Dialogue Concerning the Two Chief Views on Spacetime Rigidity

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Abstract

During their afternoon walks about the *piazza*, Sagredo and Salviati discuss the significance of spacetime rigidity: the non-existence of distinct isometries that restrict to the same isometry on an open subset of their domain. Their conversations clarify a number of foundational and philosophical results regarding general relativity.

THE FIRST DAY

Sagredo: That in general relativity (GR) there be two distinct isometries that restrict to the same isometry on an open subset of their domain: t'would be rebarbative to reason. So indeed tells us the esteemed academian, Geroch (1969). But my concern here is with more recent work by Halvorson and Manchak (2025) which builds upon Geroch's results. Those illustrious authors, I take it, have two central claims to make regarding that vexatious 'hole argument' of GR:

1. Weatherall (2018) has "convincingly argued" that the existence of distinct but isomorphic (i.e. isometric) models of GR "does not support the hole argument" (p. 295).
2. There is nevertheless a worry that, were there to exist multiple distinct isometries relating two isomorphic models of GR (that restrict to the same isometry on a proper subset of their domain), the hole argument could rearise. But in fact a theorem from Geroch (1969) rules this out, so the hole argument is thereby "closed".

I won't discuss (1) here—that, after all, has been done elsewhere (see e.g. Cheng and Read (2025), Cudek (forthcoming), and Pooley and Read (2025)). Rather, I am worried about (2) even from within the 'formalist' perspective endorsed by Halvorson and Manchak (2025) and Weatherall (2018)—a perspective which I'll also adopt here for the sake of argument.

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In particular, it does not seem to me that the rigidity result of Geroch (1969) is needed in order to underwrite the formalist argument of Weatherall (2018). To see why I think this, let me first remind you of the result presented by Halvorson and Manchak (2025, Theorem 1) which they take to be crucial to the hole argument, based upon the prior work on rigidity by Geroch:

> Let (M, g) and (M', g') be relativistic spacetimes. If φ and ψ are isometries from (M, g) to (M', g') such that $\varphi|_U = \psi|_U$ for some non-empty open subset U of M, then $\varphi = \psi$.

In addition, let me recall the definition of a spacetime isometry and, from Menon and Read (2023), two different species thereof. Letting (M, g) and (M, g') be Lorentzian manifolds and letting d be a diffeomorphism on M, we say d is an *isometry* from (M, g) to (M, g') just in case $d^*g' = g$.[1] Then:

- We say that an isometry d from (M, g) to (M, g') is an $\texttt{Isometry}_1$ just in case $g' = g$.
- We say that an isometry d from (M, g) to (M, g') is an $\texttt{Isometry}_2$ just in case $g' = d_*g$.

We see that every diffeomorphism d is an $\texttt{Isometry}_2$ from (M, g) to $(M, g') = (M, d_*g)$. But d is an $\texttt{Isometry}_1$ from (M, g) to $(M, g') = (M, g)$ only in special cases, e.g. a translation in Minkowski spacetime.

Now, the reason why I do not think that rigidity is needed in order to 'close' the hole argument by shoring up the formalist arguments of Weatherall (2018) is summarised in the following passage from Menon and Read (2023, p. 14):[2]

> [W]e in fact think that the above line—that Halvorson and Manchak close a hole in Weatherall's argument by proving the negation of `Distinct isometries`—concedes too much to Halvorson and Manchak, and not enough to Weatherall. For in fact, the negation of `Distinct isometries` is unnecessary for Weatherall's argument (i.e., what Pooley and Read (2025) dub the 'argument from mathematical structuralism', as presented in Weatherall (2018)) to proceed as intended (of course, whether Weatherall's argument is ultimately successful is another matter, to which we turn below). For even if there were to exist multiple diffeomorphisms witnessing the isometry between models $\mathcal{M} = \langle M, g_{ab} \rangle$ and $\mathcal{M}' = \langle M, \psi^* g_{ab} \rangle$, these maps would differ at most by a transformation which leaves the metric invariant (i.e., an automorphism of the metric)—in which case, a multiplicity of such maps would still not imply indeterminism. To see this, suppose that there are two pull-backs of the metric which coincide: $\psi_1^* g_{ab}(p) = \psi_2^* g_{ab}(p)$. From this, it follows that $(\psi_1 \circ \psi_2^{-1})^* g_{ab}(p) = g_{ab}(p)$—so $\psi_1 \circ \psi_2^{-1}$ is an $\texttt{Isometry}_1$ of g_{ab}. For a generic metric, these isometries are just the identity, so $\psi_1 = \psi_2$.

