A Hierarchy of Spacetime Symmetries: Holes to Heraclitus

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Abstract

We present a hierarchy of symmetry conditions within the context of general relativity. The weakest condition captures a sense in which spacetime is free of symmetry "holes" of a certain type. All standard models of general relativity satisfy the condition but we show that violations can occur if the Hausdorff assumption is dropped. On the other extreme, the strongest condition of the hierarchy is satisfied whenever a model is completely devoid of symmetries. In these "Heraclitus" spacetimes, no pair of distinct points can be mapped (even locally) into one another. We prove that such spacetimes exist. We also show a sense in which Heraclitus spacetimes are completely determined by their local properties. We close with a brief comment on the prospect of using the symmetries of a spacetime as a guide to how much "structure" it possesses.

1. Introduction

In the instructive and influential second chapter of his book *World Enough and Spacetime*, Earman ([1989]) constructs an elegant hierarchy of classical spacetime theories. The hierarchy tracks both the geometric structures involved as well as the associated spacetime symmetries. Stepping back, one finds that "as the space-time structure becomes richer, the symmetries become narrower, the list of absolute quantities increases, and more and more questions about motion become meaningful" (Earman [1989], p. 36).

Following Earman, here we also construct a hierarchy of spacetime symmetries. But instead of comparing the symmetry properties of different spacetime theories, we restrict attention to one particular spacetime theory – general relativity – and compare the symmetry properties of different spacetime models within that theory. In this way, the symmetry hierarchy we present is akin to the hierarchy of causal conditions that has long been used in the foundations of general relativity (Hawking and Ellis [1973]).

In what follows, we begin with a few mathematical preliminaries concerning spacetime isometries. We then present the hierarchy of symmetry conditions. The weakest condition captures a sense in which a spacetime is free of symmetry "holes" of a certain type. It turns out that all standard models of general relativity satisfy the condition but we show that violations can occur if the Hausdorff assumption is dropped. On the other extreme, the strongest condition of the hierarchy is satisfied whenever a model is completely devoid of symmetries. In these "Heraclitus" spacetimes, no pair of distinct points can be mapped (even locally) into one another. We prove that such spacetimes exist. We also show a sense in which Heraclitus spacetimes are completely determined by their local properties. We close with a brief comment on the prospect of using the symmetries of a spacetime as a guide to how much "structure" it possesses.

2. Isometries and Holes

Unless otherwise flagged, a spacetime is a pair (M, g_{ab}) where M is a smooth, n-dimensional (for $n \geq 2$), connected, Hausdorff manifold without boundary and g_{ab} is a smooth, Lorentzian metric on M of signature $(-, +, ..., +)$. Given a pair of spacetimes (M, g_{ab}) and (M', g'_{ab}), we say a diffeomorphism $\psi : M \to M'$ is an isometry if $\psi^*(g'_{ab}) = g_{ab}$ where ψ^* is the pull back associated with ψ.

One can identify the collection of isometries from a spacetime (M, g_{ab}) to itself by letting $(M', g'_{ab}) = (M, g_{ab})$ in the definition. This collection of isometries are the "global symmetries" of a given spacetime. (The notion of the "local symmetries" of a spacetime is more subtle and will be considered in due course.) Of course, any spacetime (M, g_{ab}) has a trivial global symmetry: the identity map $\psi : M \to M$ defined by $\psi(p) = p$ for all $p \in M$. For some spacetimes, the identity map is its only global symmetry (see the discussion of "giraffe" spacetimes below). But virtually all example spacetimes found in textbooks have additional global symmetries such as the translations, rotations, and boosts in Minkowski spacetime.

Within this context, it might be useful to consider an influential construction used in discussions of the "hole argument" (Earman and Norton [1987]). Let (M, g_{ab}) be any spacetime, let $O \subset M$ be an open set whose closure is a proper subset of M. It is well-known that, no matter what the global symmetries of (M, g_{ab}) happen to be, there always exists a diffeomorphism $\psi : M \to M$ which is non-trivial (it is not the identity map) but which acts as the identity map on the restricted domain $M - O$. It is immediate that this "hole" diffeomorphism ψ counts as an isometry between the spacetimes (M, g_{ab}) and $(M, \psi_*(g_{ab}))$ where ψ_* is the push forward map associated with ψ. But note the following facts: (i) the identity map on M is not an isometry between (M, g_{ab}) and $(M, \psi_*(g_{ab}))$ (see Weatherall 2018) and (ii) the map ψ is not an isometry from (M, g_{ab}) to itself. The latter fact tells us that a hole diffeomorphism ψ fails to be a global symmetry of the spacetime (M, g_{ab}).

Both facts (i) and (ii) follow from special cases of Theorem 1 in Halvorson and Manchak [forthcoming] which itself follows from a more general uniqueness result due to Geroch ([1969], Theorem A1). Theorem 1 states: given any pair of spacetimes (M, g_{ab}) and (M', g'_{ab}) and any pair of isometries $\psi, \varphi : M \to M'$, if ψ and φ agree on an open region of M, then $\psi = \varphi$. To see that fact (i) holds, consider the case where $M = M'$, ψ is the hole diffeomorphism given above, and $g'_{ab} = \psi_*(g_{ab})$. Since ψ is an isometry between (M, g_{ab}) and $(M, \psi_*(g_{ab}))$ and since ψ and the identity map on M agree on the open region $M - O$, we find that the identity map on M cannot be an isometry between (M, g_{ab}) and $(M, \psi_*(g_{ab}))$. To see that fact (ii) holds, consider the case where $M = M'$, $g_{ab} = g'_{ab}$, and φ is the identity map on M. Since φ is an isometry from (M, g_{ab}) to itself and since φ and the hole diffeomorphism ψ agree on the open region $M - O$, we find that ψ cannot be an isometry between (M, g_{ab}) and itself. Let us now consider an explicit example to shed additional light on facts (i) and (ii).

Example 1: Let (M, g_{ab}) be two-dimensional Minkowski spacetime where $M = \mathbb{R}^2$ in (t, x) coordinates and $g_{ab} = -\nabla_a t \nabla_b t + \nabla_a x \nabla_b x$. Let O be the $t > 0$ portion of M. Consider the hole diffeomorphism $\psi : M \to M$ defined by $\psi(t, x) = (t, x)$ for all $t \leq 0$ and $\psi(t, x) = (t + e^{-1/t}, x)$ for all $t > 0$. One can verify that ψ is smooth and has a smooth inverse. Clearly, it is non-trivial but acts as the identity on $M - O$. What is $\psi^*(g_{ab})$? Let us carefully pull back the metric and see what we find (see Malament 2012, pp. 35-42). For the region $M - O$, things are easy since ψ acts as the identity there; we have $\psi^*(g_{ab}) = g_{ab}$. To do the calculation for the region O, we first pull back the coordinate maps $t : M \to \mathbb{R}$ and $x : M \to \mathbb{R}$ to find $\psi^* t = t \circ \psi = t + e^{-1/t}$ and $\psi^* x = x \circ \psi = x$. Next, we pull back the derivatives of the coordinate maps to find $\psi^*(\nabla_a t) = \nabla_a(\psi^* t) = \nabla_a(t + e^{-1/t}) = (1 + t^{-2} e^{-1/t}) \nabla_a t$ and $\psi^*(\nabla_a x) = \nabla_a(\psi^* x) = \nabla_a x$. We now pull back the metric.

$$\begin{aligned}
\psi^*(g_{ab}) &= \psi^*(-\nabla_a t \nabla_b t + \nabla_a x \nabla_b x) \\
&= -\psi^*(\nabla_a t)\psi^*(\nabla_b t) + \psi^*(\nabla_a x)\psi^*(\nabla_b x) \\
&= -(1 + t^{-2} e^{-1/t})^2 \nabla_a t \nabla_b t + \nabla_a x \nabla_b x
\end{aligned}$$

We see immediately that $\psi^*(g_{ab}) \neq g_{ab}$ everywhere on O. Consider the point $p = (1, 0)$ for example. We find that $\psi^*(g_{ab})(p) = -(1 + 1/e)^2 \nabla_a t \nabla_b t + \nabla_a x \nabla_b x$ while $g_{ab}(p) = -\nabla_a t \nabla_b t + \nabla_a x \nabla_b x$. Since $(1 + 1/e)^2 \neq 1$, we see that $\psi^*(g_{ab})(p) \neq g_{ab}(p)$. Because $\psi^*(g_{ab}) \neq g_{ab}$, it is immediate that (ii) the map ψ is not an isometry from (M, g_{ab}) to itself. Think of it this way. The diffeomorphism ψ maps the points $o = (0, 0)$ and $p = (1, 0)$ to the points $\psi(o) = o = (0, 0)$ and $\psi(p) = (1 + 1/e, 0)$. If ψ were an isometry from (M, g_{ab}) to itself, then ψ would preserve all distances as determined by g_{ab}. But it doesn't. According to g_{ab}, the elapsed time along the timelike geodesic from o to p is 1 while the elapsed time along the timelike geodesic from $\psi(o)$ to $\psi(p)$ is $1 + 1/e$.

