On Privileged Coordinates and Kleinian Methods*

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Abstract

This paper examines two ways in which the 'privileged coordinates' of a geometric space might have significance. First, the structure of the space might be 'determined by its privileged coordinates'. Second, the space might be presentable using 'Kleinian methods'. We examine the geometric spaces for which these two conditions hold. Along the way, we investigate the relationship between these two conditions.

1 Introduction

It is often remarked that there is a close relationship between the structure of Minkowski spacetime and the 'privileged coordinates' that it admits. For example, North (2021, p. 29) writes the following about the 'Lorentz frames' — those obtained by translating, spatially rotating, reflecting, and Lorentz boosting the standard t, x, y, z coordinates:

> In special relativity, the spacetime interval between events, the spatiotemporal "distance" or separation between them, is the same in any Lorentz frame. Since the choice of Lorentz frame is an arbitrary choice in description, and since the spacetime interval between events is the same regardless of that choice, we conclude that this quantity is part of the objective, intrinsic nature of the world, according to the theory. We infer that the spacetime structure of a special relativistic world is Minkowskian, the kind of spacetime that's characterized by this interval.

If we consider the Lorentz frames to be the 'privileged coordinates' of Minkowski spacetime, one can make North's remark precise by showing that the Minkowski metric (the 'interval') is up to isometry the only metric that admits those privileged coordinates (Barrett and Manchak, 2024, Proposition 4.1.1). Minkowski spacetime is in this sense 'determined by' its privileged coordinates. Philosophers and physicists have been motivated by such cases to stress the "fundamental significance" (Fock, 1964, p. 375) of the privileged coordinates of a geometric

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space. If the case of the Minkowski spacetime is representative, then privileged coordinates provide us with a useful tool for investigating the structure of our physical theories that are formulated in geometric terms.

The aim of this paper is to examine two related ways in which this fundamental significance of privileged coordinates might manifest. Suppose that we have a geometric space X. The first way in which the privileged coordinates of X might be significant is that they might 'determine' the structure of X, as in the case of Lorentz frames and Minkowski spacetime. This kind of significance is pointed at by North (2021, p. 26), for example, when discussing the Euclidean plane:

> [...] the features or quantities that are agreed upon by all the different coordinate systems we can use for the plane, the coordinate-independent, invariant features, correspond to the intrinsic nature of the plane, to aspects of the plane itself, apart from our descriptions of it — that is, to [...] its structure.

If North is correct that the structure of a geometric space is comprised of just those features agreed upon by all privileged coordinates, then we should expect two spaces with the same privileged coordinates to have the same structure. This leads us to the following natural way to make precise the idea that the privileged coordinates of X determine its structure.

Determination. If another geometric space Y has the same privileged coordinates as X, then X and Y are the same.

We will call results like this "uniqueness results" for X. Of course, in order to prove a uniqueness result, one first needs to say what the privileged coordinates are for X and Y, and what the entailed sense of 'sameness' is. When true, a uniqueness result captures a sense in which the privileged coordinates of X 'implicitly define' or 'determine' its geometric structure. For if any other space admits the same privileged coordinate as X, it must be (in some sense) structurally the same as X. If no such result holds, that means that the privileged coordinates of X do not determine, and hence are not a perfect guide to, its underlying structure.

There is a second way in which the privileged coordinates of X might be significant. Following Wallace (2019), one can distinguish between 'Riemannian' and 'Kleinian' methods of presenting a geometric space. The Riemannian method presents the geometric space X using the standard tools of differential geometry to define a smooth manifold and lay down layers of structure on it. The Kleinian method instead presents X "by means of the coordinate systems we can use for the space and the features that are invariant under transformations of them" (North, 2021, p. 23). Wallace (2019) has recently suggested that the Kleinian method can be made precise by employing the framework of locally G-structured spaces. In brief, a locally G-structured space is a pair (S, C), where S is a set and C is a collection of maps from S to $\mathbb{R}^n$, which we can think of as the privileged coordinates on S. From the data provided by (S, C), one can recover geometric structure. This leads us to the following second way that privileged coordinates might have fundamental significance. They might allow us to present the structure of X using a locally G-structured space.

Kleinian Presentability. X can be presented in the framework of locally G-structured spaces.

Wallace (2019) remarks that locally G-structured spaces provide "a legitimate and informative way to formulate" physical theories, and he offers evidence for this claim by working through a number of cases where Kleinian Presentability holds. Kleinian Presentability is also suggested by some remarks that North makes. She does not explicitly mention locally G-structured spaces, but she does write that the standard method and the coordinate-based method are "two ways of characterizing a given structure, and two corresponding routes to learning about it" (North, 2021, p. 23).

One can show that Kleinian Presentability holds for Euclidean space and for Minkowski spacetime (Barrett and Manchak, 2024). One might thus expect that Determination holds of a geometric space X if and only if Kleinian Presentability holds of X. We will show that, perhaps surprisingly, this is not the case. In particular, Determination does not always entail Kleinian Presentability. Along the way we will examine the kinds of geometric spaces for which Determination and Kleinian Presentability hold, and we will make some suggestions about exactly what kinds of uniqueness results one needs in order to generate success for Kleinian methods.

2 Determination

We begin by isolating three simple and instructive uniqueness results. In brief, we will show that a variety of Determination holds for symplectic manifolds, flat classical spacetimes, and flat relativistic spacetimes. Each of these kinds of geometric spaces is the unique geometric space of its kind that admits its privileged coordinates.

We consider symplectic manifolds first. A **symplectic manifold** is a pair (M, ω_{ab}) where M is a smooth $2n$-dimensional manifold and ω_{ab} is a smooth tensor field on M that is closed, antisymmetric, and non-degenerate. Symplectic manifolds are of interest because they are the geometric setting for the Hamiltonian formulation of classical mechanics. (See North (2009), Curiel (2014), and Barrett (2017) for details.) We will say that a coordinate chart (U, ϕ) on M is a **Darboux coordinate chart** on (M, ω_{ab}) if

$$\omega_{ab} = \frac{1}{2}\sum_{i=1}^{n} d_a u^i d_b u^{i+n} - d_b u^i d_a u^{i+n}$$

in the region U. One has the following famous result (Lee, 2012, Theorem 22.13).

Lemma 1 (Darboux's theorem)**.** *Let* (M, ω_{ab}) *be a symplectic manifold with* $p \in M$*. Then there is a Darboux coordinate chart* (U, ϕ) *with* $p \in U$*.*

Darboux's theorem immediately implies the following uniqueness result, which is closely related to one proven by Barrett (2022). The Darboux coordinates of a symplectic manifold completely determine its structure.

Proposition 1. *Let* (M, ω_{ab}) *be a symplectic manifold. If a symplectic manifold* (M', ω'_{ab}) *admits the same Darboux coordinates as* (M, ω_{ab})*, then* $(M, \omega_{ab}) = (M', \omega'_{ab})$*.*

Proof. We first show that the underlying sets of the manifolds must be equal. Let $p \in M$. Lemma 1 implies that there is a Darboux chart (U, ϕ) on (M, ω_{ab}) that contains p. (U, ϕ) is also a Darboux chart on (M', ω'_{ab}), so $U \subset M'$ and hence $p \in M'$. One argues in the same way to show that every point in M' is in M, and hence $M = M'$. We now need to show that the two manifolds have the same atlas. Without loss of generality we show that the atlas on M' is contained in the atlas on M. Let (V, ψ) be a chart in the atlas on M'. We know that (V, ψ) is compatible with the Darboux coordinate charts on (M', ω_{ab}). One easily sees that the Darboux charts cover M (by Lemma 1), are compatible with one another, and satisfy the Hausdorff condition. One then argues as in Proposition 1.1.1 of Malament (2012) to show that (V, ψ) is compatible with all charts in the atlas on M. Since the atlas on M is maximal, this implies that (V, ψ) is a chart on M. We now show that $\omega_{ab} = \omega'_{ab}$. Let p be in M. Lemma 1 implies that is a Darboux chart (U, ϕ) around p for (M, ω_{ab}). By assumption (U, ϕ) is also a Darboux chart with respect to (M, ω'_{ab}). This implies that $\omega_{ab} = \omega'_{ab}$ at p. Since p was arbitrary, $\omega_{ab} = \omega'_{ab}$ everywhere on M. □

If one considers the 'privileged coordinates' of a symplectic manifold (M, ω_{ab}) to be its Darboux coordinates, then its privileged coordinates completely determine its structure.

