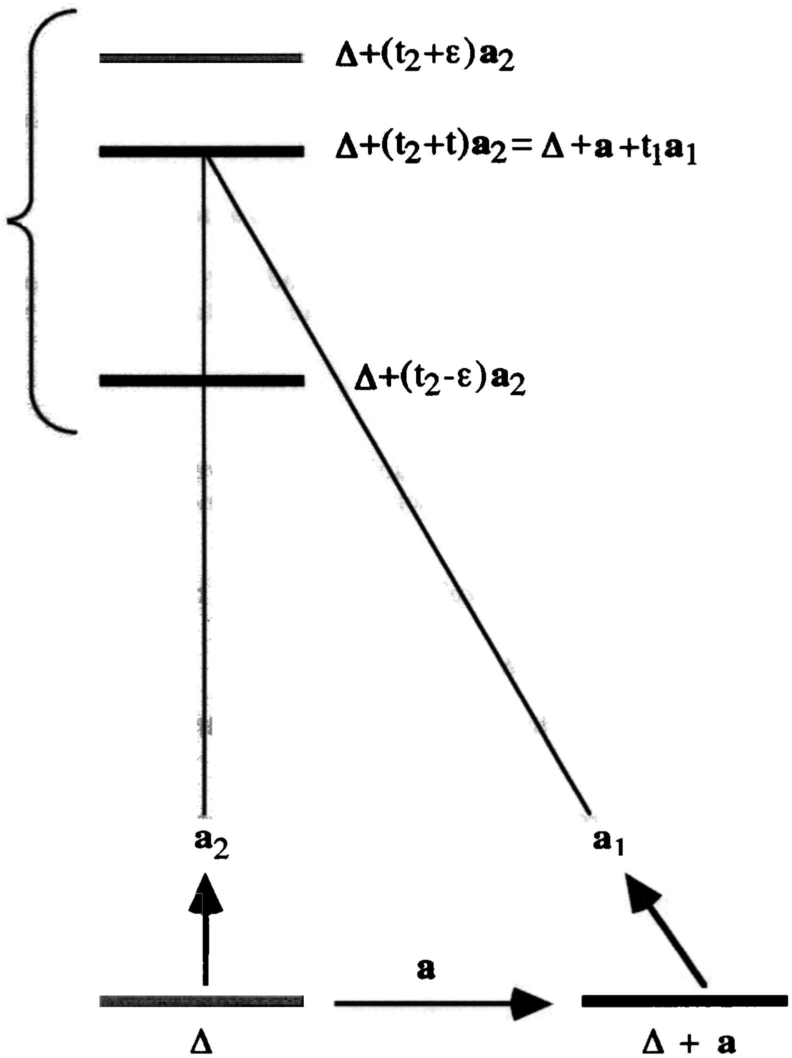

Figure 1

Next, let $\mathbf{a}_2$ be any future directed unit timelike vector. For all sufficiently large $t_2 > 0$, the set $\Delta + t_2\,\mathbf{a}_2$ is to the timelike future of $\Delta + \mathbf{a}$ (see figure 1 again). So we can certainly find a particular $t_2 > 0$, and $\varepsilon > 0$, such that $\Delta + (t_2+t)\,\mathbf{a}_2$ is to the timelike future of $\Delta + \mathbf{a}$ for all t with $|t| < \varepsilon$. Hence, if $|t| < \varepsilon$, there is a future directed, unit timelike vector $\mathbf{a}_1$ and a number t_1 such that

$$\Delta + (t_2+t)\,\mathbf{a}_2 = \Delta + \mathbf{a} + t_1\,\mathbf{a}_1.$$

Therefore, by (c), if $|t| < \varepsilon$,

$$P_\Delta\ P_{\Delta + (t_2+t)\mathbf{a}_2} = P_{\Delta + (t_2+t)\mathbf{a}_2}\ P_\Delta = \mathbf{0}$$

or, equivalently (by the translation covariance condition),

$$P_\Delta\ U(t\mathbf{a}_2)\,P_{\Delta + t_2\mathbf{a}_2}U(-t\mathbf{a}_2) = U(t\mathbf{a}_2)\,P_{\Delta + t_2\mathbf{a}_2}U(-t\mathbf{a}_2)\ P_\Delta = \mathbf{0}$$

If we now invoke the lemma again (taking $V(t) = U(t\mathbf{a}_2)$, $P_1 = P_\Delta$, and $P_2 = P_{\Delta + t_2\mathbf{a}_2}$), we may conclude that

$$P_\Delta\ U(t\mathbf{a}_2)\,P_{\Delta + t_2\mathbf{a}_2}U(-t\mathbf{a}_2) = \mathbf{0}$$

and therefore (by the translation covariance condition again)

$$P_\Delta\ U[(t+t_2)\mathbf{a}_2]\,P_\Delta\ U[-(t+t_2)\mathbf{a}_2] = \mathbf{0}$$

for *all* t. Hence (taking $t = -t_2$), $P_\Delta = P_\Delta P_\Delta = \mathbf{0}$. ∎

The University of Chicago

REFERENCES

Borchers, H. J. (1967), "A Remark on a Theorem of B. Misra", *Communications in Mathematical Physics* 4: 315–323.
Fleming, G. (1965a), "Covariant Position Operators, Spin And Locality", *Physical Review B 137*: 188–197.
Fleming, G. (1965b), "Nonlocal Properties of Stable Particles", *Physical Review B 139*: 963–968.
Hegerfeldt, G. C. (1974), "Remark on Causality and Particle Localization", *Physical Review D 10*: 3320–3321.
Hegerfeldt, G. C. (1985), "Violation of Causality in Relativistic Quantum Theory?", *Physical Review Letters 54*: 2395–2398.
Hegerfeldt, G. C. (1989), "Difficulties with Causality in Particle Localization", *Nuclear Physics B (Proc. Suppl.) 6*: 231-237.

Hegerfeldt, G. C., and S.N.M. Ruijsenaars. (1980), "Remarks on Causality, Localization, and Spreading of Wave Packets", *Physical Review D 22:* 377–384.

Jancewicz, B. (1977), "Operator Density Current and [the] Relativistic Localization Problem", *Journal of Mathematical Physics 18:* 2487-2494.

Lüders, G. (1951), "Über die Zustandsänderung durch den Messprozess", *Annalen der Physik 8:* 322–328.

Maudlin, T. (1994), *Quantum Non-Locality and Relativity*. Oxford: Blackwell.

Perez, J. F., and I. F. Wilde. (1977), "Localization and Causality in Relativistic Quantum Mechanics", *Physical Review D 16:* 315-317.

Ruijsenaars, S. N. M. (1981), "On Newton–Wigner Localization and Superluminal Propagation Speeds", *Annals of Physics 137:* 33–43.

Schlieder, S. (1971), "Zum kausalen Verhalten eines relativistischen quantenmechanischen Systems", in S.P. Dürr (ed.), *Quanten und Felder*. Braunschweig: Friedrich Vieweg + Sohn, pp. 145–160.

Skagerstam, B. K. (1976), "Some Remarks Concerning the Question of Localization of Elementary Particles", *International Journal of Theoretical Physics 15:* 213–230.

Weinberg, S. (1987), *Elementary Particles and the Laws of Physics, The 1986 Dirac Memorial Lectures*. Cambridge: Cambridge University Press.