AN EXTENDIBLE SPACETIME WITHOUT CLOSED TIMELIKE CURVES WHOSE EVERY EXTENSION CONTAINS CLOSED TIMELIKE CURVES

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Abstract. By removing a fractal from time-rolled Minkowski spacetime, we construct an extendible spacetime without closed timelike curves whose every extension contains closed timelike curves. This settles a question posed by Geroch.

1. Introduction

Geroch [1] has emphasized that a suitable definition of a "singularity" within the context of general relativity depends crutially on the definition of spacetime maximality. He has also suggested that the latter definition is far from clear given its sensitivity to a background collection of spacetimes. In recent years, there has been significant interest in better understanding this sensitivity by exploring spacetime maximality outside the standard context, e.g., C^0-maximality [2, 3, 4]. In his paper, Geroch articulated a number of "important and unsolved problems" concerning spacetime maximality – some of which remain open 50+ years later. The present work concerns one such problem.

Let $\mathscr{U}$ be the collection of all spacetimes (M, g) where M is a smooth, connected, Hausdorff manifold and g is a smooth Lorentzian metric on M. For any collection $\mathscr{P} \subseteq \mathscr{U}$, let us say that a spacetime in $\mathscr{P}$ is $\mathscr{P}$-maximal if it is not isometric to a proper subset of another spacetime (M', g') in $\mathscr{P}$. We say a spacetime in $\mathscr{P}$ is $\mathscr{P}$-extendible if it is not $\mathscr{P}$-maximal. Now consider the following condition on a collection $\mathscr{P} \subset \mathscr{U}$:

(*) Every $\mathscr{P}$-maximal spacetime is $\mathscr{U}$-maximal.

Many spacetime properties of interest do not satisfy (*). For instance, if $\mathscr{P} \subset U$ is the collection of all globally hyperbolic spacetimes, one can easily find a $\mathscr{P}$-maximal spacetime that is $\mathscr{U}$-extendible: the $t < 0$ region of Misner spacetime is one such example [5, 6]. This particular example can be used to obtain similar results for a number of other steps on the causal ladder: stable causality, strong causality, and (past and future) distinguishability. The case of causality is handled with another simple example [7]. Consider any $\mathscr{U}$-maximal spacetime (M, g) with a single closed null curve and then remove a point p on this curve. The resulting spacetime $(M - \{p\}, g)$ is causal but it has only one proper extension: the acausal spacetime (M, g). This last step relies on an intuitive but non-trivial result recently

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proved by Sbierski [4]: If (M, g) is a $\mathscr{U}$-maximal spacetime and $p \in M$, then the only proper $\mathscr{U}$-extension (up to isometry) of $(M - \{p\}, g)$ is (M, g) itself.

It can also be seen that some spacetime properties trivially satisfy (*): if $\mathscr{P} \subset U$ is the collection of all geodesically complete spacetimes, then it is immediate that each spacetime in $\mathscr{P}$ is $\mathscr{U}$-maximal and thus $\mathscr{P}$-maximal [8, 9]. But the status of (*) with respect to other spacetime properties is sometimes difficult to settle. Geroch [1] wondered about four spacetime properties in particular:

(i) "is a source-free solution to Einstein's equations"
(ii) "has no closed timelike curves"
(iii) "satisfies an energy condition"
(iv) "has a Killing vector"

Questions concerning (i) and (iv) are still open as far as we know. Question (iii) was settled by Manchak [7] who showed that (*) is not satisfied by any of the standard energy conditions.

Question (ii) is settled in the present paper. Geroch had special interest in this case, he writes in [1, p. 277]: "In fact, the status of closed timelike curves with respect to condition [(*)] would have an important bearing on whether or not the program for refining the notion of a "singular point" by using extensions can be carried out for the ideal points construction." For some time there was hope for a positive resolution, but it turns out that condition (*) is not satisfied by the collection $\mathscr{P} \subset \mathscr{U}$ of all spacetimes without closed timelike curves (CTCs).

In this paper, an example is presented which is a $\mathscr{U}$-extendible spacetime without CTCs whose every extension contains CTCs. The example is a variation of an idea that is often used in the foundations of Lorentzian causality theory: take Minkowski spacetime and "roll it up" along the time direction and then remove points in an appropriate way [10, 11]. In our case, the set of removed points needs to be quite intricate; here it is a fractal and at the same time a generalized Cantor set. We note that because of this construction, the example (M, g) should not be considered a "time machine" since all of the CTCs in every extension (M', g') are confined to the timelike past of M in M' [12, 13].

2. Puncturing time-rolled Minkowski spacetime to ban all CTCs

In this section we construct the desired spacetime for dimension 2, it will be extended to all dimensions in Section 4. Time-rolled Minkowski spacetime has plenty of CTCs, we construct a "barrier" B such that each CTC in time-rolled Minkowski spacetime has at least one point from B. Because all extensions of the example need to have CTCs, the set B cannot contain any line segment.

The idea is to build an infinite horizontal wall from fractal bricks, each of which is constructed as the Cantor set; but instead of removing middle thirds starting from the interval $[0, 1]$, we remove sets from a closed trapezoid having lightlike legs. We do this such that what remains are three disjoint closed trapezoids each similar to the original one and whose corresponding legs are on the same lightlike line. Then we continue this procedure with the small trapezoids ad infinitum. See Figure 1.