An analogous uniqueness result holds in the case of flat classical spacetimes. A classical spacetime is a tuple (M, t_a, h^{ab}, ∇), where M is a smooth, connected, four-dimensional manifold, the field t_a is a 'temporal metric', the field h^{ab} is a 'spatial metric', and ∇ is a derivative operator, which specifies which trajectories through the spacetime are geodesics. These structures are required to satisfy a few basic conditions (Malament, 2012, p. 249). A classical spacetime is flat when its derivative operator ∇ is flat. Flat classical spacetimes are the geometric setting for standard Newtonian gravitation theory. (See Malament (2012, Chapter 4) for details.) We will say that a chart (U, ϕ) on M is a **Galilean coordinate chart** on (M, t_a, h^{ab}, ∇) if the following three conditions hold.

$$t_a = d_a u^1,$$

$$h^{ab} = \left(\tfrac{\partial}{\partial u^2}\right)^a \left(\tfrac{\partial}{\partial u^2}\right)^b + \left(\tfrac{\partial}{\partial u^3}\right)^a \left(\tfrac{\partial}{\partial u^3}\right)^b + \left(\tfrac{\partial}{\partial u^4}\right)^a \left(\tfrac{\partial}{\partial u^4}\right)^b,$$

and ∇ is the coordinate derivative operator for (U, ϕ), where u^i are the coordinate maps associated with (U, ϕ). Recall that the coordinate derivative operator is defined to be the unique derivative operator ∇ on U that satisfies $\nabla_n \left(\frac{\partial}{\partial u^i}\right)^a = \mathbf{0}$ for each $i = 1, \ldots, 4$ (Malament, 2012, Proposition 1.7.11).

One can now prove the following analogue of Darboux's theorem in the case of flat classical spacetimes. We place the proof in an appendix.

Lemma 2. *Let (M, t_a, h^{ab}, ∇) be a flat classical spacetime with $p \in M$. Then there is a Galilean chart (U, ϕ) with $p \in U$.*

This lemma implies the following uniqueness result about flat classical spacetimes. If one considers the 'privileged coordinates' of a flat classical spacetime (M, t_a, h^{ab}, ∇) to be its Galilean coordinates, then once again privileged coordinates completely determine structure.

Proposition 2. *Let (M, t_a, h^{ab}, ∇) be a flat classical spacetime. If a classical spacetime $(M', t'_a, h'^{ab}, \nabla')$ has the same Galilean coordinates as (M, t_a, h^{ab}, ∇), then $(M, t_a, h^{ab}, \nabla) = (M', t'_a, h'^{ab}, \nabla')$.*

Proof. One argues as in Proposition 1 to show that $M = M'$, $t_a = t'_a$, and $h^{ab} = h'^{ab}$. We need to show that $\nabla = \nabla'$. Let α be a smooth tensor on M of arbitrary index structure. Let $p \in M$ and consider Galilean coordinates (U, ϕ) for (M, h^{ab}, t_a, ∇) with $p \in U$, which we know exist by Lemma 2. We know that $\nabla = \nabla'$ on U since (U, ϕ) is also a Galilean chart with respect to $(M, t'_a, h'^{ab}, \nabla')$. This means that $\nabla_n \alpha = \nabla'_n \alpha$ at the point p. Since p was arbitrary, $\nabla = \nabla'$ everywhere on M. □

We now turn to our final example: flat relativistic spacetimes. The situation here is perfectly analogous to the case of symplectic manifolds and the case of flat classical spacetimes. We present it because it will provide a useful illustration of how one might use uniqueness results to generate success for the Kleinian method. Recall that a **relativistic spacetime** is a pair (M, g_{ab}) where M is a smooth, n-dimensional (for $n \geq 2$), connected, Hausdorff manifold without boundary and g_{ab} is a smooth Lorentzian metric on M. We will say that a coordinate chart (U, ϕ) on M is a **Minkowskian coordinate chart** on (M, g_{ab}) if

$$g_{ab} = d_a u^1 d_b u^1 - d_a u^2 d_b u^2 - \ldots - d_a u^n d_b u^n$$

in the region U, where again u^i are the coordinate maps for the chart (U, ϕ). We now have the following guarantee about the existence of Minkowskian charts on flat relativistic spacetimes (O'Neill, 1983, p. 223).

Lemma 3. *Let (M, g_{ab}) be a flat relativistic spacetime with $p \in M$. Then there is a Minkowskian chart (U, ϕ) with $p \in U$.*

As in the two previous cases, this lemma yields the following uniqueness result. Proposition 3 follows from Lemma 3 precisely as Proposition 1 followed from Lemma 1. The proof is perfectly analogous, so we leave it to the reader. We also note that an analogous result holds of flat Riemannian manifolds.

Proposition 3. *Let* (M, g_{ab}) *be a flat relativistic spacetime. If a relativistic spacetime* (M', g'_{ab}) *admits the same Minkowskian coordinates as* (M, g_{ab})*, then* $(M, g_{ab}) = (M', g'_{ab})$.

There is a helpful way to understand these uniqueness results. Each of the lemmas is saying that all n-dimensional geometric spaces of a particular kind are 'locally' isomorphic to some fixed space of that kind that has underlying manifold $\mathbb{R}^n$. We see this clearly in the case of Lemma 3. A relativistic spacetime $(\mathbb{R}^n, g'_{ab})$ is a **representation of** (M, g_{ab}) if for every point $p \in M$, there are open sets $O \subset M$ and $O' \subset \mathbb{R}^n$ such that $p \in O$ and (O, g_{ab}) is isometric to (O', g'_{ab}). Lemma 3 is saying that Minkowski spacetime is a representation of every flat relativistic spacetime. Similarly, Lemma 1 tells us that an arbitrary $2n$-dimensional symplectic manifold is 'locally' exactly like a region of the standard symplectic manifold $(\mathbb{R}^{2n}, \omega_{ab})$ (Lee, 2012, p. 568). Lemma 2 tells us that an arbitrary flat classical spacetime is 'locally' exactly like a region of Galilean spacetime. In each case, the lemma shows that a particular class of geometric spaces (symplectic manifolds, flat classical spacetimes, flat relativistic spacetimes) has a 'common representation': a geometric space X with underlying manifold $\mathbb{R}^n$ such that each geometric space in the class is 'locally' exactly like a region of X.

It is important to emphasize that these three uniqueness results are quite strong. The sense of 'sameness' that appears in the consequent of all three results is identity, not just isomorphism. The reason that the sense of entailed sameness can be so strong is that the required sense of sameness between privileged coordinates — which is appealed to in the antecedent — is quite strong as well. In Proposition 3, for example, we are requiring that exactly the same charts are Minkowskian on our two spacetimes. One naturally wonders what kind of 'similarity' between classes of privileged coordinates will lead to isomorphic geometric spaces, rather than identical geometric spaces. The framework of locally G-structured spaces allows one to isolate a sense in which classes of privileged coordinates themselves might be 'isomorphic', and so we turn to this framework next.