Of course, the map ψ does count as an isometry from (M, g_{ab}) to the spacetime $(M, \psi_*(g_{ab}))$ where ψ_* is the push forward map associated with ψ. This follows since $\psi^*(\psi_*(g_{ab})) = g_{ab}$. Similarly, the map ψ counts as an isometry from $(M, \psi^*(g_{ab}))$ to (M, g_{ab}). Indeed, one can show that the elapsed time along the timelike geodesic from o to p according to $\psi^*(g_{ab})$ is $1+1/e$ while the elapsed time along the timelike geodesic from $\psi(o)$ to $\psi(p)$ according to g_{ab} is also $1 + 1/e$ as noted above. But one can verify that (i) the identity map $\varphi : M \to M$ is not an isometry between (M, g_{ab}) and $(M, \psi_*(g_{ab}))$. Suppose not. So $\varphi^*\psi_*(g_{ab}) = g_{ab}$. Since φ is the identity map, we have $\varphi^*(\psi_*(g_{ab})) = \psi_*(g_{ab})$. So $g_{ab} = \psi_*(g_{ab})$. Pulling back both sides with ψ^*, we find $\psi^*(g_{ab}) = \psi^*(\psi_*(g_{ab})) = g_{ab}$. But this contradicts our finding above that $\psi^*(g_{ab}) \neq g_{ab}$.

3. Symmetry Hierarchy

In what follows, six symmetry conditions will be considered (along with the conjunction of two of the conditions). As we explore them, the diagram below will be a useful guide. Arrows correspond to implication relations. If two conditions in the diagram are not connected by an arrow (or series of arrows), then the corresponding implication relation does not hold (examples will be given to show this).

$$
\begin{tikzcd}
& \text{Heraclitus} \arrow[d, Rightarrow] & \\
& \text{Locally Giraffe} \arrow[d, Rightarrow] & \\
& \text{Giraffe} \arrow[dl, Rightarrow] \arrow[d, Leftrightarrow] \arrow[dr, Rightarrow] & \\
\text{Point Rigid} \arrow[dr, Rightarrow] & \text{\&} & \text{Fixed Point} \arrow[dl, Rightarrow] \\
& \text{Rigid} &
\end{tikzcd}
$$

4. Global Symmetries

We begin with the weakest condition whose formulation draws on the "hole" construction considered in the previous section. The condition essentially requires that when global spacetime symmetries are fixed in an open region – however small – they are fixed everywhere. Following Geroch ([1969]), we will usually refer to such spacetimes as “rigid” to avoid confusion with issues related the hole argument and also with other types of spacetime “holes” related to causal determinism and prediction (Geroch [1977]; Manchak [2014]).

> Rigid: A spacetime (M, g_{ab}) is rigid if, for any open set $O \subseteq M$ and any isometry $\psi : M \to M$, if ψ is the identity map on O, then ψ is the identity map on M.

Proposition 1
Any spacetime is rigid.

A proof of the proposition is given in Halvorson and Manchak ([forthcoming], Corollary 3) which draws on a general rigidity theorem due to Geroch ([1969], Theorem A1). Given that every spacetime is rigid, the condition would seem to be quite weak. But it is worth appreciating that violations of rigidity can easily occur if the Hausdorff condition is relaxed. Consider the following example.

> Example 2: Let (M, g_{ab}) be the non-Hausdorff version of two-dimensional Minkowski spacetime where the manifold M is the “plane with two origins” $o, o' \in M$. Let $\psi : M \to M$ be the bijection which exchanges o and o' but maps any other point in M to itself (see Figure 1). One can verify that ψ counts as an isometry. Moreover, since ψ is non-trivial yet acts as the identity map on the open set $M - \{o, o'\}$, we see that (M, g_{ab}) is not rigid.

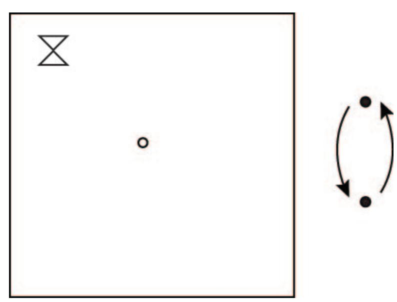

Figure 1: Example 2

The example shows that a non-Hausdorff spacetime need not be rigid: fixing symmetries in an open set does not fix the symmetries everywhere. It turns out that one can generalize the result in the natural way. Consider an arbitrary non-Hausdorff spacetime (M, g_{ab}) with any pair of non-Hausdorff witness points $p, p' \in M$. Using the bijection on M which exchanges p and p' but maps any other point in M to itself, one can show the following.

Proposition 2
Any non-Hausdorff spacetime fails to be rigid.

The rigidity condition requires that when global spacetime symmetries are fixed in an open region – however small – they are fixed everywhere. One can naturally strengthen the condition by requiring that when global spacetime symmetries are fixed at a single point, they are fixed everywhere. Consider the following.

Point Rigid: A spacetime (M, g_{ab}) is point rigid if, for any point $p \in M$ and any isometry $\psi : M \to M$, if $\psi(p) = p$, then ψ is the identity map.

Proposition 3
Any point rigid spacetime is rigid. The implication does not go in the other direction.

The proof of the first statement is immediate since any spacetime is rigid. The following example shows that some rigid spacetimes are not point rigid.

Example 3: Let $(\mathbb{R}^2, \eta_{ab})$ be two-dimensional Minkowski spacetime in standard (t, x) coordinates. By Proposition 1, it is rigid. Let $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry defined by $\psi(t, x) = (t, -x)$. So $\psi(p) = p$ for $p = (0, 0)$ but ψ is not the identity map. So the spacetime fails to be point rigid.

Another natural way to strengthen the rigidity condition is to require that, at least at some points, the global spacetime symmetries are completely fixed. Consider the following.

Fixed Point: A spacetime (M, g_{ab}) has a fixed point if there is a point $p \in M$ such that $\psi(p) = p$ for any isometry $\psi : M \to M$.

Proposition 4
Any spacetime with a fixed point is rigid. The implication does not go in the other direction.

The proof of the first statement is immediate since any spacetime is rigid. Two-dimensional Minkowski spacetime (Example 3) can be used to show that some rigid spacetimes fail to have a fixed point. Consider the isometry $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\psi(t, x) = (t + 1, x)$. Since $\psi(p) \neq p$ for all $p \in \mathbb{R}^2$, the spacetime fails to have a fixed point. Now, what is the relationship between the fixed point condition and the point rigid condition? It turns out they are independent. Consider the following.

Proposition 5
Some spacetimes with a fixed point fail to be point rigid. Some point rigid spacetimes fail to have a fixed point.