[1] For a significantly more general definition of 'isometry', see Malament (2012, p. 85).

[2] Citations have been updated in this passage.

> And in the case in which g_{ab} has non-trivial isometries (in the sense of `Isometry`$_1$), $\psi_1 \circ \psi_2^{-1}$ is *still* an automorphism of the metric, and so does not shift fields on the manifold in such a way as to lead to the possibility of the Hole Argument re-arising. Given this, the above reconstruction of the contribution of Halvorson and Manchak's results to Weatherall's argument does not seem compelling: Weatherall's arguments needed nothing like such results to begin with; the denial of `Distinct isometries` is not a crucial-but-implicit element of his reasoning. (Menon and Read 2023, p. 14)

To try to make clear what is going on in the above argument, let me put it in slightly different words. The strategy is to proceed by exhaustion: first consider the case where the model (M, g) of GR under consideration has no non-trivial isometries in the sense of `Isometry`$_1$, and then consider the case in which it does have non-trivial isometries in the sense of `Isometry`$_1$. If Menon and Read (2023) are correct, then in neither case is rigidity necessary to 'close' the hole argument.

The reasoning on the first 'horn' of the above argument proceeds as follows:

- P1. Consider a model $\mathcal{M}_1 = (M, g)$ and some isometric model $\mathcal{M}_2 = (M, g')$.
- P2. Suppose that there are multiple maps which witness the isometry between $\mathcal{M}_1$ and $\mathcal{M}_2$.
- C1. Consider two such maps, call them ψ_1 and ψ_2. Then $g' = \psi_1^* g = \psi_2^* g$, so $(\psi_2 \circ \psi_1^{-1})^* g = g$.
- P3. *Ex hypothesi*, there are no non-trivial isometries (in the sense of `Isometry`$_1$).
- C2. So $\psi_2 \circ \psi_1^{-1} = \mathrm{Id}$, so $\psi_1 = \psi_2$.

(Throughout, when I say 'isometric' without further clarification, I mean isometric in the sense of `Isometry`$_2$.) This reasoning seems perfectly sound to me—and note that rigidity is not invoked here, at least explicitly.

Let me proceed now then to the second horn of the above argument—i.e., the case in which the spacetime *does* have non-trivial isometries in the sense of `Isometry`$_1$. In this case, we can derive (as before) that $(\psi_2 \circ \psi_1^{-1})^* g = g$ at p for all $p \in M$, but now (unlike before) it might be the case that $\psi_2 \circ \psi_1^{-1} \neq \mathrm{Id}$. Now, the reasoning given by Menon and Read (2023) is that *even if* we have $(\psi_2 \circ \psi_1^{-1})^* g = g$ with $\psi_2 \circ \psi_1^{-1} \neq \mathrm{Id}$, this composition of the maps yields the same metrical value anyway (for whichever point $p \in M$ is under consideration), and so cannot lead to any underdetermination of metrical values at p which is paradigmatic of the hole argument. Note that there is nothing particular about GR here: the thought is that *whenever* we have (a) a spacetime (M, g) with non-trivial isometries in the sense of `Isometry`$_1$, and (b) distinct isometries from (M, g) to some (M, g'), we are still not going to be confronted with the hole argument.

All of this reasoning seems correct to me. So, I do not see why we need rigidity in the sense of Halvorson and Manchak (2025, Theorem 1) in order to "close" the hole argument.

Salviati: I am not convinced by either horn of this argument.

Let us explore the significance of spacetime rigidity quite generally—not just within the context of GR. Consider any spacetime collection $\mathscr{C}$ whose models have the form $(M, O_1, ..., O_n)$ where M is a (possibly non-Hausdorff) manifold and $O_1, ..., O_n$ are geometric objects on M.[3] We can articulate a general version of the Geroch (1969) rigidity result on a collection $\mathscr{C}$ of spacetimes as follows.

(Geroch) Let $(M, O_1, ..., O_n)$ and $(M', O'_1, ..., O'_n)$ be models in $\mathscr{C}$. If φ and ψ are isomorphisms from $(M, O_1, ..., O_n)$ to $(M', O'_1, ..., O'_n)$ such that $\varphi|_U = \psi|_U$ for some non-empty open subset U of M, then $\varphi = \psi$.