3 Kleinian Presentability

These simple uniqueness results capture a sense in which Determination holds of symplectic manifolds, flat classical spacetimes, and flat relativistic spacetime. It is natural to ask whether Kleinian Presentability holds of these geometric spaces. We will show in detail that Kleinian Presentability does in fact hold of flat relativistic spacetimes. We conjecture that the same holds in the other two cases, but leave careful verification to future work.

3.1 Locally G-structured spaces

We will review the framework of locally G-structured spaces here, but encourage the reader to consult Wallace (2019) and Barrett and Manchak (2024) for details.

We begin with some preliminaries. A **pseudogroup** (Kobayashi and Nomizu, 1996, p. 1) is a collection of bijective structure-preserving maps between open subsets of a topological space. One can think of it as a 'local' automorphism group of a space. The **diffeomorphism pseudogroup** of a smooth manifold M is the class of diffeomorphisms $f : U \to V$ between open sets U and V of M. If (M, g_{ab}) is a relativistic spacetime, then the **isometry pseudogroup** of (M, g_{ab}) is the class of diffeomorphisms $f : U \to V$ between open sets U and V of M such that $f^*(g_{ab}) = g_{ab}$.

Let G be a pseudogroup on $\mathbb{R}^n$ that is contained in the diffeomorphism pseudogroup of $\mathbb{R}^n$. A **locally G-structured space** is a pair (S, C), where S is a set, C is a collection of injective partial functions $c : S \to \mathbb{R}^n$, and the following conditions hold:

Cover condition. For every point $p \in S$ there is a map $c \in C$ such that $p \in \text{dom}(c)$.

Range condition. For every map $c \in C$ there is a map $g \in G$ such that $\text{ran}(c) = \text{dom}(g)$.

Compatibility condition. For any partial function $f : S \to \mathbb{R}^n$ whose range is the domain of an element of G, $f \in C$ if and only if for every $f' \in C$ such that $\text{dom}(f) \cap \text{dom}(f')$ is non-empty, $f \circ f'^{-1} \in G$.

The data provided by a locally G-structured space (S, C) allows us to recover geometric structure. (S, C) inherits smooth manifold structure in the following manner. For each $f \in C$, $(\text{dom}(f), f)$ is an n-chart on S. Let C^+ be the collection of all n-charts on S that are compatible with all these n-charts in C. One then shows that (S, C^+) is a smooth n-dimensional manifold (Barrett and Manchak, 2024, Proposition 2.2.1). One recovers various levels of geometric structure on the manifold (S, C^+) in the following way. The maps in C induce a pseudogroup on (S, C^+). This **coordinate transformation pseudogroup** Γ contains all of the maps between open subsets of S that 'transform' from one of our privileged coordinate systems in C to another one of them. Γ contains those homeomorphisms between open sets of S generated by functions of the form $f^{-1} \circ g$, where f and g are in C (Barrett and Manchak, 2024, Definition 2.2.2). Γ now implicitly defines geometric structures on (S, C^+). We will say that a smooth tensor field α (of arbitrary index structure) on a smooth manifold M is **implicitly defined** by a pseudogroup G on M just in case $h^*(\alpha) = \alpha|_U$ for all $h : U \to V$ in G. We now equip (S, C^+) with those smooth tensor fields α that are implicitly defined by the coordinate transformation pseudogroup Γ. We can thereby recover a geometric space — a smooth manifold with tensor fields on it — from a locally G-structured space.

One naturally wonders which geometric spaces are recoverable from a locally G-structured space. In the case of relativistic spacetimes, there is a natural way to build a locally G-structured space (S, C) from (M, g_{ab}). We can think of this (S, C) as our 'best bet' for recovering the structure of (M, g_{ab}). One begins by showing that every relativistic spacetime has a representation (Barrett and Manchak, 2024, Lemma 3.2.2). This fact provides us with a method of constructing a locally G-structured space from a relativistic spacetime (M, g_{ab}). Let (M, g_{ab}) be a relativistic spacetime with $(\mathbb{R}^n, g'_{ab})$ a representation of it. We then define the following:

- Let $S = M$.
- Let G be the isometry pseudogroup of $(\mathbb{R}^n, g'_{ab})$.
- Let C be the collection of isometries between open subsets of (M, g_{ab}) and open subsets of $(\mathbb{R}^n, g'_{ab})$, i.e. diffeomorphisms $c : U \to V$ where $U \subset M$ and $V \subset \mathbb{R}^n$ are open and $c^*(g'_{ab}) = g_{ab}|_U$.

This (S, C) is indeed a locally G-structured space (Barrett and Manchak, 2024, Lemma 3.2.3). We will call this (S, C) the **locally G-structured space determined by** (M, g_{ab}). This terminology is justified, for one can show that different choices of representation in our construction of (S, C) result in isomorphic locally G-structured spaces (Barrett and Manchak, 2024, Proposition 3.2.3). (See the appendix for a precise definition of isomorphism between locally G-structured spaces.)

Note that this definition is natural. In particular, for flat relativistic spacetimes it is closely related to our discussion of Minkowskian coordinates in section 2. Let (M, g_{ab}) be a flat relativistic spacetime and consider what the locally G-structured space (S, C) determined by (M, g_{ab}) is like when we pick Minkowski spacetime as the representation. One can easily verify that $c \in C$ if and only if $(\mathrm{dom}(c), c)$ is a Minkowskian coordinate chart on (M, g_{ab}). This means that, when we use Minkowski spacetime as the representation for (M, g_{ab}) in our construction, the locally G-structured space (S, C) determined by (M, g_{ab}) is such that C is just the collection of Minkowskian coordinates for (M, g_{ab}).

With this definition in hand, there is a sense in which one can recover a relativistic spacetime (M, g_{ab}) from the locally G-structured space (S, C) that it determines. This sense is given by the following theorem (Barrett and Manchak, 2024, Proposition 3.2.1).

Theorem 1. *Let (S, C) be the locally G-structured space determined by (M, g_{ab}). Then both of the following hold:*

1. *The identity map 1_M is a diffeomorphism between the manifold (S, C^+) and M.*
2. *The coordinate transformation pseudogroup Γ on S is the isometry pseudogroup of (M, g_{ab}).*

Let (M, g_{ab}) be a relativistic spacetime. Theorem 1 implies that the locally G-structured space (S, C) determined by (M, g_{ab}) recovers the manifold structure of M, in the sense that (S, C^+) and M are diffeomorphic. And (S, C) also recovers the metric g_{ab}, in the sense that Γ implicitly defines g_{ab}, since clause 2 of the theorem entails that $h^*(g_{ab}) = g_{ab}$ for each $h \in \Gamma$. There is therefore a sense in which the structure of an arbitrary relativistic spacetime is recoverable from some or other locally G-structured space.

Unfortunately, this sense is weak. It can be that g_{ab} is not the only metric recoverable from (S, C) in this manner. If (M, g_{ab}) has a small enough isometry pseudogroup, then more than one metric will be implicitly defined by the coordinate transformation pseudogroup of (S, C). In such a case, one does not know which metric structure (S, C) is giving rise to. We can make this point clear with one further result. In order to do so, we need the following two definitions. First, we will say that a relativistic spacetime (M, g_{ab}) is **determined by local isometry** if all relativistic spacetimes (M, g'_{ab}) with the same isometry pseudogroup as (M, g_{ab}) are isometric to (M, g_{ab}). Second, let (M, g_{ab}) be a relativistic spacetime that determines the locally G-structured space (S, C). We will say that (M, g_{ab}) is **locally presentable** if for any relativistic spacetime (M', g'_{ab}) (that determines the locally G'-structured space (S', C')), if (S, C) and (S', C') are isomorphic, then (M, g_{ab}) and (M', g'_{ab}) are isometric.