To see the condition for self-similarity, let S be the length of the longer base and T be the height of the trapezoid, and let λ be the ratio of similarity, see Figure 2. Then, because the legs of the trapezoids are lightlike, we have $S = \frac{3}{\lambda}S - 2T$, which

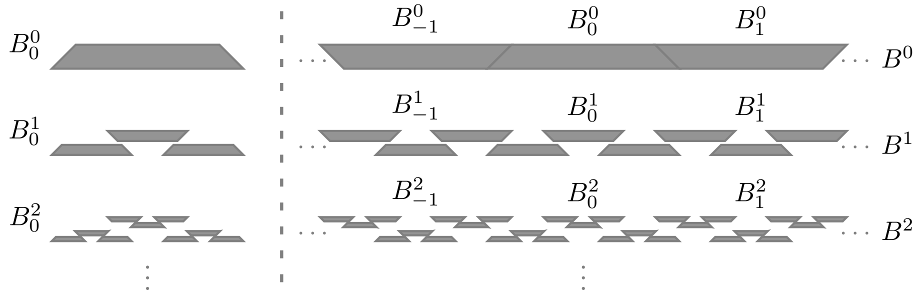

FIGURE 1. Construction of the barrier B by an infinite iteration.

can be reorganized as

$$2\lambda T = (3-\lambda)S. \tag{1}$$

If $\lambda = 3$, then $T = 0$ and our construction degenerates into a line segment. If $\lambda = 2$, then $S = 4T$ and the smaller trapezoids overlap. However, any of the cases $2 < \lambda < 3$ are equally good for our purposes.

To construct the fractal bricks, let us choose and fix some λ for which $2 < \lambda < 3$. For concreteness, choose $T = 1$, then $S = 2\lambda/(3-\lambda)$.[1] Let B^0_0 be the closed trapezoid with nodes $(0,0), (1,1), (1, S-1)$ and $(0, S)$. Then B^0_0 has horizontal bases and lightlike legs such that its height and longer base satisfy equation (1). See Figure 2.

Let B^1_0 be the union of the three closed smaller trapezoids similar to B^0_0 placed as in Figure 2, and let B^{i+1}_0 be the union of the 3^{i+1} trapezoids that we get by iterating the same replacement step with all the 3^i trapezoids of B^i_0. Let m be the length of the midsegment of trapezoid B^0_0. For each $n \in \mathbb{Z}$, let B^i_n be B^i_0 translated horizontally by nm and rotated around its center if n is odd. See Figure 1.

We now define our barrier set B as

$$B^i := \bigcup_{n\in\mathbb{Z}} B^i_n \qquad \text{and} \qquad B = \bigcap_{i=0}^{\infty} B^i. \tag{2}$$

We note that $B \subseteq [0,1] \times \mathbb{R}$ and the horizontal projection of B to the vertical interval $[(0,0),(1,0)]$ is almost the Cantor set. The only difference is that the lengths of the removed intervals here are shorter. This is so because the lengths of the removed intervals are $(1-2/\lambda)$-th of the original ones, which is less than one third since $2 < \lambda < 3$. The intersection of B with the horizontal line segment $[(0,0),(0,S)]$ is also almost the Cantor set.

Proposition 1.

(a) B is closed.

(b) The complement of B is connected.

Proof. It is easy to see that, for each i, the set $B^i = \bigcup_{n\in\mathbb{Z}} B^i_n$ is closed because it is the union of isolated-enough trapezoids. For a formal proof, B^i is the union of the locally finite collection of closed sets B^i_n, i.e., there is a neighborhood of each point

[1] $S = 6$ if $\lambda = 2.25$. The figures are constructed with $\lambda = 2.4$.

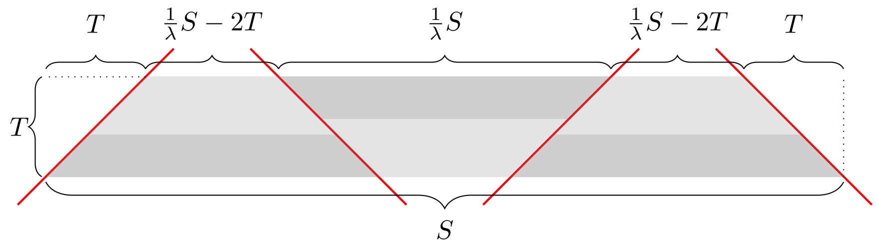

FIGURE 2. The figure illustrates the main construction step of barrier B together with the variables of equation (1) that is used to derive the condition $2<\lambda<3$ of self-similarity. The light gray part is the set that we remove from the trapezoid, and the three dark gray trapezoids are the remaining closed trapezoids which are similar to the original one. Analogously to the construction of the Cantor set, we repeat this removal step ad infinitum.

in $\mathbb{R}^2$ such that it intersects only finitely many of the closed sets [14, Cor.1.1.12]. Then B is closed because it is an intersection of closed sets by (2).

For each $i \geq 1$, the complement of B^i is clearly connected since we just removed some isolated closed trapezoids from the plane. The complement of B is the union of the complements of B^i. Hence, the complement of B is connected because it is the union of an upward directed collection of connected sets. □

The following proposition says that no causal curve can cross region $[0,1]\times\mathbb{R}$ without intersecting B. It is here where we use that the legs of the trapezoids are lightlike.

Proposition 2. *Assume that $\gamma:[0,1]\to\mathbb{R}^2$ is a broken future-directed causal curve such that $\gamma(0)=(0,x)$ and $\gamma(1)=(1,y)$ for some $x,y\in\mathbb{R}$. Then there is $0\leq t\leq 1$ such that $\gamma(t)\in B$.*

Proof. Let γ be a causal curve in Minkowski spacetime $(\mathbb{R}^2,\eta)$ as in the statement. The plan of the proof is as follows. Using induction, we are going to show that there is an $n\in\mathbb{Z}$ such that, for all natural numbers i, γ intersects one of the trapezoids of B^i_n at its longer base. Moreover, if γ intersects a trapezoid in B^i_n, then it intersects a sub-trapezoid in B^{i+1}_n. The intersection of these nested closed trapezoids is a point p of B to which the intersection points converge. Thus p has to be on γ because the latter is continuous. This will complete the proof of the proposition.