In the first horn of the argument of Menon and Read (2023), one considers spacetimes which fail to have non-trivial isometries in the sense of $\texttt{Isometry}_1$. We note that this is the generic case.[4] Consider a generalized condition on a collection $\mathscr{C}$ of spacetimes:

(Trivial) If $(M, O_1, ..., O_n)$ is a model in $\mathscr{C}$ and φ is an isomorphism from $(M, O_1, ..., O_n)$ to itself, then φ is the identity map.

We now have a simple result:[5]

Proposition. (Trivial) $\Rightarrow$ (Geroch).

To see why the proposition holds, suppose a collection $\mathscr{C}$of spacetimes satisfies (Trivial). Let $(M, O_1, ..., O_n)$ and $(M', O'_1, ..., O'_n)$ be models in $\mathscr{C}$ and let φ and ψ be isomorphisms from $(M, O_1, ..., O_n)$ to $(M', O'_1, ..., O'_n)$ such that $\varphi|_U = \psi|_U$ for some non-empty open subset U of M. Since φ and ψ are isomorphisms from $(M, O_1, ..., O_n)$ to $(M', O'_1, ..., O'_n)$, we see that $(M, O_1, ..., O_n)$ is just $(M, \varphi^* O'_1, ..., \varphi^* O'_n) = (M, \psi^* O'_1, ..., \psi^* O'_n)$. Now consider the composed map $\psi^{-1} \circ \varphi$. One can verify $(\psi^{-1} \circ \varphi)_* = \psi^* \circ \varphi_*$. It follows that $(\psi^{-1} \circ \varphi)_*(M, O_1, ..., O_n) = (M, O_1, ..., O_n)$. So $\psi^{-1} \circ \varphi$ is an isomorphism from $(M, O_1, ..., O_n)$ to itself. Since $\mathscr{C}$ satisfies (Trivial), $\psi^{-1} \circ \varphi$ must be the identity map. So $\varphi = \psi$ and therefore $\mathscr{C}$ satisfies (Geroch).

The Proposition shows a general sense in which rigidity is a necessary condition for the first horn of the argument of Menon and Read (2023). Thus, to invoke (Trivial) is to invoke a condition stronger than (Geroch).

[3] Here, let us limit attention to arbitrary tensor fields and derivative operators.

[4] See Mounoud (2015) for details concerning Lorentzian manifolds. The analogous case for Riemannian manifolds is also generic: see Ebin (1968). One wonders about other cases, e.g. the collection of all classical Newtonian spacetimes (Malament 2012, ch. 4).

[5] This is a generalization of the statement that the 'giraffe' asymmetry condition implies the 'rigidity' asymmetry condition as defined by Manchak and Barrett (forthcoming). For an extended dicsussion of the giraffe/trivial condition, see Barrett et al. (2023).

Now for the second horn. I agree with the reasoning given by Menon and Read (2023) that the composed map $\psi_2 \circ \psi_1^{-1}$ will be an isometry from the spacetime (M, g) to itself. This is an astute observation and a closely related idea was used in the proof that (Trivial) $\Rightarrow$ (Geroch).[6] But does it follow from this fact that we are certain to avoid a type of indeterminism paradigmatic of the hole argument? I do not think so. Consider the following example from Manchak and Barrett (forthcoming, Example 2):

Example. Suppose we relax the Hausdorff condition in GR. Let (M, g) be Minkowski spacetime with 'two origins' p and q which fail to have disjoint neighborhoods. Consider a worldline γ running through p. Let ψ_1 be the identity map on M and let ψ_2 be the map which acts as the identity everywhere on M except that it exchanges p and q. Both maps count as isometries from (M, g) to $(M, \psi_{1*}g) = (M, \psi_{2*}g) = (M, g)$. So the collection $\{(M, g)\}$ is not rigid, i.e. it fails to satisfy (Geroch). We also see that Menon and Read (2023) are correct: the composed map $\psi_2 \circ \psi_1^{-1}$ is an isometry from the spacetime (M, g) to itself. But there is also a type of indeterminism here given that $(\psi_2 \circ \psi_1^{-1})[\gamma] \neq \gamma$. Those endorsing the hole argument are likely to wonder: does the worldline γ run through p or q?