Two remarks will serve to unravel these definitions. First, if a spacetime (M, g_{ab}) is determined by local isometry, then its isometry pseudogroup uniquely determines the spacetime. This is because if some spacetime admits the same local isometries as (M, g_{ab}), then it must be isometric to (M, g_{ab}). We might say that if one knows the isometry pseudogroup of such a spacetime, one can know the structure of the spacetime. Second, a spacetime (M, g_{ab}) that is locally presentable is one that can be genuinely recovered from its underlying locally G-structured space (S, C), or in other words, presented in the framework of locally G-structured spaces. If some relativistic spacetime has the same underlying locally G-structured space (up to isomorphism) as (M, g_{ab}), then that spacetime is guaranteed to be isometric to (M, g_{ab}). This means that the structure of (S, C) determines the structure of (M, g_{ab}) (up to isometry). One can also put this basic idea in the following manner. If one knows (S, C), one will know (M, g_{ab}) too, since it's the only relativistic spacetime that determines a locally G-structured space isomorphic to (S, C). It is natural to think of local presentability as one way to capture the idea that a spacetime can be presented in the framework of locally G-structured spaces.

We now have the following result (Barrett and Manchak, 2024, Theorem 4.2.1).

Theorem 2. *Let (M, g_{ab}) be a relativistic spacetime. Then (M, g_{ab}) is locally presentable if and only if it is determined by local isometry.*

On the one hand, Theorem 2 implies that there is a fragment of general relativity that can be presented using locally G-structured spaces — namely, those spacetimes that are determined by local isometry. In this sense, Kleinian Presentability holds of spacetimes determined by local isometry.

On the other hand, however, Theorem 2 implies that Kleinian Presentability does not hold of all relativistic spacetimes. If a relativistic spacetime has a 'small enough' isometry pseudogroup, then it will not be locally presentable. To take an extreme case, consider a Heraclitus spacetime $(\mathbb{R}^2, g_{ab})$. This is a spacetime with a trivial isometry pseudogroup, i.e. every map in its isometry pseudogroup is an identity map (Manchak and Barrett, 2024). One can show that there is another Heraclitus spacetime $(\mathbb{R}^2, g'_{ab})$ that is not isometric to $(\mathbb{R}^2, g_{ab})$ (Barrett and Manchak, 2024, Proposition 3.2.4). This implies that $(\mathbb{R}^2, g_{ab})$ is not determined by local isometry. Theorem 2 then implies that it cannot be presented using the apparatus of locally G-structured spaces. This is just to rehearse the argument given by Barrett and Manchak (2024). Kleinian Presentability does not hold of all relativistic spacetimes.

It is worth emphasizing exactly why this is the case. At heart, the problem has to do with the kind of 'implicit definability' at work in the method of recovering a geometric space from a locally G-structured space. When a space has a particularly small collection of symmetries, many structures will be implicitly definable by these symmetries. Given a Heraclitus spacetime $(\mathbb{R}^2, g_{ab})$, we know that if some locally G-structured space (S, C) recovers g_{ab}, it must be that its coordinate transformation pseudogroup Γ is trivial. For if not, it would contain a map that does not preserve g_{ab}, and hence Γ would not implicitly define g_{ab}. But we know that there are non-isometric metrics on $\mathbb{R}^2$ that are invariant under all and only those maps in Γ. (The metric g'_{ab} is one such example.) So the data provided by (S, C) will not allow us to recover our spacetime.

This basic idea has precedent in the literature. Norton (1993, 1999, 2002) remarks that while Kleinian methods work in special relativity, they do not extend into the general relativistic setting. North (2021, p. 117) suggests that there are geometric spaces that "lie beyond the scope of Klein's program." Torretti (2016) directly writes that

> Klein's conception is too narrow to embrace all Riemannian geometries, which include spaces of variable curvature. Indeed, in the general case, the group of isometries of a Riemannian n-manifold is the trivial group consisting of the identity alone, whose structure conveys no information at all about the respective geometry.

It is important to mention, however, that the apparatus of locally G-structured spaces that we are considering is more powerful than the kind of Kleinian apparatus that these authors seem to have in mind. This can be seen by appreciating the fact that some relativistic spacetimes with trivial isometry groups — for example, the flat spacetime with trivial isometry group described by Barrett et al. (2023) — can still be presented using locally G-structured spaces. Although they have a trivial isometry group, we will shortly show that their isometry pseudogroup might still contain enough information to encode the structure of the spacetime. The full extent of the failure of Kleinian methods in general relativity has therefore not been appreciated. Even the more powerful Kleinian apparatus of locally G-structured spaces cannot present the structure of an arbitrary general relativistic spacetime.

We can now make a remark about the relationship between Determination and Kleinian Presentability. A variety of Determination follows from the fact that Kleinian Presentability holds of the relativistic spacetimes determined by local isometry. In essence, the right-to-left implication in Theorem 2 is itself a uniqueness result. Notice that the local presentability of a relativistic spacetime

(M, g_{ab}) is requiring that a kind of uniqueness result holds of (M, g_{ab}): if another spacetime has the same privileged coordinates as (M, g_{ab}) (in the sense of determining an isomorphic locally G-structured space), then it must be isometric to (M, g_{ab}). Hence if one uses local presentability to make precise the idea that a space 'can be presented in the framework of locally G-structured spaces', then when Kleinian Presentability holds, a corresponding uniqueness result is guaranteed for free. In this sense, Kleinian Presentability entails (a variety of) Determination.

3.2 Flat relativistic spacetimes

We have seen that Kleinian Presentability holds of spacetimes determined by local isometry, but one might want something more. Although Minkowski spacetime is determined by local isometry (Barrett and Manchak, 2024, Proposition 4.1.1), one conjectures that such spacetimes are rare. Even certain highly symmetric spacetimes are not determined by local isometry. Indeed, there are flat spacetimes that are not determined by local isometry. For example, let U be the $0 < t < 1$ region of Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$. One can easily verify that the two spacetimes (U, η_{ab}) and $(U, 2\eta_{ab})$ have the same isometry pseudogroup. But they cannot be isometric spacetimes, since they disagree about what the length of the longest timelike curve is. And hence (U, η_{ab}) is not determined by local isometry. There is, nonetheless, a close relationship between (U, η_{ab}) and $(U, 2\eta_{ab})$. There is no isometry between them, but there is a homothety. Recall that a diffeomorphism $f : M \to M'$ is a **homothety** between (M, g_{ab}) and (M', g'_{ab}) if there is some non-zero scalar $c \in \mathbb{R}$ such that $f : M \to M'$ is an isometry between (M, g_{ab}) and (M', cg'_{ab}). Clearly the identity map is a homothety between (U, η_{ab}) and $(U, 2\eta_{ab})$.

One can show that this example is representative. All flat spacetimes with the same isometry pseudogroup are related by homothety. We will say that (M, g_{ab}) is **determined (up to homothety) by local isometry** if for any relativistic spacetime (M, g'_{ab}) with the same isometry pseudogroup as (M, g_{ab}) there exists a homothety between (M, g_{ab}) and (M, g'_{ab}). We have the following result. An elementary proof is contained in the appendix.

Proposition 4. *Every flat relativistic spacetime is determined (up to homothety) by local isometry.*

Proposition 4 implies that one can use the framework of locally G-structured spaces to present all flat relativistic spacetimes, at least up to homothety. We will say that (M, g_{ab}) is **locally presentable (up to homothety)** if for any relativistic spacetime (M', g'_{ab}) (that determines the locally G'-structured space (S', C')), if (S, C) and (S', C') are isomorphic, then there is a homothety between (M, g_{ab}) and (M', g'_{ab}). It is natural to think of local presentability (up to homothety) a weaker way to capture the idea that a spacetime can be presented in the framework of locally G-structured spaces. It is clear that all locally presentable spacetimes are locally presentable (up to homothety). It may be that there are other conditions that capture the basic idea behind Kleinian Presentability. It is worth investigating such conditions, but we leave that for future work.

