Geometry and Spacetime
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# Model Solutions for Odd-Numbered Problems in Section 2.2

[Note: At this stage, we allow ourselves to perform simple computations with vectors (e.g., rearranging terms in a sum) without justifying every step with a direct appeal to clauses VS 1 - VS 8 in the definition of a vector space.]

**Problem 2.2.1** Show that for all points p and q in A, and all subspaces W of V, the following conditions are equivalent.

(i) q belongs to p+W

(ii) p belongs to q+W

(iii) $\overrightarrow{pq} \in W$

(iv) p+W and q+W coincide (i.e., contain the same points)

(v) p+W and q+W intersect (i.e., have at least one point in common)

**Proof** Let $p$ and $q$ be points in $A$, and let $W$ be a subspace of $V$.

**(i)** $\Rightarrow$ **(ii)** Assume that $q$ belongs to $p + W$ . Then there is a vector $u$ in $W$ such that $q = p + u$. It follows that $p = q + (-u)$. Since $u$ is in $W$ (and since $W$ is a subspace of $V$), $(-u)$ is in $W$ as well. So $p$ belong to $q + W$.

**(ii)** $\Rightarrow$ **(iii)** Assume that $p$ belongs to $q + W$. Then there is a vector $v$ in $W$ such that $p = q + v$. So $\overrightarrow{qp} = v \in W$. But $W$ is a subspace of $V$. So, since $\overrightarrow{qp}$ belongs to $W$, $-\overrightarrow{qp}$ belongs to $W$ as well. It follows that $\overrightarrow{pq} = -\overrightarrow{qp} \in W$.

**(iii)** $\Rightarrow$ **(iv)** Assume that $\overrightarrow{pq}$ belongs to $W$. We show that $(p+W) \subseteq (q+W)$. (A similar argument shows that $(q+W) \subseteq (p+W)$.) Let $r$ be a point in $p+W$. Then there is a vector $u$ in $W$ such that $r = p + u$. It follows that

$$r = (q + \overrightarrow{qp}) + u = q + (\overrightarrow{qp} + u) \in q + W$$

(since both $\overrightarrow{qp}$ and $u$ belong to $W$ and $W$ is a subspace of $V$). So $r$ is in $q+W$. Thus, $(p + W) \subseteq (q + W)$, as claimed.

**(iv)** $\Rightarrow$ **(v)** This one is trivial.

**(v)** $\Rightarrow$ **(i)** Assume there is a point $r$ that belongs to both $p + W$ and $q + W$. Then there exist vectors $u$ and $v$ in $W$ such that $r = p + u$ and $r = q + v$. It follows that

$$q = r + (-v) = (p + u) + (-v) = p + (u - v).$$

Since $u$ and $v$ are both in $W$, and since $W$ is a subspace of $V$, $(u - v)$ is in $W$. So $q$ belongs to $p + W$. $\square$

**Problem 2.2.3** Let $p, q, r, s$ be any four distinct points in $A$. Show that the following conditions are equivalent.

(i) $\overrightarrow{pr} = \overrightarrow{sq}$

(ii) $\overrightarrow{sp} = \overrightarrow{qr}$

(iii) The midpoints of the line segments $LS(p, q)$ and $LS(s, r)$ coincide, i.e.,

$$p + \frac{1}{2}\,\overrightarrow{pq} \;=\; s + \frac{1}{2}\,\overrightarrow{sr}.$$

**Proof**

**(i) $\Rightarrow$ (ii)** Assume $\overrightarrow{pr} = \overrightarrow{sq}$. Then

$$\overrightarrow{sp} \;=\; \overrightarrow{sq} + \overrightarrow{qr} + \overrightarrow{rp} \;=\; \overrightarrow{pr} + \overrightarrow{qr} + \overrightarrow{rp} \;=\; \overrightarrow{qr} + (\overrightarrow{pr} + \overrightarrow{rp}) \;=\; \overrightarrow{qr} + \mathbf{0} \;=\; \overrightarrow{qr}.$$

So we have (ii).

**(ii) $\Rightarrow$ (iii)** Assume $\overrightarrow{sp} \;=\; \overrightarrow{qr}$. Then

$$\begin{aligned}
p + \frac{1}{2}\,\overrightarrow{pq} &= (s + \overrightarrow{sp}) + \frac{1}{2}\,(\overrightarrow{pr} + \overrightarrow{rq}) \\
&= s + \frac{1}{2}\,(\overrightarrow{sp} + \overrightarrow{sp} + \overrightarrow{pr} + \overrightarrow{rq}) \\
&= s + \frac{1}{2}\,(\overrightarrow{sp} + \overrightarrow{qr} + \overrightarrow{pr} + \overrightarrow{rq}) \\
&= s + \frac{1}{2}\,((\overrightarrow{sp} + \overrightarrow{pr}) + (\overrightarrow{qr} + \overrightarrow{rq})) \\
&= s + \frac{1}{2}\,(\overrightarrow{sr} + \mathbf{0}) = s + \frac{1}{2}\,\overrightarrow{sr}.
\end{aligned}$$

So we have (iii).