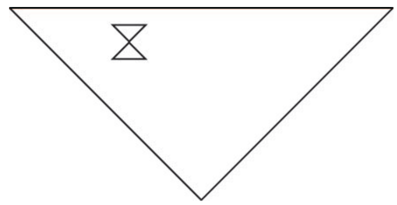

Figure 2: Example 4

Example 4: Let $(\mathbb{R}^2, \eta_{ab})$ be two-dimensional Minkowski spacetime in standard (t, x) coordinates. Consider (M, η_{ab}) where $M = \{(t, x) : 0 < t < 1, x^2 < t^2\}$ (see Figure 2). Aside from the identity map, there is only one other isometry $\psi : M \to M$ defined by the reflection $\psi(t, x) = (t, -x)$. So for any isometry $\psi : M \to M$, we have $\psi(p) = p$ for the point $p = (1/2, 0)$ showing that the spacetime has a fixed point. But since the identity map is not the only isometry such that $\psi(p) = p$, the spacetime is not point rigid.

Example 5: Let $(\mathbb{R}^2, \eta_{ab})$ be two-dimensional Minkowski spacetime in standard (t, x) coordinates. For each integer n, excise the compact region enclosed by the points $(0, n)$, $(1/2, n)$, and $(0, n + 1/2)$. Let the resulting spacetime be (M, η_{ab}) (see Figure 3). For each integer n, there is an isometry $\psi_n : M \to M$ defined by $\psi_n(t, x) = (t, x + n)$. But these are the only isometries by construction. It follows that the spacetime fails to have a fixed point but is point rigid.

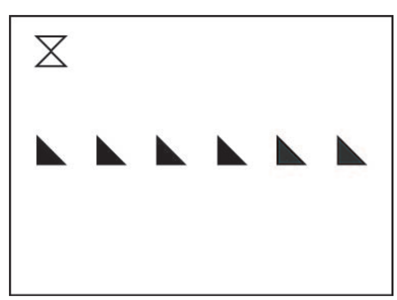

Figure 3: Example 5

Now we come to the strongest condition concerning global symmetries: the requirement that they are completely fixed. Consider the following.

Giraffe: A spacetime (M, g_{ab}) is giraffe if the only isometry $\varphi : M \to M$ is the identity map.

David Malament has suggested an elegant way to construct a giraffe spacetime: take Minkowski spacetime and excise a compact region “shaped like a giraffe” (Barrett et al. [forthcoming]). The shape of a sufficiently asymmetric giraffe ensures that there are no global symmetries. A less interesting but more tractable giraffe spacetime will be constructed later on (Example 6). How strong is the giraffe condition? It has been claimed that “everyone knows” giraffe spacetimes are generic in some sense (D’Ambra and Gromov [1991], p. 21). But the meaning of “generic” is not made precise and a general proof remains elusive (Mounoud [2015]). How is the giraffe condition related to the other symmetry conditions considered so far? We have the following.

Proposition 6
Any giraffe spacetime is point rigid and has a fixed point. The implications do not go in the other direction.

The proof of the first statement is immediate from the definitions. Example 4 is a spacetime with a fixed point which fails to be giraffe. Example 5 is a point rigid spacetime which fails to be giraffe. When considered separately, both the point rigid and the fixed point conditions are strictly weaker than the giraffe condition. However, the conjunction of these conditions turns out to be strong enough to imply the giraffe condition. Consider the following.

Proposition 7
A spacetime is giraffe if and only if it is both point rigid and has a fixed point.

Proof
One direction is trivial. Suppose a spacetime (M, g_{ab}) is both point rigid and has a fixed point. Let $\psi : M \to M$ be any isometry. Since the spacetime has a fixed point, there is a point $p \in M$ such that $\psi(p) = p$. Because the spacetime is point rigid, we know that for all $q \in M$, if $\psi(q) = q$, then ψ is the identity map. Since $\psi(p) = p$, it follows that ψ is the identity map. $\square$

5. Local Symmetries

We now turn to the notion of the “local” symmetries of spacetime. There are a number of conditions one might consider. For example, one might explore those involving the non-existence of the “infinitesimal isometries” associated with Killing vector fields. Given a spacetime (M, g_{ab}), we say a smooth vector field λ^a on M is a Killing field if $\pounds_\lambda g_{ab} = \mathbf{0}$. Here, the Lie derivative term $\pounds_\lambda g_{ab}$ represents the “rate of change” of the metric along the local flow maps determined by λ^a. Now the local flow maps need not be globally defined (Malament [2012], p. 175). But if they are, then the spacetime admits a non-trivial global isometry and must therefore fail to be giraffe. But now consider a spacetime (M, g_{ab}) which contains no “local Killing fields” in the sense that for every open connected set $O \subseteq M$, the spacetime (O, g_{ab}) has no Killing fields aside from the zero tensor. One might be tempted to declare such a spacetime free of local symmetries. But one must keep in mind that the full collection of spacetime symmetries "may include some discrete isometries (such as reflections in a plane) which are not generated by Killing vector fields" (Hawking and Ellis [1973], p. 44). This will be important later on.

Another, more general, approach to the "local" symmetries of spacetime makes use of the machinery built up so far concerning global symmetries. A natural condition along these lines is the requirement that any open connected region has only trivial global symmetries when considered as a spacetime in its own right. Consider the following.

Locally Giraffe: A spacetime (M, g_{ab}) is locally giraffe if, for any connected open set $O \subseteq M$ the spacetime (O, g_{ab}) is giraffe.

Proposition 8
Any locally giraffe spacetime is giraffe. The implication does not go in the other direction.

The proof of the first statement is immediate from the definitions. The following example is giraffe but not locally giraffe.

Example 6: Let $(\mathbb{R}^2, \eta_{ab})$ be two-dimensional Minkowski spacetime in standard (t, x) coordinates. Consider (M, η_{ab}) where $M = \{(t, x) : 0 < t < 1, 0 < x, x^2 < t^2\}$ (see Figure 4). This spacetime is just the $x > 0$ portion of Example 4. The identity map is the only isometry showing the spacetime is giraffe. But consider the connected open set $O = \{(t, x) \in M : t + x < 1\}$ which is the region below the dotted line. The spacetime (O, η_{ab}) is not giraffe since there is an isometry $\psi : O \to O$ defined by $\psi(t, x) = (-t + 1, x)$ which reflects O about the $t = 1/2$ line.

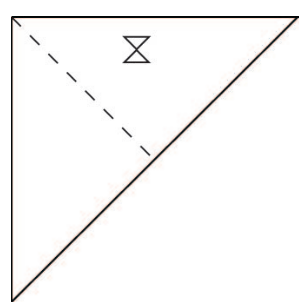

Figure 4: Example 6

We now come to the strongest condition in the symmetry hierarchy which requires that no pair of distinct points can be isometrically mapped – even locally – into one another. Consider the following.

Heraclitus: A spacetime (M, g_{ab}) is Heraclitus if, for any distinct points $p, q \in M$, and any open neighborhoods $O_p, O_q \subseteq M$ of p and q respectively, there is no isometry $\psi : O_p \to O_q$ such that $\psi(p) = q$.

A Heraclitus spacetime is utterly devoid of symmetries – global and local. Since any neighborhoods of any distinct points fail to be isometric, each event is unlike any other. One might say that in such a spacetime "it is impossible to step in the same river twice." One can show that any Heraclitus spacetime is locally giraffe but not the other way around. We have the following.

Proposition 9
Any Heraclitus spacetime is locally giraffe. The implication does not go in the other direction.

Proof
Let (M, g_{ab}) be a spacetime which fails to be locally giraffe. Then there is some connected open set $O \subseteq M$ such that (O, g_{ab}) is not giraffe. So there is an isometry $\psi : O \to O$ which is not the identity map. So for some point $q \in O$, we have $\psi(q) = r$ where $r \neq q$. So there are distinct points $q, r \in M$ and open neighborhoods $O_q = O$ and $O_r = O$ of q and r respectively such that there is an isometry $\psi : O_q \to O_r$ with $\psi(q) = r$. So (M, g_{ab}) fails to be Heraclitus. For a locally giraffe spacetime which fails to be Heraclitus, see Example 7 below. □

Clearly, the Heraclitus condition is quite strong in the sense that it sits atop the hierarchy presented here. On the other hand, perhaps Heraclitus spacetimes are "generic" in the same sense that giraffe spacetimes seem to be due to their asymmetries. Stepping back, there is a general question of interest here: just how "physically reasonable" is the Heraclitus condition?