For now, we have the following result. The proof is exactly analogous to the proof of Theorem 2, and has been placed in the appendix.

Theorem 3. *Let (M, g_{ab}) be a relativistic spacetime. Then (M, g_{ab}) is locally presentable (up to homothety) if and only if it is determined (up to homothety) by local isometry.*

Theorem 3 and Proposition 4 together yield the following corollary.

Corollary 1. *All flat relativistic spacetimes are locally presentable (up to homothety).*

Insofar as one takes local presentability (up to homothety) to capture the idea that a geometric space is presentable using locally G-structured spaces, Kleinian Presentability holds of flat relativistic spacetimes. We are therefore able to successfully parlay the simple uniqueness result presented in Proposition 3 into a Kleinian formulation of this fragment of general relativity. One conjectures that the same is true in the cases of symplectic manifolds and flat classical spacetimes. It seems that, at least in some cases, uniqueness results can be understood as evidence for Kleinian Presentability. It is natural then to wonder what it is about Proposition 3 — and one conjectures, Propositions 1 and 2 as well — that results in a successful Kleinian presentation of a class of geometric objects. We will return to this question later.

We close this section by remarking that not all relativistic spacetimes are locally presentable (up to homothety). This is because they are not in general determined (up to homothety) by local isometry.

Lemma 4. *There are Heraclitus spacetimes $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ such that no homothety exists between them.*

Proof. Consider the Heraclitus spacetime (M, g_{ab}) constructed by Manchak and Barrett (2024). Here $M = \{(t, x) \in \mathbb{R}^2 : t, x > 0 \text{ and } t^2 > x^2\}$ and $g_{ab} = \Omega^2[-d_a t d_b t + d_a x d_b x]$ where $\Omega : M \to \mathbb{R}$ is defined by $\Omega(t, x) = (t^2 + x^2)^{-1}$. One can verify that the Ricci tensor R_{ab} on M comes out as $\alpha\Omega^2[-d_a t d_b t + d_a x d_b x]$ where $\alpha : M \to \mathbb{R}$ is defined by $\alpha(t, x) = 4(t^2 - x^2)$.

Consider the region $N \subset M$ for which $1/2 < \Omega^2 < 1$. Now let U be the subset of N for which $0 < \alpha < 1/2$ and let V be the subset of N for which $1 < \alpha < 2$. We find that that both U and V are non-empty open subsets of M. Clearly $\alpha\Omega^2 < 1/2$ on U while $\alpha\Omega^2 > 1/2$ on V. Now consider the spacetimes (U, g_{ab}) and (V, g_{ab}). Since the the Ricci tensor R_{ab} in each spacetime is $\alpha\Omega^2[-d_a t d_b t + d_a x d_b x]$, our construction ensures that there is no diffeomorphism $f : U \to V$ that preserves R_{ab}. Since any homothety must preserve R_{ab} (O'Neill 1983, p. 92), we see that there is no homothety between the spacetimes (U, g_{ab}) and (V, g_{ab}). Since U and V are both star-shaped regions of $\mathbb{R}^2$, each region is diffeomorphic to $\mathbb{R}^2$. Let $f : U \to \mathbb{R}^2$ and $h : V \to \mathbb{R}^2$

be diffeomorphisms. We find that the spacetimes $(\mathbb{R}^2, f_*(g_{ab}))$ and $(\mathbb{R}^2, h_*(g_{ab})$ are Heraclitus and not related by a homothety. □

With this lemma in hand, one can now see that neither $(\mathbb{R}^2, g_{ab})$ nor $(\mathbb{R}^2, g'_{ab})$ is determined (up to homothety) by local isometry. They have the same trivial isometry pseudogroup, but no homothety exists between them. Theorem 3 implies that they are not locally presentable up to homothety. This means that the variety of Kleinian Presentability that holds of flat relavistic spacetimes does not hold of arbitrary relativistic spacetimes.

4 Determination in general relativity

We now ask whether Determination holds of arbitrary (not necessarily flat) relativistic spacetimes. We will here provide two uniqueness results for general relativity. The first we will call "piecemeal" and the second "pointwise". In brief, we will show that while one can prove uniqueness results for arbitrary relativistic spacetimes, these results differ in important conceptual ways from the uniqueness results in section 2, and for this reason do not lead to Kleinian Presentability for all relativistic spacetimes.

4.1 A piecemeal presentation

We mentioned above that every relativistic spacetime has a representation. Let (M, g_{ab}) be a relativistic spacetime. Notice that a representation $(\mathbb{R}^n, g'_{ab})$ of it allows us to build a privileged collection of maps from M to $\mathbb{R}^n$. We can consider those diffeomorphisms $f : U \to V$ where $U \subset M$ and $V \subset \mathbb{R}^n$ are open and $f^*(g'_{ab}) = g_{ab}$. Suppose we take these maps to be the privileged coordinates of (M, g_{ab}). (Indeed, this is exactly what was suggested by the definition of the locally G-structured space determined by (M, g_{ab}).) More precisely, let (M, g_{ab}) be a relativistic spacetime with representation $(\mathbb{R}^n, g'_{ab})$. We will call the coordinates (U, ϕ) on M such that $\phi_*(g_{ab}) = g'_{ab}$ the **representative coordinates of** (M, g_{ab}) **with respect to** $(\mathbb{R}^n, g'_{ab})$. The following lemma is immediate from the definition of a representation.

Lemma 5. *Let (M, g_{ab}) be a relativistic spacetime with $p \in M$. Let $(\mathbb{R}^n, g'_{ab})$ be a representation of (M, g_{ab}). Then there are representative coordinates (U, ϕ) of (M, g_{ab}) with respect to $(\mathbb{R}^n, g'_{ab})$ with $p \in U$.*

One can move from Lemma 5 to the following uniqueness result in a similar manner as one moved from Lemmas 1, 2, and 3 to Propositions 1, 2, and 3.

Proposition 5. *Let (M, g_{ab}) be a relativistic spacetime with representation $(\mathbb{R}^n, g'_{ab})$. If (M', g''_{ab}) admits the same representative coordinates with respect to $(\mathbb{R}^n, g'_{ab})$ as (M, g_{ab}), then $(M, g_{ab}) = (M', g''_{ab})$.*

Proof. Suppose that (M', g''_{ab}) admits the same representative coordinates with respect to $(\mathbb{R}^n, g'_{ab})$ as (M, g_{ab}). One argues as in Proposition 1 to show that the two manifolds M and M' are equal. Now let $p \in M$. We know by Lemma 5 that there are representative coordinates (U, ϕ) on (M, g_{ab}) with respect to $(\mathbb{R}^n, g'_{ab})$ with $p \in U$. By assumption, this means that (U, ϕ) is also representative on (M, g''_{ab}) with respect to $(\mathbb{R}^n, g'_{ab})$. So $\phi_*(g''_{ab}) = g'_{ab}$ and $\phi_*(g_{ab}) = g'_{ab})$ which implies that $g_{ab} = g''_{ab}$ on U, since $\phi : U \to \phi[U]$ is a diffeomorphism. Hence $g_{ab} = g''_{ab}$ at p. Since p was arbitrary it must be that $(M, g_{ab}) = (M', g''_{ab})$. □

Each relativistic spacetime is in this sense determined by a class of maps to a spacetime with underlying manifold $\mathbb{R}^n$. We call this a 'piecemeal' uniqueness result because we have defined 'privileged coordinates' for each relativistic spacetime on a 'case-by-case' basis. For each relativistic spacetime one picks a representation and then defines its privileged coordinates relative to that representation.