By definition, B^0 is $[0,1]\times\mathbb{R}$, so $\gamma(0)$ is on one of the bases of some B^0_n. If $\gamma(0)$ is on the shorter base of B^0_n, then $\gamma(1)$ is on the longer base of B^0_n, because the legs of B^0_n are lightlike and γ is broken future-directed causal. See Figure 3. Thus either $\gamma(0)=(0,x)$ or $\gamma(1)=(1,y)$ is on the longer base of B^0_n for some $n\in\mathbb{Z}$. This shows the base case $i=0$ of the induction.

Assume now that we have already seen that γ intersects one of the trapezoids of B^i_n at its longer base. By symmetry, without loss of generality, we can assume that this longer base is at the bottom. Then there are three cases: either γ intersects the left closed, middle open or the right closed part of this bottom base. In the first and third cases, γ intersects the longer base of the left or the right smaller trapezoid of the next iteration step. In the second case, γ has to intersect the longer base

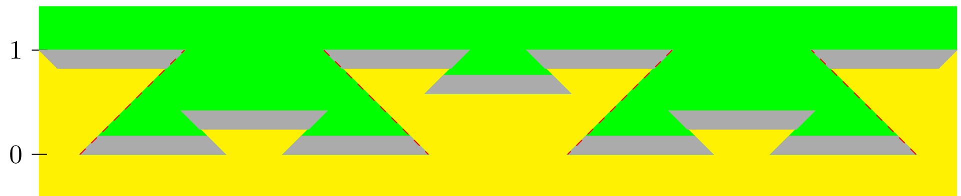

FIGURE 3. No causal curve can cross region $[0, 1] \times \mathbb{R}$ without intersecting B, cf. Proposition 2 below.

of the middle trapezoid because the common legs of these trapezoids are lightlike. Hence, γ intersects one of the included trapezoids of B_n^{i+1} at its longer base, and this is what we wanted to show.

Let now p_i be the intersection points of γ with the longer bases and let p be the unique point in the intersection of the nested trapezoids γ intersects. Then $p \in B$ by definition. Also, the p_i converge to p because each open neighbourhood O of p contains one of these nested trapezoids as a subset, since their diameters tend to 0 as i tends to infinity. But then O contains all p_j, $j \geq i$ for some i. We have seen that the p_i converge to p. Since γ is continuous, then $\gamma(t) = p$ for some $0 \leq t \leq 1$ because for all i, we have that $p_i = \gamma(t_i)$ for some $0 \leq t_i \leq 1$. □

By Proposition 1, we have that $(\mathbb{R}^2 \setminus B, \eta)$ is a spacetime. We can roll it up by choosing a large enough $\ell \in \mathbb{Z}$ and gluing points $(-\ell, x)$ and (ℓ, x) together, for all $x \in \mathbb{R}$. For concreteness, choose an $\ell > \lambda/(3 - \lambda) = S/2$ (cf. equation (1)) and let $\mathbf{M}^- = (M^-, \eta)$ be the spacetime that we get this way. By Proposition 2, there are no closed timelike curves (CTCs) in $\mathbf{M}^-$. Clearly, $\mathbf{M}^-$ is extendible, because, e.g., Minkowski spacetime rolled up at the same ℓ is an extension of $\mathbf{M}^-$. We can call $\mathbf{M}^-$ *punctured time-rolled Minkowski spacetime.*

In the following, we state three lemmas about B that we will use in the next section when proving that each proper extension of $\mathbf{M}^-$ does have a CTC. In the rest of this section, we will work in $(\mathbb{R}^2, \eta)$.

Because the diameters of the trapezoids tend to 0, every point of B can be described by a *choice sequence* that starts with an integer and continues with an infinite series of decision of $\mathcal{L}$eft, $\mathcal{M}$iddle, $\mathcal{R}$ight. In other words, there is a one-to-one correspondence between points of B and $\mathbb{Z} \times \{\mathcal{L}, \mathcal{M}, \mathcal{R}\}^\omega$.

We call a point $e \in B$ *eventually middle* iff its choice sequence contains only finitely many $\mathcal{L}$ and $\mathcal{R}$ choices, in other words, it becomes constant $\mathcal{M}$ after some time.

Lemma 1. *The set of eventually middle points is everywhere dense in B, i.e., for every $\varepsilon > 0$ and for every $p \in B$ there is an eventually middle point e such that $|p - e| < \varepsilon$.*

Proof. This follows easily from the construction because every point of B is in a small enough closed trapezoid of the construction and every such trapezoid contains (infinitely many) eventually middle points. □

The statement of the next lemma is illustrated in Figure 4.

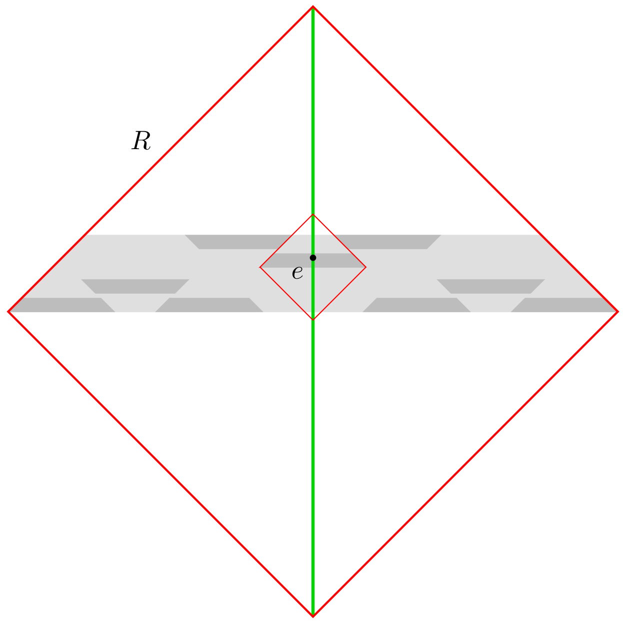

FIGURE 4. Inside the smallest light rhombus containing a trapezoid used in the construction of B, the vertical centerline intersects B only in one point, which is an eventually middle point.