We close this section by giving an equivalent definition of a Heraclitus spacetime which does not make reference to points and their neighborhoods. This will be useful in what follows.

Heraclitus*: A spacetime (M, g_{ab}) is Heraclitus* if, for any open sets $U, V \subseteq M$ and any isometry $\psi : U \to V$, it follows that (i) $U = V$ and (ii) ψ is the identity map.

Proposition 10
A spacetime is Heraclitus if and only if it is Heraclitus*.

Proof
Suppose a spacetime (M, g_{ab}) fails to be Heraclitus. So there are distinct points $p, q \in M$, and open neighborhoods $O_p, O_q \subseteq M$ of p and q respectively, such that there is an isometry $\psi : O_p \to O_q$ with $\psi(p) = q$. Let U and V be the open sets O_p and O_q respectively. If $U \neq V$ then (M, g_{ab}) fails to satisfy (i) in the definition of a Heraclitus* spacetime. Suppose then that $U = V$. Since $p, q \in U$ are distinct and $\psi(p) = q$, then (M, g_{ab}) fails to satisfy (ii) in the definition of

a Heraclitus* spacetime. So (M, g_{ab}) fails to be Heraclitus*.

Now suppose (M, g_{ab}) fails to be Heraclitus*. So for some open sets $U, V \subseteq M$ there is an isometry $\psi : U \to V$ such that either (i) $U \neq V$ or (ii) ψ fails to be the identity map. Suppose (i) $U \neq V$. So either there is a point $p \in U$ which fails to be in V or there is a point $r \in V$ which fails to be in U. Suppose the first possibility obtains (an analogous argument can be made for the other case). So $\psi(p) = q$ for some point $q \neq p$. So there are distinct points p and q and open neighborhoods $O_p = U$ and $O_q = V$ of p and q respectively, such that there is an isometry $\psi : O_p \to O_q$ with $\psi(p) = q$. So (M, g_{ab}) fails to be Heraclitus. Now suppose that $U = V$ but (ii) ψ fails to be the identity map. Then there will be distinct points $p, q \in U$ such that $\psi(p) = q$. So there are open neighborhoods $O_p = U$ and $O_q = U$ of p and q respectively, such that there is an isometry $\psi : O_p \to O_q$ with $\psi(p) = q$. So (M, g_{ab}) fails to be Heraclitus. □

6. Heraclitus Existence

There is a vast literature on "inhomogeneous cosmology" which investigates a variety of asymmetric models of the universe (Ellis [2011]). Even so, it seems that all of the examples considered make use of various "symmetries which are sufficiently strong to render the field equations tractable" (Collins and Szafron [1979] p. 2347). One might therefore wonder about the possibility of finding a spacetime without any symmetries at all. Do Heraclitus spacetimes even exist?

In a paper entitled "A Metric with No Symmetries or Invariants," Koutras and McIntosh [1996] present a peculiar spacetime. Consider the manifold $M = \mathbb{R}^4$ in (u, w, x, y) coordinates and let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary smooth function. The metric g_{ab} on M is given by the following.

$$\begin{aligned} g_{ab} \quad &= \quad 2x\nabla_{(a}w\nabla_{b)}u - 2w\nabla_{(a}u\nabla_{b)}x \\ &\quad\ +[2f(u)x(x^2+y^2) - w^2]\nabla_a u\nabla_b u - \nabla_a x\nabla_b x - \nabla_a y\nabla_b y \end{aligned}$$

One can show that the spacetime (M, g_{ab}) admits no local Killing fields. This is a remarkable property. But here it is instructive to recall that there are discrete isometries that are not generated by Killing fields. In the present case, one can easily verify that (M, g_{ab}) has a global isometry $\psi : M \to M$ defined by the reflection $\psi(u, w, x, y) = (u, w, x, -y)$. So not only does the spacetime fail to be Heraclitus, it isn't even point rigid. Stepping back, it may be that restricting attention to the $y > 0$ portion of M will result in a Heraclitus spacetime for an appropriate choice of the function $f(u)$. In any case, here we present a simple Heraclitus example in order to get a better grip on the condition. Our example is significantly different from the one presented by Koutras and McIntosh [1996] in the sense that the latter has vanishing scalar polynomial curvature invariants while the former does not. Indeed, it is the special the behavior of these non-vanishing scalar curvature invariants in our example that will allow us to prove an even stronger existence result later on.

To help the reader along, we begin with a conceptual overview of what follows. First, consider two-dimensional Minkowski spacetime $(\mathbb{R}^2, \eta_{ab})$ in standard

(t,x) coordinates where $\eta_{ab} = -\nabla_a t \nabla_b t + \nabla_a x \nabla_b x$. We let $f = t^2 + x^2$ be the Euclidean distance from the origin. Next we consider the conformally flat spacetime (M, g_{ab}) such that $M = \mathbb{R}^2 - \{(0,0)\}$ and $g_{ab} = \Omega^2 \eta_{ab}$ where $\Omega = f^{-1}$. We then consider two scalar curvature invariants associated with (M, g_{ab}): the Ricci scalar R and the scalar Q defined by $g^{ab}(\tilde{\nabla}_a R)\tilde{\nabla}_b R$ where $\tilde{\nabla}$ is the unique derivative operator associated with g_{ab}. A lemma shows that $R = 8(t^2 - x^2)$ and $Q = -32Rf^2$. We then consider the spacetime (N, g_{ab}) where N is the portion of M for which $t, x > 0$ and $t^2 > x^2$. On this region we have $R > 0$. Since R and Q are both invariant scalar functions, it follows that any local isometry between open sets of N must map a point p to a point q with the same R and Q values. Since $R > 0$ and $Q = -32Rf^2$, we find that p and q must have the same f values as well. Finally, because of the way N is truncated, p and q can have the same R and f values only if $p = q$ (see Figure 5). So (N, g_{ab}) must be Heraclitus.

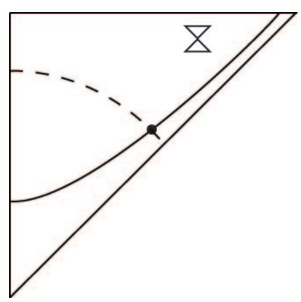

Figure 5: The spacetime (N, g_{ab}).

Lemma 1
(M, g_{ab}) is such that $R = 8(t^2 - x^2)$ and $Q = -32Rf^2$.

Proof
Consider the manifold $\mathbb{R}^2$ in (t,x) coordinates and let ∇ be the associated coordinate derivative operator. Let $(\mathbb{R}^2, \eta_{ac})$ be two-dimensional Minkowski spacetime where $\eta_{ac} = -\nabla_a t \nabla_c t + \nabla_a x \nabla_c x$. So $\nabla_a \eta_{nm} = \mathbf{0}$. (Here, our choice of index notation will adhere to (Wald [1984], p. 446) which will be useful for the following calculation.) For convenience let $t^a = (\frac{\partial}{\partial t})^a$ and $x^a = (\frac{\partial}{\partial x})^a$. Let χ^a be the "position field" on $\mathbb{R}^2$ relative to ∇ and the origin $(0,0)$; this is the unique, smooth vector field on $\mathbb{R}^2$ that vanishes at the origin and satisfies the condition $\nabla_a \chi^n = \delta_a^n$ (Malament [2012], p. 66). At any point $(t,x) \in \mathbb{R}^2$, one can verify that $\chi^a = tt^a + xx^a$. Let $h_{ac} = \nabla_a t \nabla_c t + \nabla_a x \nabla_c x$. We find that $\nabla_a h_{nm} = \mathbf{0}$ and $h_{nm}\xi^n\xi^m \geq 0$ for all vectors ξ^n. Let $M = \mathbb{R}^2 - \{(0,0)\}$ and let $\Omega : M \to \mathbb{R}$ be the smooth, strictly positive function f^{-1} where $f = h_{nm}\chi^n\chi^m$. One can verify that $h_{nm}\chi^n\chi^m = t^2 + x^2$ and so $\Omega = (t^2 + x^2)^{-1}$. We have the

following facts which will be useful later on.