Sagredo: With regard to the first horn of the argument, I concede the point. Of course if $X \to Y$ then Y is necessary for X. Your proof shows that a spacetime having no non-trivial isometries in the sense of $\texttt{Isometry}_1$ implies that it is rigid, so of course rigidity is necessary for the first horn. The fact that rigidity is not invoked explicitly in the argument from Menon and Read (2023) is irrelevant to this point. On this we are in agreement, so I concede that rigidity is necessary on the first horn.

On the second horn, I also agree that the example from non-Hausdorff GR shows that it would be too fast to infer from $(\psi_2 \circ \psi_1^{-1})^* g = g$ (with $\psi_2 \circ \psi_1^{-1} \neq \mathrm{Id}$) that such transformations will not generate hole-type indeterminism. I do have some worries that we need to be clearer and more explicit about exactly what we mean by 'determinism' and 'indeterminism' here, but I will leave those for the time being; I might return to them later.

Concerning the second horn, I think that there is still more to clarify when it comes to the relationship between rigidity and Weatherall's argument. We know that the (Geroch) condition is satisfied by any collection of (standard) GR spacetimes. So *in GR*, there is a sense in which rigidity is necessary for Weatherall's argument. But as we have seen, the role of rigidity can be clarified by looking to a broader class of spacetime theories (e.g. non-Hausdorff

[6] In the proof for (Trivial) $\Rightarrow$ (Geroch), the composed map $\psi^{-1} \circ \varphi$ is used to construct an isomorphism from $(M, O_1, ..., O_n)$ to itself. Note that if $\psi \circ \varphi^{-1}$ were used instead, it would be an isomorphism from $(M', O_1', ..., O_n')$ to itself. The key general idea is: if there is a pair of isomorphisms from one model to another model, one can compose one map with the inverse of the other to construct an isomorphism from a model to itself.

GR). I would like to better understand the relationship between rigidity and Weatherall's argument in a more general sense.

Salviati: Let me see if I can articulate a general sense in which the work of Weatherall (2018) rests on rigidity. There is no doubt that a key formal claim of his paper is this: if (M, g) is a spacetime and φ is a non-trivial hole diffeomorphism on M, then the identity map on M is not an isometry from (M, g) to $(M, \varphi_* g)$.[7] In the event that I were called upon to construct a proof of this statement, I would surely appeal to the foundational rigidity result of Geroch (1969). Indeed, a generalized Weatherall (2018) statement is a simple corollary of the (Geroch) condition. Consider the following condition on a collection $\mathscr{C}$ of spacetimes:

(Weatherall) If $(M, O_1, ..., O_n)$ is a model in $\mathscr{C}$ and φ is a non-trivial hole diffeomorphism on M, then the identity map on M is not an isomorphism from $(M, O_1, ..., O_n)$ to $(M, \varphi_* O_1, ..., \varphi_* O_n)$.

It is not difficult to see that (Geroch) implies (Weatherall) for any collection $\mathscr{C}$. Suppose (Weatherall) fails. So there is a model $(M, O_1, ..., O_n)$ in $\mathscr{C}$ and a non-trivial hole diffeomorphism φ on M such that the identity map on M is an isomorphism from $(M, O_1, ..., O_n)$ to $(M, \varphi_* O_1, ..., \varphi_* O_n)$. Since φ is a hole diffeomorphism on M, its action must agree with the identity map on some non-empty open set U in M, i.e. the region 'outside the hole'. Since φ is non-trivial, it is distinct from the identity map. So (Geroch) fails. We have shown the following:

Proposition. (Geroch) $\Rightarrow$ (Weatherall).

Does not this Proposition capture a general sense in which a key formal statement of Weatherall (2018) rests on the Geroch (1969) rigidity result?

Sagredo: This result is of course very interesting. But I do not think we are yet warranted in concluding, on the basis of what has been said thus far, that rigidity is *necessary* in order to underwrite Weatherall's argument, in the more general class of spacetime theories upon which you have invited us to focus. For what you have shown here is that rigidity is *sufficient* to rule out the second horn of the argument from Menon and Read (2023), not that it is necessary. In other words, perhaps there are classes of generalised spacetimes where Weatherall's formalist results regarding the hole argument go through, but which are not rigid. I would very much like to see a result of the form '(Weatherall) $\Rightarrow$ (Geroch)'.