Lemma 2. *Let e be an eventually middle point. Inside the smallest light rhombus R containing the trapezoid after which we always choose $\mathcal{M}$ to reach e, the vertical centerline of R intersects B only in e.*

Proof. This follows by self-similarity, see Figure 4. In more detail, let e be an eventually middle point and let the big grey trapezoid of the figure illustrate the one after which we always choose $\mathcal{M}$ to reach e. Let R be the smallest rhombus with lightlike sides which contains this trapezoid. By the construction, the parts of the green vertical center-line-segment of R which are outside the big trapezoid is outside B, for checking see Figure 3. Similarly, by the construction, the parts connecting the midpoints of the corresponding bases of the big and small trapezoids are outside B, for checking see Figure 4. The part of B covered by the big trapezoid and the part of B covered by the small trapezoid in the middle are similar and can be transformed into each other by a homogeneous dilation with center e. This can be seen as follows. For B_0^0, using the summing formula of geometric series, we get $e = \left(S/2, \left(1-\frac{1}{\lambda}\right)\left(1+\frac{1}{\lambda^2}+\frac{1}{\lambda^4}+\dots\right)\right) = \left(\frac{\lambda}{3-\lambda}, \frac{\lambda}{\lambda+1}\right)$. The bottom left corner of the middle trapezoid is $\left(\frac{S}{\lambda}-1+\frac{S}{\lambda^2}-\frac{1}{\lambda}, 1-\frac{1}{\lambda}\right) = \left(\frac{\lambda-1}{\lambda}\frac{\lambda+1}{3-\lambda}, \frac{\lambda-1}{\lambda}\right) = \frac{\lambda^2-1}{\lambda^2}\cdot e$. This calculation confirms that the center of the homogeneous dilation is indeed e. Hence, except e, every point of the vertical center-line-segment of R is outside B. □

The statement of the next lemma is illustrated in Figures 5, 6.

Lemma 3. *For every eventually middle point $e \in B$, there are timelike curves $\tau, \tau' : [0,1] \to \mathbb{R}^2$ such that*

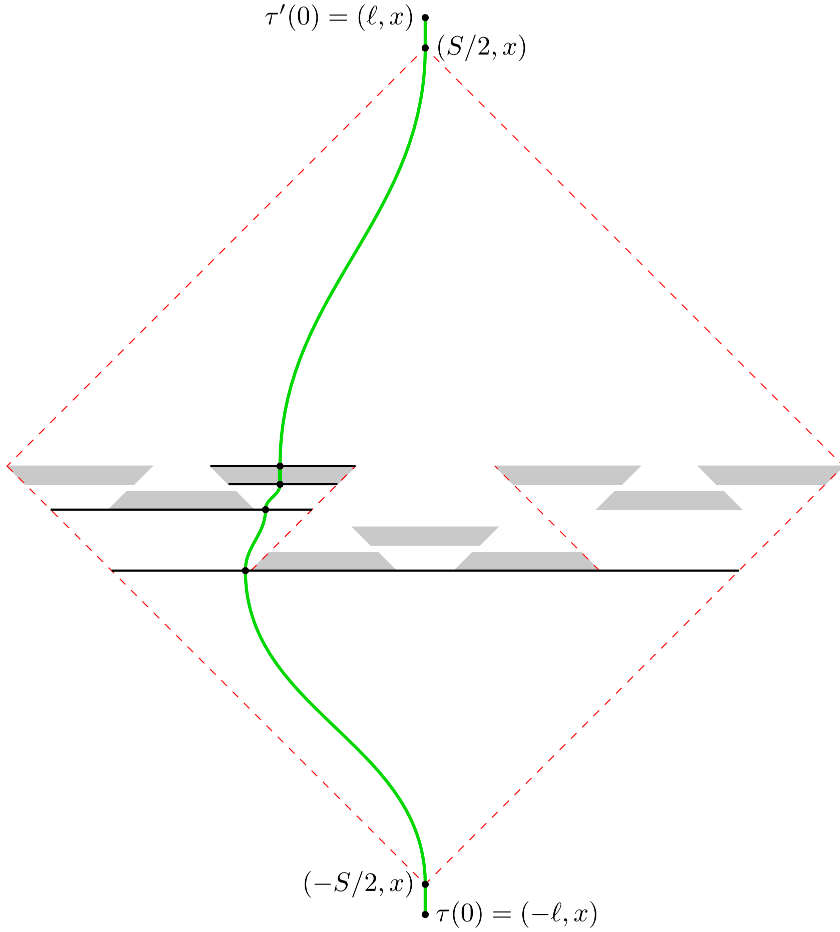

FIGURE 5. Through every eventually middle point e there are timelike curves τ and τ' respectively connecting e to points $(-\ell, x)$ and (ℓ, x) for some $x \in \mathbb{R}$ such that these curves contain no other point of B apart from e, cf. Lemma 3.

(a) $\tau(1) = \tau'(1) = e$, $\tau(0) = (-\ell, x)$, $\tau'(0) = (\ell, x)$ *for some* $x \in \mathbb{R}$,
(b) the ranges of τ *and* τ' *become part of the vertical line through* e *after a while, and*
(c) both τ *and* τ' *intersect* B *only in* e.