$$\begin{aligned}
\nabla_a f &= \nabla_a[h_{nm}\chi^n\chi^m] = h_{nm}[\chi^n\delta_a^m + \chi^m\delta_a^n] = 2h_{an}\chi^n \\
\nabla_a\nabla_c f &= \nabla_a[2h_{cn}\chi^n] = 2h_{cn}\delta_a^n = 2h_{ac} \\
\nabla_a\Omega &= \nabla_a f^{-1} = -f^{-2}\nabla_a f = -2f^{-2}h_{an}\chi^n = -2\Omega^2 h_{an}\chi^n \\
\nabla_a\nabla_c\Omega &= \nabla_a[-f^{-2}\nabla_c f] = 2f^{-3}(\nabla_a f)\nabla_c f - f^{-2}\nabla_a\nabla_c f. \\
&= 8\Omega^3 h_{an}h_{cm}\chi^n\chi^m - 2\Omega^2 h_{ac} \\
\eta^{de}h_{dn}h_{em} &= [-t^d t^e + x^d x^e][\nabla_d t\nabla_n t + \nabla_d x\nabla_n x][\nabla_e t\nabla_m t + \nabla_e x\nabla_m x] \\
&= [-t^d t^e + x^d x^e][\nabla_d t\nabla_n t\nabla_e t\nabla_m t + \nabla_d t\nabla_n t\nabla_e x\nabla_m x \\
&\quad +\nabla_d x\nabla_n x\nabla_e t\nabla_m t + \nabla_d x\nabla_n x\nabla_e x\nabla_m x] \\
&= -\nabla_n t\nabla_m t + \nabla_n x\nabla_m x = \eta_{nm} \\
\eta^{de}h_{de} &= [-t^d t^e + x^d x^e][\nabla_d t\nabla_e t + \nabla_d x\nabla_e x] = -1 + 1 = 0 \\
\eta^{de}(\nabla_d\Omega)\nabla_e\Omega &= \eta^{de}[-2\Omega^2 h_{dn}\chi^n][-2\Omega^2 h_{em}\chi^m] \\
&= 4\Omega^4\eta_{nm}\chi^n\chi^m \\
\eta^{de}\nabla_d\nabla_e\Omega &= \eta^{de}[8\Omega^3 h_{dn}h_{em}\chi^n\chi^m - 2\Omega^2 h_{de}] \\
&= 8\Omega^3\eta_{nm}\chi^n\chi^m
\end{aligned}$$

Because (M, g_{ac}) is conformally flat and two-dimensional, we find that $R = -2\Omega^{-2}\eta^{ac}\nabla_a\nabla_c \ln\Omega$ where ∇ is the coordinate derivative operator compatible with η_{ac} (see Wald [1984], p. 446). Let $\tilde{\nabla}$ be the derivative operator compatible with g_{ac} and note that $\tilde{\nabla}_a R = \nabla_a R$ since the action of any two derivative operators agree on a scalar field. Using the facts from above, we have the following as claimed.

$$\begin{aligned}
R &= -2\Omega^{-2}\eta^{ac}\nabla_a\nabla_c\ln\Omega \\
&= -2\Omega^{-2}\eta^{ac}[-\Omega^{-2}(\nabla_a\Omega)\nabla_c\Omega + \Omega^{-1}\nabla_a\nabla_c\Omega] \\
&= 2\Omega^{-4}\eta^{ac}(\nabla_a\Omega)\nabla_c\Omega - 2\Omega^{-3}\eta^{ac}\nabla_a\nabla_c\Omega \\
&= 8\Omega^{-4}\Omega^4\eta_{nm}\chi^n\chi^m - 16\Omega^{-3}\Omega^3\eta_{nm}\chi^n\chi^m \\
&= 8\eta_{nm}\chi^n\chi^m - 16\eta_{nm}\chi^n\chi^m = -8\eta_{nm}\chi^n\chi^m = 8[t^2 - x^2] \\
\tilde{\nabla}_a R &= \nabla_a R = -8\nabla_a[\eta_{nm}\chi^n\chi^m] = -8\eta_{nm}[\chi^n\delta_a^m + \chi^m\delta_a^n] = -16\eta_{an}\chi^n \\
Q &= g^{ac}(\tilde{\nabla}_a R)\tilde{\nabla}_c R = \eta^{ac}\Omega^{-2}[-16\eta_{an}\chi^n][-16\eta_{cm}\chi^m] \\
&= 256\Omega^{-2}[\delta_n^c\chi^n][\eta_{cm}\chi^m] = 256\Omega^{-2}\eta_{cm}\chi^c\chi^m = -32Rf^2 \;\Box
\end{aligned}$$

Proposition 11
There exists a Heraclitus spacetime.

Proof
Let (M, g_{ab}) be defined as in Lemma 1: $M = \mathbb{R}^2 - \{(0,0)\}$ and $g_{ab} = \Omega^2\eta_{ab}$ where $\eta_{ab} = -\nabla_a t\nabla_b t + \nabla_a x\nabla_b x$ and $\Omega = f^{-1}$ for $f = (h_{nm}\chi^n\chi^m)$. We will show that the spacetime (N, g_{ab}) is Heraclitus where $N = \{(t,x) \in M : t > 0, x > 0, t^2 > x^2\}$. Let $p = (t_p, x_p)$ and $q = (t_q, x_q)$ be any distinct points in N and

let $O_p, O_q \subseteq N$ be open neighborhoods of p and q respectively. Suppose there were an isometry $\psi : O_p \to O_q$ such that $\psi(p) = q$. We show a contradiction.

Consider the Ricci scalar $R : N \to \mathbb{R}$ associated with g_{ab} and the scalar $Q : N \to \mathbb{R}$ defined by $Q = g^{ab}(\tilde{\nabla}_a R)\tilde{\nabla}_b R$ where $\tilde{\nabla}$ is the derivative operator associated with g_{ab}. From Lemma 1, we see that $R = 8(t^2 - x^2) > 0$ on N which we will need later on. Since R, $\tilde{\nabla}$, and g^{ab} are all invariant under the isometry ψ, we know that Q is also invariant under ψ. Since $\psi(p) = q$, we have $R(p) = R(q)$ and $Q(p) = Q(q)$. In what follows, we will show that $Q(p) = Q(q)$ implies $f(p) = f(q)$ which, together with $R(p) = R(q)$, will require that $p = q$.

From Lemma 1, we know that $R = 8(t^2 - x^2)$ and $Q = -32R\Omega^{-2}$. Since $R(p) = R(q)$ we know (i) $t_p^2 - x_p^2 = t_q^2 - x_q^2$. Since $Q(p) = Q(q)$ we know $R(p)\Omega(p)^{-2} = R(q)\Omega(q)^{-2}$. Since $R(p) = R(q)$ and $R > 0$ on N, we know $\Omega(p)^{-2} = \Omega(q)^{-2}$. Since $\Omega > 0$ on N, we know $\Omega^{-1}(p) = \Omega(q)^{-1}$ and therefore $f(p) = f(q)$. So (ii) $t_p^2 + x_p^2 = t_q^2 + x_q^2$. Using equations (i) and (ii), a bit of algebra shows $t_p^2 = t_q^2$ and $x_p^2 = x_q^2$. Since both $t > 0$ and $x > 0$ on N, we have $t_p = t_q$ and $x_p = x_q$. So $p = (t_p, x_p) = (t_q, x_q) = q$ which is impossible since p and q are distinct. So there is no isometry $\psi : O_p \to O_q$ such that $\psi(p) = q$. So (N, g_{ab}) is Heraclitus. $\square$

We wish to highlight two natural questions concerning the Heraclitus spacetime (N, g_{ab}) just constructed. First, the example is two-dimensional; do there exist four-dimensional Heraclitus spacetimes? Second, the example is "extendible" in the sense that it can be properly and isometrically embedded into another spacetime; do there exist Heraclitus spacetimes which are not extendible? Both questions are open.