Salviati: At the moment, I must confess my doubts about (Weatherall) $\Rightarrow$ (Geroch). To me, the (Weatherall) condition seems to be a mere special case of

[7] Here, a non-trivial hole diffeomorphism is one which is distinct from the identity map but agrees with the identity map on some non-empty open set, i.e. the region 'outside the hole'.

the more general (Geroch) condition. Note that the latter condition concerns arbitrary isomorphisms from a model $(M, O_1, ..., O_n)$ to a model $(M', O'_1, ..., O'_n)$, while the former condition limits attention to cases in which $M = M'$ and isomorphisms are hole diffeomorphisms. But permit me to retire awhile to ponder this question.

THE SECOND DAY

Salviati: Good to see you again, dear Sagredo. I have been thinking more about your question. Last night, over fine wines and sweetmeats, I discussed it with our mutual friend, the academian Jim Weatherall. Together, we were able to construct a proof of something quite general that clarifies not only how the (Geroch) and (Weatherall) conditions are related but also how those two conditions are related to the work of Halvorson and Manchak (2025).

From the Geroch (1969) result, we have an immediate corollary (Halvorson and Manchak 2025): if (M, g) is a spacetime and φ is a non-trivial hole diffeomorphism on M, then φ is not an isometry from (M, g) to itself.[8] In other words, a non-trivial hole diffeomorphism is never an $\texttt{Isometry}_1$. This corollary is central for much of the discussion in the paper of Halvorson and Manchak (2025). Let us generalize it to construct a condition on a collection $\mathscr{C}$ of spacetimes:

(H&M) If $(M, O_1, ..., O_n)$ is a model in $\mathscr{C}$ and φ is a non-trivial hole diffeomorphism on M, then φ is not an isomorphism from $(M, O_1, ..., O_n)$ to itself.

We can now state the general result:

Theorem. (Geroch) $\Leftrightarrow$ (Weatherall) $\Leftrightarrow$ (H&M).

The proof uses an idea that we have already considered twice yesterday: once in the second horn of the argument of Menon and Read (2023) and again in the proof that (Trivial) $\Rightarrow$ (Geroch). This is the fact that, whenever one has isomorphisms φ and ψ from a model $(M, O_1, ..., O_n)$ to a model $(M', O'_1, ..., O'_n)$, the composed map $\psi^{-1} \circ \varphi$ will be an isomorphism from $(M, O_1, ..., O_n)$ to itself, i.e. the analogue of an $\texttt{Isometry}_1$ for the more general class of spacetime theories which is now our focus.

We know that (Geroch) $\Rightarrow$ (Weatherall). And it is immediate that (Geroch) $\Rightarrow$ (H&M). To prove the Theorem, we need only show that a failure of (Geroch) implies a failure of both (Weatherall) and (H&M). Suppose (Geroch) fails for some collection $\mathscr{C}$ of spacetimes. So there are models $(M, O_1, ..., O_n)$ and $(M', O'_1, ..., O'_n)$ in $\mathscr{C}$ and distinct isomorphisms φ and ψ from $(M, O_1, ..., O_n)$ to $(M', O'_1, ..., O'_n)$ such that $\varphi|_U = \psi|_U$ for some non-empty open subset U of M.

[8] One can verify that this statement is equivalent to Corollary 2 given by Halvorson and Manchak (2025, p. 309): "Let (M, g) be a relativistic spacetime, and let O be a subset of M such that $M \setminus O$ has non-empty interior. If $\varphi : (M, g) \to (M, g)$ is an isometry that is the identity outside of O, then φ is also the identity inside O."

Now consider the composed map $\psi^{-1} \circ \varphi$. It must be a non-trivial hole diffeomorphism on M. (It is a hole diffeomorphism since it acts as the identity on the open set U. But $\psi^{-1} \circ \varphi$ cannot be the identity map: because φ and ψ are distinct, their inverses must be distinct as well.) Now we invoke the idea that was used twice yesterday: $\psi^{-1} \circ \varphi$ is an isomorphism from $(M, O_1, ..., O_n)$ to itself. Since $\psi^{-1} \circ \varphi$ is a non-trivial hole diffeomorphism from $(M, O_1, ..., O_n)$ to itself, (H&M) fails. It was at this stage that my proof stalled for some time yesterday. But in my discussions with Weatherall later on, he kindly pointed out: since the identity map is an isomorphism from $(M, O_1, ..., O_n)$ to itself and $(M, O_1, ..., O_n) = (\psi^{-1} \circ \varphi)_*(M, O_1, ..., O_n) = (M, (\psi^{-1} \circ \varphi)_* O_1, ..., (\psi^{-1} \circ \varphi)_* O_n)$, the (Weatherall) condition fails. I did chide myself for not seeing this afore.