We have seen that Kleinian Presentability does not hold of all relativistic spacetimes. A simple example illustrates why Proposition 5 is consistent with this. We know that there are non-isometric Heraclitus spacetimes $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ (Barrett and Manchak, 2024, Proposition 3.2.4). They are both representations of themselves. Since $(\mathbb{R}^2, g_{ab})$ is Heraclitus, one can easily check that the representative coordinates (U, ϕ) on $(\mathbb{R}^2, g_{ab})$ with respect to $(\mathbb{R}^2, g_{ab})$, are just the pairs (U, ϕ) where $U \subset \mathbb{R}^2$ is open and $\phi : U \to U$ is an identity map. The same is true of the representative coordinates (U, ϕ) on $(\mathbb{R}^2, g'_{ab})$ with respect to $(\mathbb{R}^2, g'_{ab})$. If we think of these coordinates as maps to $\mathbb{R}^2$, therefore, both $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ admit precisely the same privileged coordinates, even though their structure differs. We are, of course, simply rehearsing the argument given above for why Kleinian Presentability does not hold of arbitrary general relativistic spacetimes. The spacetimes $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ determine the same locally G-structured space, despite the fact that they are not isometric.

Proposition 5 has the form of a uniqueness result, but it is worth considering whether it is genuinely establishing Determination. It cannot be telling us that every relativistic spacetime can be characterized by a class of coordinates, insofar as coordinates are understood as mere smooth maps to $\mathbb{R}^n$. The privileged coordinates of the spacetimes $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ are the same maps to $\mathbb{R}^2$ — the identity maps — despite the fact that the two spacetimes are non-isometric. Rather, Proposition 5 tells us that a class of maps to a fixed representation of our spacetime determines the structure of our spacetime. $(\mathbb{R}^2, g_{ab})$ and $(\mathbb{R}^2, g'_{ab})$ clearly admit different isometries to a representation of the former. In order for these maps to determine the structure of our spacetime, we must treat them not merely as maps to $\mathbb{R}^n$; we must treat them as maps to another spacetime that has underlying manifold $\mathbb{R}^n$. This means that whether Proposition 5 establishes Determination for an arbitrary relativistic spacetime depends crucially on exactly what one means by 'coordinates'. If one takes coordinates to be merely maps to $\mathbb{R}^n$, the Proposition 5 does not establish Determination. But if one thinks of coordinates as isometries to a spacetime with underlying manifold $\mathbb{R}^n$, then Proposition 5 does establish a variety of Determination.

4.2 A pointwise presentation

There is a more natural uniqueness result available. It relies on the existence of Lorentz normal coordinates. Let (M, g_{ab}) be a relativistic spacetime with $p \in M$. We will say that a coordinates (U, ϕ) with $p \in U$ are **Lorentz normal coordinates** at p if $\phi(p) = (0, \ldots, 0) \in \mathbb{R}^n$ and both the metric g_{ab} and its associated derivative operator 'take a simple form' at p in (U, ϕ) coordinates, in the sense that the 'Christoffel symbols' of the derivative operator vanish at p and the metric g_{ab} takes the Minkowskian form

$$g_{ab} = d_a u^1 d_b u^1 - d_a u^2 d_b u^2 - \ldots - d_a u^n d_b u^n$$

at the point p, where u^i are the coordinate maps associated with (U, ϕ).

We now have the following guarantee (O'Neill, 1983, p. 71–73).

Lemma 6. *Let (M, g_{ab}) be a relativistic spacetime with $p \in M$. Then there are Lorentz normal coordinates (U, ϕ) with $p \in U$.*

It is important to note that if (U, ϕ) are Lorentz normal coordinates at p, then the metric g_{ab} will in general take Minkowskian form only at p in U; at other points in U, g_{ab} will not necessarily look Minkowskian. It is for this reason that we call the following a 'pointwise' uniqueness result. Its proof proceeds exactly as the proofs of Propositions 1, 2, and 3, so we leave it to the reader.

Proposition 6. *Let (M, g_{ab}) be a relativistic spacetime. If a relativistic spacetime (M', g'_{ab}) admits the same Lorentz normal coordinates as (M, g_{ab}), then $(M, g_{ab}) = (M', g'_{ab})$.*

This result establishes a variety of Determination for general relativity. Lorentz normal coordinates determine the structure of an arbitrary relativistic spacetime. But once again, Proposition 6 does not yield Kleinian Presentability. One can see this in the following manner (Barrett and Manchak, 2024). Suppose that we use Lorentz normal coordinates to build a locally G-structured space (S, C) from a relativistic spacetime (M, g_{ab}). In particular, this will mean that for each Lorentz normal coordinate chart (U, ϕ) on (M, g_{ab}), $\phi \in C$. The resulting coordinate transformation pseudogroup Γ will not in general implicitly define g_{ab}. To again take an extreme case, suppose that (M, g_{ab}) is Heraclitus. Let $p, q \in M$ be distinct points and suppose that we have Lorentz normal coordinates (U, ϕ) about p and (V, ψ) about q. By assumption, both ϕ and ψ are in C. This means that the 'coordinate transformation' map $\psi^{-1} \circ \phi$ is in Γ. Since $\psi^{-1} \circ \phi(p) = \psi^{-1}(0, \ldots, 0) = q$, we know that $\psi^{-1} \circ \phi$ is not the identity map. Since (M, g_{ab}) is Heraclitus, $\psi^{-1} \circ \phi$ cannot be contained in its isometry pseudgroup, and $(\psi^{-1} \circ \phi)^*(g_{ab}) \neq g_{ab}$. This means that the pseudogroup Γ resulting from (S, C) will not always implicitly define g_{ab}, if we include all Lorentz normal coordinates in C.

Coordinate transformations between Lorentz normal coordinates do not necessarily preserve the metric. This means that the way in which the Lorentz normal coordinates of a relativistic spacetime allow one to recover its structure is not the method of recovery presented in section 3. Proposition 6 therefore captures a variety of what Barrett and Manchak (2024) call “revision”. The idea is that a proponent of privileged coordinate approaches to our physical theories might revise the method by which we recover a geometric space from a locally G-structured space, and thereby avoid the difficulties discussed in section 3. Proposition 6 provides one way to do so. Indeed, consider how one would go about recovering the metric g_{ab} from the class of Lorentz normal coordinates of (M, g_{ab}). One would take a point $p \in M$, and find coordinates (U, ϕ) in this class in which $\phi(p) = (0, \ldots, 0)$. This guarantees that (U, ϕ) are Lorentz normal coordinates at p, rather than at some other point in U. One then would stipulate that the metric at p is $d_a u^1 d_b u^1 - d_a u^2 d_b u^2 - \ldots - d_a u^n d_b u^n$ in (U, ϕ) coordinates. One does this for each point $p \in M$.

While it may shed light on the significance of privileged coordinates, this variety of revision does not represent a victory for proponents of Kleinian methods. This is because there is an important sense in which the resulting method of recovering g_{ab} is much more ‘Riemannian’ than Kleinian. Indeed, in recovering the metric in this way one is simply saying what the metric g_{ab} is at each point in M. Such a presentation is conceptually the same as the standard Riemannian method of presenting (M, g_{ab}), in which one defines the tensor g_{ab} on M (in the usual way). Kleinian methods are distinctive because they employ a variety of implicit definability, looking to those structures that are ‘invariant under symmetry’. Norton (2002, p. 259) writes that under the Kleinian method a “geometric theory would be associated with a class of admissible coordinate systems and a group of transformations that would carry us between them. The cardinal rule was that physical significance can be assigned just to those features that were invariants of this group.” Similarly, North (2021, p. 48) writes that “Klein suggested that any geometry can be identified by means of the transformations that preserve the structure, likewise by the quantities that are invariant under the group of those transformations.” Wallace (2019, p. 135) remarks that the Kleinian method involves characterizing spaces “via the invariance groups of the geometry under transformations.” The method of presentation suggested by Proposition 6 is not Kleinian in this sense.