Stepping back a bit, the behavior of the scalar curvature invariants in the example Heraclitus spacetime given above suggests a special type of curvature condition which implies the Heraclitus symmetry condition (David Malament, private communication). Consider the following definition.

Separates Points: Let (M, g_{ab}) be a spacetime and let $\mathcal{S}$ be the collection of invariant scalar curvature functions on M. We say (M, g_{ab}) separates points if there do not exist distinct points $p, q \in M$ such that $f(p) = f(q)$ for all $f \in \mathcal{S}$.

If a spacetime fails to be Heraclitus, then it will contain distinct points which, by virtue of the local isometry between them, will have the same values for all scalar curvature invariants. Thus, the spacetime must also fail to separate points. We have the following.

Proposition 12
If a spacetime separates points, it is Heraclitus.

What about the other direction? This is an open question. In any case, the special behavior of the scalar curvature invariants of the Heraclitus spacetime given above ensures the following stronger existence result.

Proposition 13
A spacetime which separates points exists.

7. Heraclitus Properties

Here we show a sense in which Heraclitus spacetimes are completely determined by their local properties. Once we have defined "local property" in this context, we will present a recovery result: given any collection of local spacetime properties, there is at most one Heraclitus spacetime (up to isometry) with exactly those local properties. Let us say that spacetimes (M, g_{ab}) and (M', g'_{ab}) are locally isometric if each point $p \in M$ has an open neighborhood O which is isometric to some open set $O' \subseteq M'$ and, correspondingly, with the roles of (M, g_{ab}) and (M', g'_{ab}) interchanged. One can use this definition to make precise the notion of local spacetime properties. Consider the collection $\mathscr{U}$ of all spacetimes. In the natural way, a spacetime property can be regarded as a sub-collection of $\mathscr{U}$. We now have the following (Manchak [2009]).

> Local Property: A spacetime property $\mathscr{P} \subseteq \mathscr{U}$ is local if, for any locally isometric spacetimes $(M, g_{ab}), (M', g'_{ab}) \in \mathscr{U}$, we have $(M, g_{ab}) \in \mathscr{P}$ if and only if $(M', g'_{ab}) \in \mathscr{P}$.

We now show that if a pair of Heraclitus spacetimes are locally isometric, they must be isometric. Consider the following lemma (O'Neill [1983], p. 5).

Lemma 2
Let M and N be manifolds. For each index $\alpha \in A$, let O_α be an open set on M and let $\psi_\alpha : O_\alpha \to N$ be a smooth map. If, for all $\alpha, \beta \in A$, $\psi_\alpha = \psi_\beta$ on $O_\alpha \cap O_\beta$, then the unique map $\psi : \bigcup O_\alpha \to N$ defined such that $\psi_{|O_\alpha} = \psi_\alpha$ for all $\alpha \in A$ must be smooth.

Given a pair of locally isometric Heraclitus spacetimes, we can use the lemma to "patch together" the local isometries to construct a unique global isometry. The process is analogous to putting together a puzzle where the picture is so asymmetric that one knows exactly where each piece must go.

Proposition 14
If Heraclitus spacetimes are locally isometric, then they are isometric.

Proof
Let (M, g_{ab}) and (M', g'_{ab}) be locally isometric Heraclitus spacetimes. Because the spacetimes are locally isometric, for each point $p \in M$, we can fix once and for all an associated open neighborhood $O_p \subseteq M$ and an isometry $\psi_p : O_p \to O'_p$ where O'_p is an open set in M'. Let p, q be any points in M and suppose there is a point $r \in O_p \cap O_q$. Let $U' = \psi_p[O_p \cap O_q]$ and $V' = \psi_q[O_p \cap O_q]$. Since ψ_p and

ψ_q are isometries, we know that $\psi_q \circ \psi_p^{-1} : U' \to V'$ is an isometry which maps $\psi_p(r)$ to $\psi_q(r)$. Since (M', g'_{ab}) is Heraclitus, it follows that $\psi_p(r) = \psi_q(r)$. So $\psi_p = \psi_q$ on the region $O_p \cap O_q$ for any $p, q \in M$. Since $\bigcup O_p = M$, it follows from Lemma 2 that the unique map $\psi : M \to M'$ defined such that $\psi_{|O_p} = \psi_p$ for all $p \in M$ must be smooth.

Next we show that ψ is a bijection. Let p, q be any points in M and suppose that $\psi(p) = \psi(q)$. So $\psi_p(p) = \psi_q(q)$ where $\psi_p : O_p \to O'_p$ and $\psi_q : O_q \to O'_q$ are the isometries associated with p and q. Let $U = \psi_p^{-1}[O'_p \cap O'_q]$ and $V = \psi_q^{-1}[O'_p \cap O'_q]$. Since ψ_p and ψ_q are isometries, we know $\psi_q^{-1} \circ \psi_p : U \to V$ is an isometry which maps p to q. Since (M, g_{ab}) is Heraclitus, it follows that $p = q$ and thus ψ is injective. Now let p' be any point in M'. Because the spacetimes are locally isometric, there is an isometry $\varphi : N' \to N$ where N' is an open neighborhood of p' and N is an open set in M. Let $p \in M$ be the point $\varphi(p')$ and consider its associated isometry $\psi_p : O_p \to O'_p$. Let $U' = \varphi^{-1}[O_p \cap N]$ and $V' = \psi_p[O_p \cap N]$. Since φ and ψ_p and are isometries, we know that $\psi_p \circ \varphi : U' \to V'$ is an isometry which maps p' to $\psi_p(p)$. Since (M', g'_{ab}) is Heraclitus, it follows that $p' = \psi_p(p)$. Because $\psi_p(p) \in \psi[M]$, we know $p' \in \psi[M]$ and thus ψ is surjective. So ψ is a bijection.

Next we show that ψ^{-1} is smooth. For each $p \in M$, we can consider the inverse of its associated isometry: $\psi_p^{-1} : O'_p \to O_p$. Let p, q be any points in M. Suppose there is a point $r' \in O'_p \cap O'_q$. The map ψ is defined such that $\psi_{|O_p} = \psi_p$ for all $p \in M$. So ψ must send the point $\psi_p^{-1}(r') \in O_p$ to the point $r' \in O'_p$. Similarly, ψ must send the point $\psi_q^{-1}(r') \in O_q$ to the point $r' \in O'_q$. Since ψ is injective, we know $\psi_p^{-1}(r') = \psi_q^{-1}(r')$. So $\psi_p^{-1} = \psi_q^{-1}$ on the region $O'_p \cap O'_q$ for any $p, q \in M$. Since ψ is surjective, it follows that $\bigcup O'_p = M'$. So ψ^{-1} is the unique map from $\bigcup O'_p = M'$ to M defined such that $\psi^{-1}_{|O'_p} = \psi_p^{-1}$ for all $p \in M$. By Lemma 2, ψ^{-1} must be smooth.

Since ψ is a smooth bijection with a smooth inverse, it is a diffeomorphism. The final step is to verify that it is an isometry. Consider any point $p \in M$ and its associated isometry $\psi_p : O_p \to O'_p$. We know $\psi_p^*(g'_{ab}) = g_{ab}$ on the region O_p where ψ_p^* is the pull-back associated with ψ_p. Since $\psi_{|O_p} = \psi_p$, we know $\psi^*(g'_{ab}) = g_{ab}$ on O_p where ψ^* is the pull-back associated with ψ. Since p was chosen arbitrarily, $\psi^*(g'_{ab}) = g_{ab}$ on all of M and thus ψ is an isometry. $\square$

From Proposition 14, we have the following result which captures a sense in which Heraclitus spacetimes are completely determined by their local properties.