Moreover, there are $r_n, t_n \in (0, 1)$ *tending to* 1 *when* n *tends to infinity, and there are curves* $\lambda_n \subset \mathbb{R}^2 \setminus B$ *connecting* $\tau'(r_n)$ *and* $\tau(t_n)$ *such that the Euclidean length of* λ_n *tends to* 0 *when* n *tends to infinity.*

Proof. Let $e \in B$ be an eventually middle point. There is a series of nested trapezoids corresponding to the choice sequence of e. Let B_n^0 be the first trapezoid of this sequence. Let x be such that $(-\ell, x)$ and (ℓ, x) are on the vertical centerline of B_n^0. Then the edges of B_n^0 at its longer base are lightlike related to points $(-S/2, x)$ and $(S/2, x)$, cf. Figure 5.

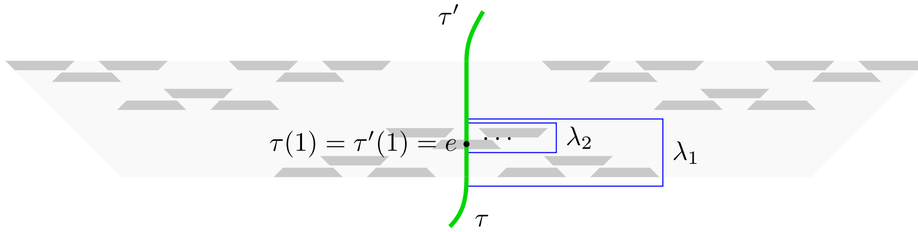

FIGURE 6. There is an infinite sequence of curves $\lambda_n \subset \mathbb{R}^2 \setminus B$ whose Euclidean length tends to 0 as n tends to infinity and who "witnesses" in $\mathbb{R}^2 \setminus B$ that the two halves of the centerline meet at e, cf. Lemma 3.

First by induction, we show that: *Each point p of the lower bases of the trapezoids from B_n^0 are reachable from below from point $(-\ell, x)$ by a timelike curve intersecting B at most this point p; and analogously, the upper bases of these trapezoids are reachable from above from (ℓ, x) by a timelike curve avoiding B in this sense.*

By symmetry, it is enough to show the first part of this statement. The base case, when p is on the bottom base of B_n^0 is easy because we have that the line segment from $(-\ell, x)$ to p is timelike since point $(-S/2, x)$ and the endpoints of the bottom base of B_n^0 are lightlike separated, and $\ell > S/2$. Since there is no point of B below this bottom base, we can easily reach p by an appropriate timelike curve from $(-\ell, x)$.

To see the induction step, it is enough to observe that the bottom base of each trapezoid is either part of the bottom base of the trapezoid of the previous iteration step, and then we have already reached it with an appropriate timelike curve; or it is reachable from a point of that base which is not in B by a timelike curve avoiding B; and hence, by continuing the curve given by the induction hypothesis with this one, we get the timelike curve we need. The above property used in the induction step is easy to confirm by the self-similarity of the construction, see Figure 5.

Consider now the trapezoid after which we always choose $\mathcal{M}$ to reach e. By Lemma 2, inside this trapezoid, the vertical centerline intersects B only in the point e, see Figure 6.

Now take a timelike curve from $(-\ell, x)$ to the bottom point of this centerline and continue it with the bottom part until e, with an appropriate parametrization this gives curve τ, and a completely analogous way we can find a τ' connecting (ℓ, x) and e.

By self-similarity, to see the existence of curves λ_n, it is enough to see the existence of r_1, t_1 and λ_1, which is easy because there is plenty of space outside B but inside the first middle trapezoid to go around the next middle trapezoid connecting some point $\tau(r_1)$ and $\tau'(t_1)$. Now, we can recursively define λ_n by scaling down λ_{n-1} by scaling factor $1/\lambda$. That the parameter points r_n and t_n of τ and τ' tend to 0, as well as, that the Euclidean length of λ_n tends to 0 when n tends to infinity follows from the fact that, in each step, we scaled down by factor $1/\lambda < 1/2$. $\square$

3. All extensions of the punctured time-rolled Minkowski spacetime have closed timelike curves

The idea of the proof is that any proper extension of $\mathbf{M}^-$ has to fill in a point of B since time-rolled Minkowski spacetime is geodesically complete. Once the extension fills in a point, it fills in nearby points, of B, too (see the next lemma). To any point of B, arbitrarily close there are eventually middle points, and those are the only missing points of some CTCs in $\mathbf{M}^-$. Thus, the extension will have at least these CTCs.

We begin by proving a general lemma about extensions. We say that an extension fills in a limit for a curve living in the smaller spacetime if the curve converges to a point in the extension (but it may not converge to any point in the smaller spacetime). The next lemma says, intuitively, that if an extension fills in a limit, it also fills in nearby limits witnessed by short curves in a coordinate system, and two such new limits coincide if a system of short coordinate curves witness their coincidence.

Some notation: For a chart ψ of $\mathbf{S}$ and broken curve $\delta : [0,1) \to S$, we denote the Euclidean coordinate-length of $\psi(\delta)$ by $|\delta|_\psi$. We say that δ convereges to q in S if there is a broken curve $\delta' : [0,1] \to S$ such that $\delta'(1) = q$ and $\delta'(x) = \delta(x)$ for all $0 \leq x < 1$. We say that δ can be continued if there is $\delta' : [0, y) \to S$ with $y > 1$ and $\delta'(x) = \delta(x)$ for all $0 \leq x < 1$. By $\delta \subseteq X$ we mean that the range of δ is a subset of X.