5 Conclusion

The results presented here are closely related to the recent debate concerning the extent to which special relativity is ‘locally valid’ in general relativity. (See Fletcher (2020, 2021), Linnemann et al. (2024), Fletcher and Weatherall (2023a,b) for discussion.) As has been noted, an arbitrary relativistic spacetime (M, g_{ab}) is not ‘locally like’ Minkowski spacetime in the most straightforward sense. It is not always the case that each point $p \in M$ is contained in an open neighborhood that is isometric to some open neighborhood of Minkowski spacetime. In other words, Minkowski spacetime is not in general a representation of (M, g_{ab}). At the end of section 2 we mentioned that there was a helpful way of understanding the crucial lemmas behind the uniqueness results in Propositions

1, 2, and 3. Each of the lemmas was saying that all geometric spaces of a particular kind (and particular dimension) have a fixed representation. For example, Lemma 3 is saying that Minkowski spacetime is a representation of every flat spacetime. It was precisely this that allowed us to present flat spacetimes using Kleinian methods. But only flat spacetimes have this property.

One therefore wonders whether there is any spacetime — Minkowski spacetime or not — that is a representation of all relativistic spacetimes. It is easy to see that there is not. We will say that a **common representation** for a class C of relativistic spacetimes is a spacetime $(\mathbb{R}^n, g'_{ab})$ that is a representation for each spacetime in C.

Proposition 7. *The class of n-dimensional relativistic spacetimes has no common representation.*

Proof. Suppose for contradiction that there is such a representation $(\mathbb{R}^n, g'_{ab})$. Since it is second countable, we know that there can only be countably many open disjoint sets. For each real number $r \in (0, 1)$ consider a variant of n-dimensional de Sitter spacetime with Ricci scalar r. Since $(\mathbb{R}^n, g'_{ab})$ is a representation of the class of n-dimensional relativistic spacetimes, for each of these de Sitter spacetimes there must be an isometry from an open set to $(\mathbb{R}^n, g'_{ab})$. But the images of these open sets under these isometries must be disjoint since the Ricci scalar differs in each. So there must be uncountably many disjoint open sets in $(\mathbb{R}^n, g'_{ab})$. □

This means that not only is an arbitrary spacetime not 'locally like' Minkowski spacetime (in the most straightforward sense), there is no fixed spacetime that all arbitrary spacetimes are 'locally like'. And this precludes the possibility of proving a lemma in the case of arbitrary relativistic spacetimes that is analogous to Lemmas 1, 2, and 3. Indeed, it is easy to see that Lemmas 5 and 6 are conceptually different from Lemmas 1, 2, and 3. We first compare Lemma 3 to Lemma 5. Lemma 3 is true because there is a *fixed* common representation for all flat relativistic spacetimes — namely, Minkowski spacetime. Lemma 5, on the other hand, is true because each relativistic spacetime has a representation; but different spacetimes will admit different representations. Now compare Lemma 3 to Lemma 6. Lemma 3 illustrates that there are coordinates (U, ϕ) about p in which the metric takes Minkowskian form on the entirety of U. Lemma 6 only illustrates that there are coordinates (U, ϕ) about p in which the metric takes Minkowskian form at p.

One small remark is worth making regarding Proposition 7. We will say that an n-dimensional spacetime is **maximally symmetric** if its vector space of Killing fields has dimension $\frac{1}{2}n(n+1)$. Each of the de Sitter spacetimes considered in the proof is maximally symmetric. This means that the proof of Proposition 7 has actually established something stronger than its statement indicates. Even the class of maximally symmetric spacetimes has no common representation. We conjecture, however, that each maximally symmetric spacetime is determined (up to homothety) by local isometry, and is thus presentable using Kleinian methods. It would furthermore be interesting to know exactly which other symmetry conditions — isotropy, homogeneity, etc. — spacetimes that are determined (up to homothety) by local isometry might have or lack. (The reader is invited to consult Belot (2023, Chapter 3) for details.)

In sum, we have demonstrated that varieties of Determination hold of symplectic manifolds (Proposition 1), flat classical spacetimes (Proposition 2), and flat relativistic spacetimes (Proposition 3). And even arbitrary relativistic spacetimes satisfy a kind of Determination (Propositions 5 and 6). We also demonstrated that while Kleinian Presentability does not hold of arbitrary relativistic spacetimes, it does hold of flat spacetimes. And because of the conceptual similarity between Propositions 1, 2, and 3, we conjectured that it also holds of symplectic manifolds and flat classical spacetimes.

In light of the fact that one cannot use locally G-structured spaces to present the entirety of general relativity (an argument which we rehearsed in section 3), Barrett and Manchak (2024) suggest three possible routes forward for the proponent of privileged coordinate approaches: restriction, revision, and reservation. In addition to the variety of revision suggested by Proposition 6, our results take steps forward along the restriction and reservation routes as well. First, a proponent of the significance of privileged coordinates might argue that locally G-structured spaces suffice to present geometric space within some 'restricted' domain; this is the idea behind the restriction route. We have here demonstrated a sense in which this is the case. Locally G-structured spaces can be used to present flat classical spacetimes; all flat classical spacetimes are locally presentable up to homothety. And we conjecture that one can take other steps along the restriction route by showing that analogous results hold for symplectic manifolds and flat classical spacetimes. Nevertheless, when taking this variety of restriction it is important to not over-generalize. Geroch and Horowitz (1979, p. 215) remark that "we are still somewhat over-conditioned to Minkowski spacetime." It is always tempting to try to understand how, for example, flat relativistic spacetimes work, and then generalize the lessons into the arbitrary case. But it is important to guard against this temptation when considering the significance of privileged coordinates. Kleinian methods work when spacetimes are flat; they do not work in general.

The reservation route involves being more 'reserved' or 'modest' about the significance of privileged coordinates. Barrett and Manchak (2024) discuss a few varieties of reservation; our results here suggest an additional one. We remarked above that there is a sense in which Kleinian Presentability entails Determination. Our discussion of uniqueness results in general relativity demonstrates that Determination need not entail Kleinian Presentability. Heraclitus spacetimes are determined by their Lorentz normal coordinates, for example, even though they are not presentable using the apparatus of locally G-structured spaces. Determination will only entail Kleinian Presentability if the established uniqueness result takes the right form. In particular, it must be that the privileged coordinates appealed to in the uniqueness result yield a coordinate transformation pseudogroup Γ that implicitly defines all and only the structures of the geometric space under consideration. The kind of privileged coordinates singled out in Proposition 5 were such that the resulting Γ might implicitly define more than just the metric on our spacetime. And the kind of privileged coordinates singled out in Proposition 6 were such that the resulting Γ might not even implicitly define the metric on our spacetime. Determination and Kleinian Presentability are therefore distinct theses that one might hold about the significance of privileged coordinates. And this opens up room for a variety of reservation. In certain cases, privileged coordinates do in a sense 'determine' the structure of a geometric space, but they do not allow for an alternative Kleinian presentation of that space.

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6 Appendix

Lemma 2. *Let (M, h^{ab}, t_a, ∇) be a flat classical spacetime with $p \in M$. Then there is a Galilean chart (U, ϕ) with $p \in U$.*

Proof. Let $p \in M$ and consider an orthonormal basis for h^{ab} at p, given by the covectors $t_a, \overset{2}{\sigma}_a, \overset{3}{\sigma}_a, \overset{4}{\sigma}_a$. We know that such a basis exists by the signature condition on h^{ab} and Proposition 4.1.1a of Malament (2012). Now we extend these vectors to smooth fields $t_a, \overset{2}{\sigma}_a, \overset{3}{\sigma}_a, \overset{4}{\sigma}_a$ on some neighborhood O of p such that the fields are everywhere constant with respect to ∇. We can do this because ∇ is flat and hence parallel transport is path independent in a small enough region around p. One can easily verify that since the fields $t_a, \overset{2}{\sigma}_a, \overset{3}{\sigma}_a, \overset{4}{\sigma}_a$ are constant on O and an orthonormal basis for h^{ab} at p, they form an orthonormal basis for h^{ab} everywhere in O.