Corollary 1

Given any collection of local spacetime properties, there is at most one Heraclitus spacetime (up to isometry) with exactly those local properties.

Proof

Given a collection of local properties, suppose there were non-isometric Heraclitus spacetimes (M, g_{ab}) and (M', g'_{ab}) with exactly those local properties. Proposition 14 requires that the spacetimes are not locally isometric. Let $\mathscr{P} \subset \mathscr{U}$

be the collection of spacetimes locally isometric to (M, g_{ab}). We find that $\mathscr{P}$ is a local property possessed by (M, g_{ab}) but not (M', g'_{ab}). So the spacetimes cannot have the same collection of local properties: a contradiction. $\square$

Stepping back, one might wonder whether the Heraclitus property in Proposition 14 can be weakened somewhat: are locally giraffe spacetimes which are locally isometric also required to be isometric? The following example shows they need not be. The example also makes good on our earlier claim that there exist locally giraffe spacetimes which fail to be Heraclitus (recall Proposition 9).

Example 7: Let (N, g_{ab}) be the Heraclitus spacetime constructed in Proposition 11. Let (N_i, g_i) for $i = 1, 2, 3$ be three copies of the spacetime (N, g_{ab}). In copies (N_1, g_1) and (N_2, g_2) cut slits $S = \{(t, x) : t = 3, 1 \leq x \leq 2\}$ and, excluding the boundary points, identify the "top edge" of slit S in (N_1, g_1) with the "bottom edge" of slit S in (N_2, g_2) (but not vice versa). In copies (N_2, g_2) and (N_3, g_3) cut slits $S' = \{(t, x) : t = 4, 1 \leq x \leq 2\}$ and, excluding the boundary points, identify the "top edge" of slit S' in (N_2, g_2) with the "bottom edge" of slit S' in (N_3, g_3) (but not vice versa). Let the resulting spacetime be (N', g'_{ab}) (see Figure 6). Because of the "missing" slit boundary points, we know that (N', g'_{ab}) is not isometric to (N, g_{ab}). But the two spacetime do count as locally isometric despite the fact that there are points "missing" from (N', g'_{ab}). Consider, for example, the points $p = (3, 1)$ and $q = (4, 1)$ in (N, g_{ab}). Although p is "missing" from copies (N_1, g_1) and (N_2, g_2) in (N', g'_{ab}) due to the S slits, p remains in place in copy (N_3, g_3); similarly, although q is "missing" from copies (N_2, g_2) and (N_3, g_3) due to the S' slits, it remains in place in copy (N_1, g_1). So we see why at least three linked copies are needed to ensure that each small region of (N, g_{ab}) is isometrically reproduced somewhere in (N', g'_{ab}). Of course, each small region of (N', g'_{ab}) is isometrically reproduced somewhere in (N, g_{ab}) since the former is constructed from copies of the latter. Finally, let us note that since (N, g_{ab}) is Heraclitus, it must be locally giraffe. But although (N', g'_{ab}) is not Heraclitus (due to the multiple copies of some of its small regions) one can verify that it is locally giraffe.

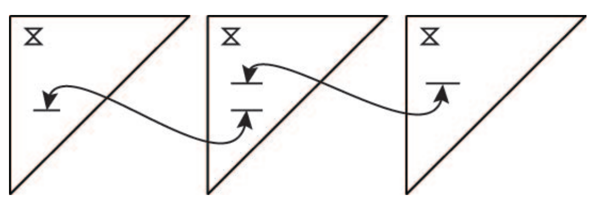

Figure 6: The spacetime (N', g'_{ab}).

We close with a remark concerning the example just given. The cut-and-paste procedure used to transform the Heraclitus spacetime (N, g_{ab}) into the locally isometric but non-isometric locally giraffe spacetime (N', g'_{ab}) is quite similar to the "clothesline construction" introduced by Malament [1977] to produce certain non-isometric spacetimes which nonetheless share all the same local properties. Let us say that a spacetime (M, g_{ab}) is observationally indistinguishable from a spacetime (M', g'_{ab}) if, for each point $p \in M$, there is a point $p' \in M'$ such that the observational pasts $I^-(p)$ and $I^-(p')$ are isometric. The clothesline construction is used to show the following general underdetermination theorem (Manchak [2009]): For any spacetime (M, g_{ab}) without closed timelike curves, there is a counterpart spacetime (M', g') such that (i) (M, g) is observationally indistinguishable from (M', g'_{ab}) and (ii) (M, g_{ab}) and (M', g'_{ab}) share all of the same local properties and yet (iii) (M, g_{ab}) and (M', g'_{ab}) are not isometric.

But now we see that something curious follows from Corollary 1. If (M, g_{ab}) is Heraclitus, then the counterpart spacetime (M', g'_{ab}) guaranteed by the theorem cannot also be Heraclitus. Indeed, we find that the clothesline construction used to secure the result has the effect of introducing some local symmetries into (M', g'_{ab}) just like the construction used in Example 7. Another way to put the point: if attention is restricted only to Heraclitus spacetimes, then the underdetermination theorem no longer goes through since, in that case, conditions (ii) and (iii) cannot both obtain.

8. Symmetry and Structure

We have isolated a number of precise senses in which general relativity has models with "few symmetries". It is worth making a brief remark about how these results come to bear on a recent debate about symmetry and structure in spacetime theories.

There is a dogma in foundations of spacetime theories that says that the symmetries of a spacetime are a guide to its amount of structure. This rough idea can be traced back at least to the passage from Earman ([1989], p. 36) in the introduction: "As the space-time structure becomes richer, the symmetries become narrower". In addition, North ([2009], p. 87) writes that "stronger structure. . . admits a smaller group of symmetries." And more recently North ([2021], p. 50) says that one of the litmus tests for the presence of more structure on an object is that the "associated group of structure-preserving transformations becomes narrower". This idea is behind the scenes in many contemporary discussions of symmetry and structure (Bradley and Weatherall [2020]; Wilhelm [2021]; Barrett [forthcoming]; Barrett, et al. [forthcoming]).

One might take this rough idea that the symmetries of a spacetime are a guide to its amount of structure to mean that the relations "has more structure than" and "has at least as much structure as" can be "defined" or "explicated" through appeal to only the symmetries of the spacetimes X and Y under consideration. More precisely, we might understand the dogma to mean that one can find a relation $\mathfrak{R}$ between the collections of symmetries of spacetimes X and Y such that an explication of one of the following forms is adequate:

($\star$): X has more structure than Y if and only if the collection of symmetries of X stands in relation $\mathfrak{R}$ to the collection of symmetries of Y.

($\star\star$): X has at least as much structure as Y if and only if the collection of symmetries of X stands in relation $\mathfrak{R}$ to the collection of symmetries of Y.

If some such explication is adequate, then the symmetries of a spacetime would provide us with a *complete* guide to its structure. A number of explications of form ($\star$) and ($\star\star$) have recently been proposed and critically discussed. The relation $\mathfrak{R}$ should capture some sense in which the one collection of symmetries is "smaller than" the other. Swanson and Halvorson [unpublished] consider the cases where $\mathfrak{R}$ is "has lower dimension than" and "is a proper subset of"; others have considered cases where $\mathfrak{R}$ is "is a subset of", "is isomorphic to a proper subgroup of", and "is isomorphic to a subgroup of" (Barrett [forthcoming]; Wilhelm [2021]; Barrett et al. [forthcoming]). Our results here put pressure on all explications of form ($\star$) and ($\star\star$).

It is worth considering two specific explications in detail. We begin with the following natural one of form ($\star$).

(1): X has more structure than Y if and only if the collection of symmetries of X is a proper subset of the collection of symmetries of Y.