Stepping back, we see that the Theorem clarifies things considerably. Key formal statements in the papers of Weatherall (2018) and Halvorson and Manchak (2025) turn out to be equivalent—not just in GR but in all the spacetime theories which we have considered here. Moreover, the (Weatherall) and (H&M) conditions are not mere corollaries of the (Geroch) condition as I had originally thought. All three conditions are quite general rigidity statements expressing the same idea in different ways.

Sagredo: This is true insight, friend, and I am most grateful to you for the result. I myself was puzzled by why Halvorson and Manchak (2025) direct their attention to $\texttt{Isometry}_1$ in e.g. their Corollary 2, and as in your (H&M). For it is not *prima facie* clear what this has to do with the hole argument, which on standard presentations (see e.g. Norton et al. (2023)) treats principally with isometries in the sense of $\texttt{Isometry}_2$. But the Theorem shows this to be equivalent to the non-existence of distinct isometries in the sense of $\texttt{Isometry}_2$ (which agree on an open subset of their domain)—i.e. (Geroch)—and also to (Weatherall). As such, the Theorem renders transparent the significance of (H&M) for formalist responses to the hole argument.

In their paper, Menon and Read (2023, pp. 12–13) raise the following complaint about Corollary 2 in Halvorson and Manchak (2025):

> Despite its name, Corollary 2 is consistent with `Hole isometry`, because the corollary states that any isometry from $\langle M, g_{ab} \rangle$ *to itself* must be the identity everywhere, so that non-trivial isometries (including hole isometries) relating $\langle M, g_{ab} \rangle$ to itself cannot exist. However, `Hole isometry` states that there exist two *distinct* models $\langle M, g_{ab} \rangle$ and $\langle M, \psi^* g_{ab} \rangle$ where ψ is a non-trivial map which witnesses those models' being isometric—and this, of course, is perfectly consistent with Corollary 2. Since Corollary 2 regards maps from $\langle M, g_{ab} \rangle$ to itself, both it, and any claims regarding the non-existence/triviality of hole isomorphisms with which it is associated, are—we contend—irrelevant for discussions of the Hole Argument as standardly construed, since those discussions trade on there being distinct models $\langle M, g_{ab} \rangle$ and $\langle M, \psi^* g_{ab} \rangle$.

But what your Theorem exhibits is that this charge of irrelevance is too strong, given the above three-way equivalence. This is a very helpful result.

Nevertheless, there is one issue which yet perturbs me. Weatherall (2020, p. 83) has drawn our attention to the fact that, in the context of classical 'Leibnizian' spacetimes (M, t, h) (where t is a degenerate temporal metric of signature $(1, 0, 0, 0)$ and h is a degenerate spatial metric of signature $(0, 1, 1, 1)$—see Malament (2012, ch. 4)), there were historically two different views regarding whether theories set on this spacetime are deterministic, attributable respectively to Stein (1977) and Earman (1977):

> On Stein's view, a deterministic Leibnizian dynamics would determine future states up to isomorphisms of Leibnizian space-time. This is because to determine future states up to isomorphism is precisely to determine only those facts that are expressible within Leibnizian space-time.
>
> Earman, by contrast, appears to demand more than this of a (deterministic) Leibnizian dynamics. [Footnote suppressed.] He seems to require that such a dynamics determine, at least, unique worldlines for particles. In other words, two histories, agreeing on the distances and angles between all particles over time, would be distinct if they disagreed on the locations occupied by the bodies over time, where "location" is characterized by the points occupied by the bodies' worldlines/worldtubes. This is strictly stronger than what Stein requires: location in this sense is not invariant under automorphisms of Leibnizian space-time, and so Earman's notion of determinism would require a Leibnizian dynamics to determine future states in a way that distinguishes between isomorphic possibilities.

The relevant point here for my purposes is this. Consider isometries of a Leibnizian spacetime in the sense of $\texttt{Isometry}_1$: i.e., maps from (M, h, t) to itself. For Stein (1977), such maps will never implicate theories set on Leibnizian spacetime in indeterminism. But for Earman (1977), one must also consider e.g. worldlines γ—and one must accordingly *enrich* the spacetime with such structures. Then, non-trivial maps $\psi \neq \text{Id}$ which are isometries of the Leibnizian spacetime in the (generalised) sense of $\texttt{Isometry}_1$ might nevertheless implicate such theories in indeterminism, because it might be the case that $\psi^*[\gamma] \neq \gamma$: particle trajectories, in other words, are changed by these transformations.