The next lemma is interesting when $p \notin N$. Lemma 4(ii) below is somewhat similar to Proposition 5.1. of [4].

Lemma 4. *Let $m \geq 2$, let $\mathbf{S} = (S, g)$ be an m-dimensonal Lorentzian manifold, let O be an open set of $\mathbf{S}$, and let $\psi : N \to \mathbb{R}^m$ be a chart of $\mathbf{S}$ such that the components g_{ij} and $\partial_k g_{ij}$ are bounded in the range of ψ. Then for all $p \in O$ there is $\varepsilon \in \mathbb{R}$ such that (i) and (ii) below hold for all broken curves $\delta, \delta' : [0,1) \to S$ starting at p (i.e., $\delta(0) = \delta'(0) = p$). Let δ^- denote the curve δ without its starting point, i.e., $\delta^- = \delta \setminus \{p\}$ and similarly for other broken curves starting at p.*

- **(i):** *If $\delta^- \subseteq N$ and $|\delta^-|_\psi < \varepsilon$ then $\delta \subset O$ and δ can be continued in O.*
- **(ii):** *If δ, δ' are as in (i) and $\psi(\delta^-), \psi(\delta'^-)$ converge to the same point in $\mathbb{R}^m$ such that this is "witnessed by a vanishing ψ-ladder", then they converge to the same point in $\mathbf{S}$, too. In more detail: Let $r_n, t_n \in (0,1)$ be such that they tend to 1 when n tends to infinity. Let the curves $\lambda_n \subset N$ connect $\delta(r_n)$ with $\delta'(t_n)$ such that $|\lambda_n|_\psi$ converges to 0 when n tends to infinity. Then δ and δ' converge to the same point in O.*

Proof. Let $\mathbf{S}$, ψ, O, p and δ be as in the lemma such that $\delta \setminus \{p\} \subset N$. Let $\xi : D \to \mathbb{R}^m$ be a chart in $\mathbf{S}$ such that $p \in D$. Since the range of ξ is an open set in $\mathbb{R}^m$, we may assume that the range of ξ is an open ball G of radius r around $\xi(p)$. By taking r to be small enough, we may assume that, in ξ, the components g_{ij} of g as well as the components $\partial_k g_{ij}$ of the derivatives of g are bounded by a number in G. Let C_g be a common bound for the components of g and its derivatives in the coordinate systems ξ and ψ.

First we prove (i). Assume that δ is not a subset of D. There is $0 < a < 1$ such that $\delta(a) \notin D$ but $\delta(x) \in D$ for all $0 \leq x < a$. Let γ be δ till this point, i.e., $\gamma : [0, a) \to S$ such that $\gamma(x) = \delta(x)$ for $0 \leq x < a$. We are going to show that γ

cannot be too short, i.e., there is $\varepsilon \in \mathbb{R}$ such that $|\gamma^-|_\psi > \varepsilon$. Therefore, $\delta \subset D \subseteq O$ if $|\delta^-|_\psi < \varepsilon$ and also δ can be continued in D in this case.

Let $k \in \mathbb{R}$ be arbitrary and assume that

$$|\gamma^-|_\psi < k. \tag{1}$$

Let $q = \delta(a)$ and let ρ denote the curve γ taken from q till p but such that $p \notin \rho$, i.e., $\rho : [0, a) \to S$ is defined by $\rho(x) = \gamma(a - x)$ for $x \in [0, a)$. Then $|\rho|_\psi = |\gamma^-|_\psi$ and $\rho \subset N$, $\gamma \subset O$.

From here on in the proof of (i), we will extensively rely on Section 4 of [4]. Take an orthonormal basis $e = (e_i : i < n)$ in $T_q\mathbf{S}$. Let $gap(\rho)$ denote the general-affine-paramater length of ρ with respect to e. This is denoted by $L_{gap,e}(\rho)$ in [4], its definition is recalled at the beginning of Section 4 of [4]. By Lemma 4.2 of [4], there is $0 < b < \omega$, depending only on k, e and C_g, such that

$$gap(\rho) < b \cdot |\rho|_\psi \qquad \text{if} \quad |\rho|_\psi < k.$$

We have $|\rho|_\psi < k$ by $|\gamma^-|_\psi < k$.

Let now $f = (f_i : i < n)$ in $T_p\mathbf{S}$ be the orthonormal basis which we get if we parallel transport e along δ from q to p. By the definition of gap-length, then $gap(\gamma) = L_{gap,f}(\gamma)$ taken with this basis f is the same as $gap(\rho) = L_{gap,e}(\rho)$. Now we can apply Corollary 4.12 of [4] to the chart ξ, since $\gamma \subset D$. It says that there are $c, d \in \mathbb{R}$ depending only on C_g and f such that if $gap(\gamma) < c$ then $|\gamma|_\xi < d \cdot gap(\gamma)$. We got

$$|\gamma|_\xi < d \cdot gap(\gamma) = d \cdot gap(\rho) < d \cdot b \cdot |\rho|_\psi = d \cdot b \cdot |\gamma^-|_\psi,$$

i.e.,

$$|\gamma|_\xi < d \cdot b \cdot |\gamma^-|_\psi \tag{2}$$

whenever $gap(\gamma) < c$. The latter holds if

$$|\gamma^-|_\psi < c \cdot b^{-1}. \tag{3}$$

Now we use that the range of ξ is a ball with radius r. This implies that $|\gamma|_\xi$ cannot be shorter than r because it is a curve starting at the center of the ball and leaving the ball. Hence $|\gamma|_\xi \geq r$ and so

$$|\gamma^-|_\psi \geq r \cdot (d \cdot b)^{-1}. \tag{4}$$

Let $\varepsilon = min\{k, c \cdot b^{-1}, r \cdot (d \cdot b)^{-1}\}$. Taking this ε makes (i) true. We also got in the proof that there is a bound $K \in \mathbb{R}$ such that

$$|\delta|_\xi < K \cdot |\delta^-|_\psi \tag{5}$$

whenever $|\delta^-|_\psi < \varepsilon$ (namely, by (2), we can take $K = b \cdot d$).