Of course, much turns here on what one means by "collection of symmetries". But our results demonstrate that (1) is an inadequate explication on some of the most natural ways of understanding what a symmetry is. Suppose first that we take the collection of symmetries of a spacetime to be its collection of automorphisms, i.e. the isometries from that spacetime to itself. (In this case, (1) becomes the criterion SYM*, which has already been discussed extensively in the literature; indeed, this "triviality" complaint against it that we level here has already been mentioned (Barrett [forthcoming]; Barrett et al. [forthcoming]).) The existence of a giraffe spacetime (see Example 6) captures a sense in which (1) is inadequate. One can simply take the giraffe spacetime and add to it an arbitrary tensor field that is not definable in terms of the metric. One wants to say that the resulting spacetime has more structure than the giraffe spacetime that we began with, but it has the same collection of automorphisms, since the only automorphism of the giraffe spacetime was the identity map. The new collection of symmetries is therefore not a proper subset of the collection of symmetries that we began with.

In light of this result, one might attempt to salvage (1) by moving to a more general notion of "symmetry". A local automorphism of a spacetime (M, g_{ab}) is a smooth map $f : O \to O$ that preserves the metric g_{ab} on M, where O is some open subset of M. The definition generalizes in the natural way to spacetimes with structures in addition to g_{ab}. Local automorphisms are simply the automorphisms of local regions of the spacetime. If we consider the local automorphisms of a spacetime to be its collection of symmetries, then more maps count as symmetries, and so we provide ourselves with more information with which to compare amounts of structure between spacetimes. But even local automorphisms do not provide a complete guide to the amount of structure that a spacetime has, since two spacetimes with different amounts of structure might nonetheless have the same local automorphisms. The existence of spacetimes that are locally giraffe (implied by Propositions 9 and 11) demonstrates that (1) is inadequate on this understanding. One simply adds to a locally giraffe spacetime an arbitrary tensor field that is not definable in terms of the metric. One again wants to say that the resulting spacetime has more structure than the one that we began with, but it has the same local automorphisms, since it follows from the local giraffe condition that the only local automorphisms were identity maps to begin with. Once again (1) is inadequate.

There is yet another way one might try to salvage the explication (1). We will say that a local homomorphism of a spacetime (M, g_{ab}) is a smooth map $f : O_1 \to O_2$ that preserves the metric g_{ab} on M, where O_1 and O_2 are open subsets of M. As above, this definition naturally generalizes to spacetimes with additional structures. All local automorphisms are local homomorphisms, but not vice versa. If we consider the local homomorphisms of a spacetime to be its collection of symmetries, then we allow ourselves to appeal to even more maps in order to compare amounts of structure. Once again, however, (1) is inadequate. The existence of a Heraclitus spacetime (Proposition 11) shows precisely this. One simply adds to a Heraclitus spacetime an arbitrary tensor field that is not definable in terms of the metric. The resulting spacetime has more structure than the one that we began with, but it has the same local homomorphisms, since the only local homomorphisms of a Heraclitus spacetime were identity maps to begin with.

The analogous explication of form ($\star\star$) fails for precisely the same reasons.

(2): X has at least as much structure as Y if and only if the collection of symmetries of X is a subset of the collection of symmetries of Y.

Suppose we again take the collection of symmetries of a spacetime to be its automorphisms. The existence of a giraffe spacetime shows that (2) is inadequate on this reading. One again takes the giraffe spacetime and adds to it an arbitrary tensor field that is not definable in terms of the metric. We want to say that the giraffe spacetime that we began with does not have at least as much structure as the resulting spacetime, despite the fact that the two have the same collection of automorphisms. If one tries to salvage (2) by taking the local automorphisms or local homomorphisms to be the collection of symmetries of a spacetime, our results lead to the same difficulty in precisely the same manner as they did for (1). So (2) is inadequate as well.

In fact, our results imply that a broad class of explication of form ($\star$) and ($\star\star$) are inadequate. Suppose that one takes the collection of symmetries of a spacetime to be its class of automorphisms, or its class of local automorphisms, or its class of local homomorphisms. In each of these three cases, we have provided examples of spacetimes X and Y such that one wants to say that X has more structure than Y (and correspondingly, Y does not have at least as much structure as X), but X and Y have the same collection of symmetries. Given such an example, both ($\star$) and ($\star\star$) have undesirable consequences. We begin with ($\star$). Since X has more structure than Y, ($\star$) implies that the collection of symmetries of X stands in relation $\mathfrak{R}$ to the collection of symmetries of Y. The collection of symmetries of X is equal to the collection of symmetries of Y, so it must be that the collection of symmetries of X stands in relation $\mathfrak{R}$ to itself. ($\star$) therefore entails that X has more structure than X, an undesirable consequence. This means that no explication of form ($\star$) will be entirely adequate. And ($\star\star$) fares no better. Since Y does not have at least as much structure as X, ($\star\star$) implies that the collection of symmetries of Y does not stand in relation $\mathfrak{R}$ to the collection of symmetries of X. The collection of symmetries of Y is equal to the collection of symmetries of X, so it must be that the collection of symmetries of Y does not stand in relation $\mathfrak{R}$ to itself. ($\star\star$) therefore entails that Y does not have at least as much structure as Y, another undesirable consequence. One therefore cannot have an entirely adequate explication of form ($\star$) or ($\star\star$) while still making the right verdicts in our examples.

There are two possible routes to salvage our dogma from above – that the symmetries of a spacetime are a guide to its amount of structure – that are worth discussing. First, one could propose a weaker "partial definition" or "partial explication" of amounts of structure in terms of symmetries. Consider, for example, the following two candidate principles:

> (3): If X has more structure than Y, then the collection of symmetries of X is a subset of the collection of symmetries of Y.
>
> (4): If the collection of symmetries of X is a proper subset of the collection of symmetries of Y, then X has more structure than Y.

Our discussion does not provide counterexamples to either of these principles (only to their converses). One therefore might put forward a collection of principles (3) and (4) that dictate how symmetries tie to amounts of structure, and in this way recover a form of the dogma. This route is worth exploring further, but there is one brief remark about it that is worth making now. (3) and (4) are by themselves quite weak. (3) gives one no way to say when X has more structure than Y, while (4) gives one no way to say when X does not have more structure than Y. Of course, one could adopt both (3) and (4). But that provides one with no way to compare amounts of structure between spacetimes X and Y when they have the same collection of symmetries. And since we have provided cases where spacetimes with the same collection of symmetries differ in their amount of structure, the conjunction of (3) and (4) will mean that symmetries only provide a partial guide to amounts of structure. (And moreover, note that no principle like (4) that appeals only to the collection of symmetries of X and Y will be able to make the correct verdict in these cases without leading to absurdity. This is for precisely the same reason why ($\star$) and ($\star\star$) failed.) Something like this may well be the most compelling version of the dogma that one can recover, but it is weaker than the version of the dogma one would have recovered with an explication of form ($\star$) or ($\star\star$).

There is one final route that one might take to salvage a strong form of the dogma in the face of the difficulties posed by Heraclitus spacetimes. It has recently been suggested that when considering amounts of structure we should change the question we are asking (Barrett [forthcoming]). Instead of asking whether one object has more structure than another, we should ask whether one kind of object has more structure than another kind of object. For example, instead of asking whether a Heraclitus spacetime has more or less structure than a Heraclitus spacetime with additional tensor field on it, we ask whether a manifold with metric has more or less structure than a manifold with metric and an additional tensor field. In order to answer this new question, one looks to the entire classes of objects of those two kinds, along with all of the structure-preserving maps between them. In essence, we are once again liberalizing what we mean by "symmetry"; instead of thinking only of automorphisms, local automorphisms, or local homomorphisms, one now considers all structure-preserving maps between objects of the same kind as "symmetries". One conjectures that this move will save the dogma – indeed, it has been emphasized elsewhere that symmetries in this most general sense do suffice to capture facts about definability (Barrett [2018], [forthcoming]) – but it requires a stark conceptual revision. In order to judge amounts of structure, one is now not just looking at maps from the spacetime to itself. Rather, one has to take a much more holistic approach. Only by looking at maps from our spacetime to and from other spacetimes of the same kind (and maps between these other spacetimes too) can one hope to use symmetries as a complete guide to amounts of structure.

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