This brings me back to the point which I raised before setting aside yesterday in the context of your example of non-Hausdorff GR and indeterminism. It seems to me that this distinction between 'Steinian' versus 'Earmannian' approaches to (in)determinism in the face of isometries in the sense of $\texttt{Isometry}_1$ is relevant to the second horn of the argument from Menon and Read (2023). Previously, you convinced me that on the second horn we still have $(\psi_2 \circ \psi_1^{-1})[\gamma] \neq \gamma$, and so there is still indeterminism here; rigidity is required in order to avoid this result. But it is surely true that there is only indeterminism here in the Earmannian sense—*not* in Steinian sense. So while I can embrace all of our above conclusions when one fixes on Earmannian determinism, I'm still unconvinced that rigidity is required for the arguments of Weatherall (2018) if one adopts the Steinian sense of determinism; in that case, it in fact still seems to me that the original arguments from Menon and Read (2023) hold on the second horn.

Salviati: You are quite right to highlight that the non-rigid Leibnizian spacetime is 'indeterministic' only on a particularly strong Earman-style definition of determinism. You are also quite right to point out affinities with the Example (non-Hausdorff Minkowski spacetime), which I employed yesterday to generate a potential problem for the second horn of the argument of Menon and Read (2023).

To better understand the situation, let us explore a series of three definitions of determinism of increasing strength.[9] We start with the two weakest due to Butterfield (1989) and Belot (1995). Consider the following general conditions on a collection $\mathscr{C}$ of spacetimes:

(De Dicto) Let $(M, O_1, ..., O_n)$ and $(M', O'_1, ..., O'_n)$ be models in $\mathscr{C}$ and let $U \subset M$ and $U' \subset M'$ be non-empty open sets. If there is an isomorphism φ from $(U, O_1, ..., O_n)$ to $(U', O'_1, ..., O'_n)$, then there is an isomorphism ψ from $(M, O_1, ..., O_n)$ to $(M', O'_1, ..., O'_n)$.

(De Re) Let $(M, O_1, ..., O_n)$ and $(M', O'_1, ..., O'_n)$ be models in $\mathscr{C}$ and let $U \subset M$ and $U' \subset M'$ be non-empty open sets. If there is an isomorphism φ from $(U, O_1, ..., O_n)$ to $(U', O'_1, ..., O'_n)$, then there is an isomorphism ψ from $(M, O_1, ..., O_n)$ to $(M', O'_1, ..., O'_n)$ such that $\psi_{|U} = \varphi$.

Following Dewar (2025), I have presented these definitions in their most general form. But note that attention is often limited to situations in which the models $(M, O_1, ..., O_n)$ are sufficiently nice to allow one to make sense of 'initial segments' $U \subset M$. In GR for example, one might restrict attention to globally hyperbolic spacetimes in which an initial segment is defined as the timelike past of any Cauchy surface.

Clearly, (De Re) $\Rightarrow$ (De Dicto). The implication does not go in the other direction, however: the 'collapsing columns' of Wilson (1993) is one famous counterexample. The two definitions have long played a central role in discussions of the hole argument. Is one the 'true' definition of determinism and the other an imposter? I believe this to be a misguided question. I concur with Dewar (2016) and Pooley (2021) that one ought to espouse a type of pluralism with respect to these definitions of determinism.[10] In a recent paper, Dewar (2025, p. 15) writes:

[9] The three definitions of determinism considered here (as well as several others) are investigated more fully in the papers of Halvorson et al. (2025) and Manchak et al. (forthcoming).

[10] Others advancing pluralist views concerning determinism more generally include Doboszewski (2019) and Smeenk and Wüthrich (2021).

> [T]here is not a productive debate about whether determinism is better captured by [(De Dicto)] or [(De Re)]. Rather, the two definitions are held to capture slightly different senses in which a theory might be deterministic—two senses that correspond, in fact, to the two species of possibility for which these definitions are named.

One can verify that both Leibnizian spacetime and non-Hausdorff Minkowski spacetime (considered as singleton collections) satisfy the (De Re) condition and hence the (De Dicto) condition as well. Thus, neither condition is strong enough to capture an Earman (1977) style definition of determinism. Indeed, both conditions are variations of determinism as understood by Stein (1977).