Proof of (ii): Let δ, δ' and λ_n be as in the statement of (ii). By (i), both δ and δ' converge to points of O, say to q and q'. These q and q' may not belong to N. We want to show $q = q'$. Let the broken curves γ_n be defined as δ from p till $\delta(r_n)$ and then continued with λ_n till $\delta'(t_n)$. Then by our conditions, for large enough n, the broken curve γ_n satisfies the conditions for (i), i.e., it starts at p, $\gamma_n^- \subset N$, and $|\gamma_n^-|_\psi < \varepsilon$. Thus there is n_0 such that $\gamma_n \subset O$, in particular $\lambda_n \subset O$ for all $n \geq n_0$.

Let us consider now curves starting at q. We have $q \in O$, by (i). Let $\varepsilon_0 \in \mathbb{R}$ be the bound that exists for q according to (i). Let the broken curves ρ_n be defined as starting from q then going in reverse direction along δ till $\delta(r_n)$, continuing along λ_n till $\delta'(t_n)$ and then continuing along δ' till its end (so that $q' \notin \rho_n$). Then $\rho_n^- \subset N$

and $|\rho_n^-|_\psi$ tends to 0 as n tends to infinity. Let $n_1 \geq n_0$ be such that $|\rho_n^-|_\psi < \varepsilon_0$ for all $n \geq n_1$.

Let K_0 be the bound that exists for q by the proof of (i), i.e., we have $|\rho_n|_\xi < K_0 \cdot |\rho_n^-|_\psi$ for all $n \geq n_1$. Thus $|\rho_n|_\xi$ tends to 0 as n tends to infinity. Since $\xi(\rho_n)$ starts at $\xi(q)$ and converges to $\xi(q')$ for all $n \geq n_1$, this means that $\xi(q) = \xi(q')$. Hence $q = q'$ since ξ is a bijection, and we are done. □

We are ready for proving the main property of $\mathbf{M}^-$, namely, that it is maximal among the spacetimes that do not contain CTCs.

Proposition 3. *Each proper extension of* $\mathbf{M}^-$ *contains closed timelike curves.*

Proof. Assume that $\mathbf{S} = (S, g)$ is a proper extension of $\mathbf{M}^- = (M^-, \eta)$. We may assume that $\mathbf{M}^-$ is the restriction of $\mathbf{S}$ to $M^- \subset S$ and M^- is an open set in $\mathbf{S}$.

From now on, we will work in $\mathbf{S}$.

Step 1: We show[2] that there is geodesic $\gamma \subset M^-$ converging to some $p \in S \setminus M^-$.

Let $q \in M^-$ and $p \in S \setminus M^-$ be arbitrary, there are such points. There is a broken geodesic γ that connects them in $\mathbf{S}$ because $\mathbf{S}$ is connected. By the properties of the real numbers, and because $p \in M^-$ and $q \notin M^-$, there is a first point in γ that is not in M^-. Thus, we may assume that $\gamma \subset M^-$ is a geodesic that converges to $p \notin M^-$ (in $\mathbf{S}$ of course), by letting γ be the last portion of the broken geodesic that lies in M^- and taking p to be the first point that is not in M^-. We may assume that $\gamma \subset \mathbb{R}^2$.

Step 2: We show that γ converges, in $\mathbb{R}^2$, to a point $p' \in B$.

This is so because γ has to converge to some point we left out from rolled-up Minkowski spacetime as the latter is geodesically complete.

We are ready to apply Lemma 4. Let $N = ((-1, 2) \times \mathbb{R}) \setminus B$. Then N is an open subset of M^- as well as it is a subset of $\mathbb{R}^2$. Let $\psi : N \to \mathbb{R}^2$ be defined to be the identity, i.e., $\psi(x, y) = (x, y)$ for all $(x, y) \in N$. Then ψ is a chart of $\mathbf{M}^-$, so it is a chart of $\mathbf{S}$, too, because $\mathbf{S}$ is an extension of $\mathbf{M}^-$.

We are going to apply Lemma 4 to $\mathbf{S}$ and ψ by taking O to be any open neighborhood of p (in $\mathbf{S}$, of course). By using this scenario and the properties of our concrete example $\mathbf{M}^-$, we are going to show that an eventually middle point in B is "filled in" in O, and this will bring in the CTC whose only missing point in $\mathbf{M}^-$ was this middle point.

The conditions of Lemma 4 hold because on chart ψ the components g_{ij} are 0 or 1 and all of $\partial g_{ij} = 0$ since here the metric is the standard Minkowski metric. Let $\varepsilon \in \mathbb{R}$ be as given by Lemma 4 for p.

Step 3: There are broken curves $\delta, \delta' : [0, 1) \to S$ starting at p and short enough according to chart ψ such that apart from their starting points they stay within $N \cap O$, and on chart ψ, they both converge to some eventually middle point $e \in B$ from opposite vertical directions.

Except from the initial ones, around every trapezoid used in the construction of B, there is a surrounding rectangle not intersecting B. By Lemma 1, there is an eventually middle point $e \in B$ arbitrarily close to p'. Let e be so close to p' that they are in a non-initial trapezoid whose surrounding rectangle R has sides less than $\varepsilon/6$, see Figure 7. Since γ tends to p' here, we can assume without

[2] We give a reason, but this is well-known, see, e.g., Lemma A.6 in [4].