**(ii) $\Rightarrow$ (i)** Assume $p + \frac{1}{2}\,\overrightarrow{pq} \;=\; s + \frac{1}{2}\,\overrightarrow{sr}$. Then

$$\begin{aligned}
p &= s + \frac{1}{2}\,\overrightarrow{sr} + \left(-\frac{1}{2}\,\overrightarrow{pq}\right) \\
&= s + \frac{1}{2}\,(\overrightarrow{sr} - \overrightarrow{pq}) \\
&= s + \frac{1}{2}\,((\overrightarrow{sp} + \overrightarrow{pr}) - (\overrightarrow{ps} + \overrightarrow{sq})) \\
&= s + \frac{1}{2}\,((\overrightarrow{sp} - \overrightarrow{ps}) + (\overrightarrow{pr} - \overrightarrow{sq})) \\
&= s + \frac{1}{2}\,(2\,\overrightarrow{sp} + (\overrightarrow{pr} - \overrightarrow{sq})) \\
&= s + \left(\overrightarrow{sp} + \frac{1}{2}\,(\overrightarrow{pr} - \overrightarrow{sq})\right).
\end{aligned}$$

So $\overrightarrow{sp} = \overrightarrow{sp} + \frac{1}{2}(\overrightarrow{pr} - \overrightarrow{sq})$. It follows that $(\overrightarrow{pr} - \overrightarrow{sq}) = \mathbf{0}$ and, hence, that $\overrightarrow{pr} = \overrightarrow{sq}$. So we have (i). $\square$

**Problem 2.2.5** Let $(V, \mathbf{A}, +)$ be a two-dimensional affine space. Let $\{p_1, q_1, r_1\}$ and $\{p_2, q_2, r_2\}$ be two sets of non-collinear points in $A$. Show that there is a unique affine space isomorphism $\varphi : A \to A$ such that $\varphi(p_1) = p_2$, $\varphi(q_1) = q_2$, and $\varphi(r_1) = r_2$.

**Proof**

Let $\{p_1, q_1, r_1\}$ and $\{p_2, q_2, r_2\}$ be two sets of non-collinear points in $A$. Then the vectors $\overrightarrow{p_1q_1}$ and $\overrightarrow{p_1r_1}$ are linearly independent and, so, form a basis for $V$. Similarly, $\overrightarrow{p_2q_2}$ and $\overrightarrow{p_2r_2}$ form a basis for $V$. It follows that there is a unique isomorphism $\Phi: V \to V$ such that

$$\begin{aligned} \Phi(\overrightarrow{p_1q_1}) &= \overrightarrow{p_2q_2} \\ \Phi(\overrightarrow{p_1r_1}) &= \overrightarrow{p_2r_2}. \end{aligned}$$

Now consider the map $\varphi: A \to A$ defined by

$$\varphi(s) = p_2 + \Phi(\overrightarrow{p_1s}). \tag{1}$$

It follows from proposition 2.2.6 that $\varphi(p_1) = p_2$, that $\varphi$ is a bijection, and that

$$\varphi(s) = \varphi(t) + \Phi(\overrightarrow{ts}). \tag{2}$$

for all $s$ and $t$ in $A$. Thus $\varphi$ qualifies as an affine space isomorphism. And it further follows from (1) that

$$\begin{aligned} \varphi(q_1) &= p_2 + \Phi(\overrightarrow{p_1q_1}) = p_2 + \overrightarrow{p_2q_2} = q_2 \\ \varphi(r_1) &= p_2 + \Phi(\overrightarrow{p_1r_1}) = p_2 + \overrightarrow{p_2r_2} = r_2, \end{aligned}$$

as required.

To establish uniqueness, suppose that $\varphi' : A \to A$ is an affine space isomorphism such that $\varphi'(p_1) = p_2$, $\varphi'(q_1) = q_2$, and $\varphi'(r_1) = r_2$. Suppose that $\Phi': V \to V$ is the corresponding vector space isomorphism. So we have

$$\varphi'(s) = \varphi'(t) + \Phi'(\overrightarrow{ts}). \tag{3}$$

for all $s$ and $t$ in $A$. It now follows by (3) and (1) that

$$\Phi'(\overrightarrow{p_1q_1}) = \overrightarrow{\varphi'(p_1)\varphi'(q_1)} = \overrightarrow{p_2\,q_2} = \overrightarrow{\varphi(p_1)\varphi(q_1)} = \Phi(\overrightarrow{p_1q_1}).$$

Similarly, we have

$$\Phi'(\overrightarrow{p_1r_1}) = \Phi(\overrightarrow{p_1r_1}).$$

So the isomorphisms $\Phi$ and $\Phi'$ agree in their action on the elements of a basis for $V$. It follows that they are agree in their action on all vectors in $V$, i.e., $\Phi' = \Phi$. From this, in turn, it follows that $\varphi$ and $\varphi'$ must be equal. For by (3) and (1) again, we have

$$\begin{aligned}
\phi'(s) &= \phi'(p_1) + \Phi'(\overrightarrow{p_1 s}) \\
&= p_2 + \Phi'(\overrightarrow{p_1 s}) \\
&= \phi(p_1) + \Phi(\overrightarrow{p_1 s}) \\
&= \phi(s)
\end{aligned}$$

for all $s$ in $A$. $\square$