This brings us to a third definition due to Belot (1995) which was unfortunately ignored for thirty years. Consider the following condition on a collection $\mathscr{C}$ of spacetimes:

(De Re*) Let $(M, O_1, ..., O_n)$ and $(M', O'_1, ..., O'_n)$ be models in $\mathscr{C}$ and let $U \subset M$ and $U' \subset M'$ be non-empty open sets. If there is an isomorphism φ from $(U, O_1, ..., O_n)$ to $(U', O'_1, ..., O'_n)$, then there is a unique isomorphism ψ from $(M, O_1, ..., O_n)$ to $(M', O'_1, ..., O'_n)$ such that $\psi_{|U} = \varphi$.

The only difference between the (De Re) and the (De Re*) conditions is the uniqueness clause in the latter. Clearly, (De Re*) $\Rightarrow$ (De Re). But the implication does not go in the other direction; Leibnizian spacetime and non-Hausdorff Minkowski spacetime are both counterexamples. Moreover, one can verify that all such counterexamples must fail to be rigid. This is because the uniqueness clause that distinguishes the (De Re*) and (De Re) conditions amounts to the requirement of rigidity (Manchak et al. forthcoming). Given the Theorem above, it is a simple excercise to show the following:

Corollary. (De Re*) $\Leftrightarrow$ ((De Re) $\wedge$ ((Geroch) $\vee$ (Weatherall) $\vee$ (H&M))).

Because Leibnizian spacetime and non-Hausdorff Minkowski spacetime both fail to satisfy (De Re*), this condition is strong enough to capture an Earman (1977) sense in which these spacetimes are indeterministic. Moreover, using the Geroch (1969) rigidity result along with the foundational theorem of Choquet-Bruhat and Geroch (1969), one can show that significant sectors of GR satisfy the (De Re*) condition. This was noticed only just recently. It was mentioned briefly in papers by Landsman (2023, footnote 23) and Cudek (forthcoming, footnote 41) and explored more fully in the work of Manchak et al. (forthcoming).

Just as I think, along with Dewar (2016) and Pooley (2021), that one ought to be a pluralist with respect to the (De Dicto) and (De Re) conditions, I also think that one ought extend this pluralism to the (De Re*) condition as well. There is indeed a sense in which Leibnizian spacetime is indeterministic just as Earman (1977) claimed. The same goes for non-Hausdorff Minkowski spacetime. Of course, this sense is not the only one. If various sectors of GR had merely satisfied (De Re) and not (De Re*), there still would have been a significant Steinian sense in which we would have considered GR to be a 'deterministic' spacetime theory. But the fact is: various sectors of GR satisfy the much stronger Earman inspired (De Re*) condition. This is *also* significant.

Sagredo: I endorse your pluralism about notions determinism, which seems to me both irenic and laudable.

Stepping back, I see now many things in sharp relief. It is clear that the first horn of the argument of Menon and Read (2023)—which obtains generically—implicitly assumes rigidity. Regarding the second horn, if one construes determinism in the sense of Stein (1977), then Menon and Read (2023) are correct that one does not require rigidity. But if one construes determinism in the stronger sense of Earman (1977), then the second horn of the argument does not suffice to show what it purports to show. When one understands determinism in that latter way, your Theorem shows—for the broad class of spacetime theories which has been our focus—that rigidity (i.e. (Geroch)) is both necessary and sufficient for a key formal statement in the paper of Weatherall (2018) regarding the hole argument (i.e. (Weatherall)). I see clearly now that the general notion of rigidity, which can be expressed in terms of $\texttt{Isometry}_2$ in the conditions of (Geroch) and (Weatherall), can also be expressed in terms of $\texttt{Isometry}_1$ in the equivalent condition (H&M). Finally, it is clear that the Earmanian (De Re*) type of determinism is equivalent to the conjunction of the Steinian (De Re) type of determinism and any one of the following rigidity conditions: (Geroch), (Weatherall), or (H&M).

I am truly satisfied with the destination at which we have arrived: if only all exchanges could be so fruitful! But come: let us head to the harbour to see the fishermen returning from their ships as they reap the bounty of the sea.

Acknowledgements

For very helpful discussions we are grateful to our esteemed colleagues Thomas Barrett, Jeremy Butterfield, Frank Cudek, Neil Dewar, Henrique Gomes, Hans Halvorson, David Malament, Eleanor March, Tushar Menon, Oliver Pooley, and Jim Weatherall.

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