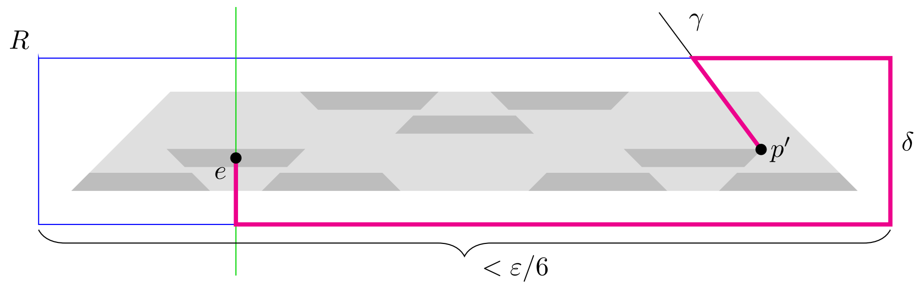

FIGURE 7. This figure illustrates the construction of the curves used in Step 3.

loss of generality that γ intersects R. Let $\delta : [0, 1) \to S$ be the broken curve defined as follows and as illustrated by Figure 7: We go from p along γ backwards until the intersection point of γ and R. From this point, we go around along R until the intersection point of the vertical line through e and R. Finally, we go along the vertical line until e. Analogously, we get δ' going to e in the opposite direction after the first breaking point. The coordinate lengths $|\delta|_\psi$ and $|\delta'|_\psi$ are both less than ε because the coordinate length of the part along γ is shorter than the diagonal of R, which is less than $2 \cdot \varepsilon/6$, and the rest is less than $4 \cdot \varepsilon/6$ because it contains at most 4 segments each of which is shorter than the longest sides of R. By their construction and Lemma 2, both δ and δ' stay within N apart from point $p = \delta(0) = \delta'(0)$. Hence, by (i) of Lemma 4, we have that, apart from p, they are also in O.

Step 4: In $\mathbf{S}$, $\delta, \delta' : [0, 1) \to S$ converge and have the same limit point.

By (i) of Lemma 4, δ and δ' can even be continued in O, and hence they have limit points in $\mathbf{S}$. After a while, the ranges of curves δ and δ' overlap with those of timelike curves τ and τ' given by Lemma 3. Hence, by Lemma 3, we have the "vanishing ψ-ladder" witnessing that δ and δ' converge to the same point in chart ψ required to apply (ii) of Lemma 4 and to conclude that they converge to the same point in O, and hence in $\mathbf{S}$.

Step 5: Through this common limit point of δ and δ', there is a CTC in $\mathbf{S}$.

Since after a while the ranges of curves δ and δ' overlap with those of τ and τ' given by Lemma 3, the common limit point of δ and δ' is also a common limit point for timelike curves τ and τ'. Hence, going forwards in τ and backwards in τ' gives us the desired CTC in $\mathbf{S}$, because the starting points of τ and τ' are glued together in $\mathbf{M}^-$. □

4. HIGHER DIMENSIONS

The barrier $B \times \mathbb{R}^{d-2}$ works in d-dimension in an analogous construction. We have that $B \times \mathbb{R}^{d-2}$ is closed and its complement is connected because the direct product of closed sets is closed and the direct product of connected sets is connected. Hence removing $B \times \mathbb{R}^{d-2}$ from d-dimensional time-rolled Minkowski spacetime gives an analogous d-dimensional punctured time-rolled Minkowski spacetime $\mathbf{M}_d^-$.

The set $B \times \mathbb{R}^{d-2}$ can also be constructed analogously to B by intersecting d-dimensional closed bars that have trapezoids as 2-dimensional cross sections. Hence that no broken future-directed causal curve can cross $[0,1] \times \mathbb{R}^{d-1}$ without intersecting $B \times \mathbb{R}^{d-2}$ can be proven the same way as Proposition 2. The only difference is that, instead of a point, the intersection of the corresponding nested bars give a horizontal $d-1$-dimensional subspace[3], which is contained in $B \times \mathbb{R}^{d-2}$ and crosses the causal curve trying to go through region $[0,1] \times \mathbb{R}^{d-2}$.

Let us note that if we intersect $B \times \mathbb{R}^{d-2}$ with any plane parallel to the plane $\{(t,x,0,\ldots,0) \in \mathbb{R}^d : t,x \in \mathbb{R}\}$, we get back B in these 2-dimensional vertical cross sections. Hence, even though in this higher dimensional construction instead of eventually middle points we have eventually middle $d-1$-dimensional subspaces, every point of these eventually middle subspaces is an eventually middle point of some B from some 2-dimensional vertical cross section, and these eventually middle points also satisfy Lemmas 1, 2 and 3 generalized to $\mathbb{R}^d$ replacing B with $B \times \mathbb{R}^{d-2}$.

We have that $\mathbf{M}_d^-$ is maximal among spacetimes that do not contain CTCs because each step of the proof of Proposition 3 goes through in this modified construction. The same way as we did in Step 1 and Step 2, we can find a point p from the extension corresponding to a removed point. Because the 2-dimensional vertical cross section of $\mathbf{M}_d^-$ through this point p is isomorphic to $\mathbf{M}_2^-$, curves δ, δ', λ_n's, τ and τ' of the two 2-dimensional construction exist also in $\mathbf{M}_d^-$. So, since Lemma 4 works in any dimension, we can repeat the same proof with these curves and find the CTC we are searching for.

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Declarations

Conflict of Interest The authors declare no conflict of interest.

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[3]Line, plane, etc. depending on the dimension d.

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