Since the fields $t_a, \overset{2}{\sigma}_a, \overset{3}{\sigma}_a, \overset{4}{\sigma}_a$ are constant, they must be closed. So they are exact, and hence there are scalar fields u^1, u^2, u^3, u^4 on O such that $d_a u^1 = t_a, d_a u^2 = \overset{2}{\sigma}_a, d_a u^3 = \overset{3}{\sigma}_a$, and $d_a u^4 = \overset{4}{\sigma}_a$. We now let

$$\phi(q) = (u^1(q), u^2(q), u^3(q), u^4(q))$$

for all $q \in O$. We know that $\phi : O \to \mathbb{R}^4$ is smooth since each of the u^i are smooth. We show that ϕ_* has trivial kernel at p. Suppose that $\phi_*(\lambda^a) = \mathbf{0}$ for some vector λ^a at p. This means that for all smooth $f : \mathbb{R}^4 \to \mathbb{R}$, $\phi_*(\lambda^a)(f) = 0$, and hence $\lambda^a(f \circ \phi) = 0$. Since $u^i = x^i \circ \phi$, this implies that $\lambda^a(u^i) = 0$. But $\lambda^a(u^1) = \lambda^a d_a u^1 = \lambda^a t_a$, meaning that $\lambda^a t_a = 0$. One similarly shows that $\lambda^a \overset{i}{\sigma}_a = 0$ for each i. Since $t_a, \overset{2}{\sigma}_a, \overset{3}{\sigma}_a, \overset{4}{\sigma}_a$ form a basis for the cotangent space at p, this means that $\lambda^a = \mathbf{0}$, and so ϕ_* has trivial kernel at p. The Inverse Function Theorem (Lee, 2012, p. 166) therefore implies that there is an open neighborhood $U \subset O$ of p such that $\phi : U \to \phi[U]$ is a diffeomorphism. And this implies that (U, ϕ) is an n-chart on M.

We now show that the three required conditions hold of (U, ϕ). The first is trivial since we have $t_a = d_a u^1$ by construction. We show that $h^{ab} = \big(\frac{\partial}{\partial u^2}\big)^a \big(\frac{\partial}{\partial u^2}\big)^b + \big(\frac{\partial}{\partial u^3}\big)^a \big(\frac{\partial}{\partial u^3}\big)^b + \big(\frac{\partial}{\partial u^4}\big)^a \big(\frac{\partial}{\partial u^4}\big)^b$ by computing that the two sides have the same action on any pairs of basis vectors $t_a, \overset{2}{\sigma}_a, \overset{3}{\sigma}_a, \overset{4}{\sigma}_a$. We compute for example that

$$\begin{aligned} h^{ab}\overset{2}{\sigma}_a\overset{3}{\sigma}_b &= 0 \\ &= \Big(\big(\frac{\partial}{\partial u^2}\big)^a\big(\frac{\partial}{\partial u^2}\big)^b + \big(\frac{\partial}{\partial u^3}\big)^a\big(\frac{\partial}{\partial u^3}\big)^b + \big(\frac{\partial}{\partial u^4}\big)^a\big(\frac{\partial}{\partial u^4}\big)^b\Big) d_a u^2 d_b u^3 \\ &= \Big(\big(\frac{\partial}{\partial u^2}\big)^a\big(\frac{\partial}{\partial u^2}\big)^b + \big(\frac{\partial}{\partial u^3}\big)^a\big(\frac{\partial}{\partial u^3}\big)^b + \big(\frac{\partial}{\partial u^4}\big)^a\big(\frac{\partial}{\partial u^4}\big)^b\Big)\overset{2}{\sigma}_a\overset{3}{\sigma}_b \end{aligned}$$

And lastly, we show that ∇ is the coordinate derivative operator on (U, ϕ). We know that $\nabla_n d_a u^i = \mathbf{0}$ for each i, since each of the fields $t_a, \overset{2}{\sigma}_a, \overset{3}{\sigma}_a, \overset{4}{\sigma}_a$ is constant on U. This means that

$$\mathbf{0} = \nabla_n\Big(\big(\frac{\partial}{\partial u^i}\big)^a d_a u^j\Big) = d_a u^j \nabla_n\big(\frac{\partial}{\partial u^i}\big)^a + \big(\frac{\partial}{\partial u^i}\big)^a \nabla_n d_a u^j = d_a u^j \nabla_n\big(\frac{\partial}{\partial u^i}\big)^a$$

for each $j = 1, 2, 3, 4$. The first equality follows since $(\frac{\partial}{\partial u^i})^a d_a u^j$ is constant, the second by properties of the derivative operator, and the third since $\nabla_n d_a u^j = \mathbf{0}$. Since $d_a u^1, \ldots, d_a u^4$ form a basis for the cotangent space at each point, this immediately implies that $\nabla_n(\frac{\partial}{\partial u^i})^a = \mathbf{0}$, and therefore ∇ is the coordinate derivative operator for (U, ϕ). □

Lemma 7. *If (M, g_{ab}) and (M, g'_{ab}) have the same isometry pseudogroup, then for every open set $U \subset M$ and vector field λ^a on U, λ^a is a Killing field of (U, g_{ab}) if and only if it is a Killing field of (U, g'_{ab}).*

Proof. Let $U \subset M$ and λ^a a vector field on U. We show that if λ^a is a Killing field of (U, g_{ab}), then it is a Killing field of (U, g'_{ab}). The other direction follows analogously. Since $\mathcal{L}_\lambda g_{ab} = \mathbf{0}$, Proposition 1.6.6 of Malament (2012) implies that for all local one-parameter groups of diffeomorphisms $\{\Gamma_t : V \to \Gamma_t[V]\}_{t\in I}$ generated by λ^a, and all $t \in I$, $(\Gamma_t)^*(g_{ab}) = g_{ab}$. This means that for all local one-parameter groups of diffeomorphisms $\{\Gamma_t : V \to \Gamma_t[V]\}_{t\in I}$ generated by λ^a, and all $t \in I$, Γ_t is in the isometry pseudogroup of (M, g_{ab}). And hence it is in the isometry pseudogroup of (M, g'_{ab}), which immediately implies that $(\Gamma_t)^*(g'_{ab}) = g'_{ab}$. Proposition 1.6.6 of Malament (2012) then implies that $\mathcal{L}_\lambda g'_{ab} = \mathbf{0}$, and hence λ^a is a Killing field of (U, g'_{ab}). □

Proposition 4. *Every flat relativistic spacetime is determined (up to homothety) by local isometry.*

Proof. Let (M, g_{ab}) be a flat relativistic spacetime and suppose that (M, g'_{ab}) is a relativistic spacetime with the same isometry pseudogroup as (M, g_{ab}). Let $p \in M$ with ξ^a a vector at p. Since (M, g_{ab}) is flat, we can extend ξ^a to a constant field (with respect to ∇, the derivative operator associated with g_{ab}) on some open set O containing p. We call this extended field ξ^a. One can easily see that it is a Killing field of (M, g_{ab}). By Lemma 7, it is also a Killing field of (M, g'_{ab}). We now compute the following:

$$\mathbf{0} = \mathcal{L}_\xi g'_{ab} = \xi^n \nabla_n g'_{ab} + g'_{nb} \nabla_a \xi^n + g'_{an} \nabla_b \xi^n = \xi^n \nabla_n g'_{ab}$$

on O. The first equality holds since ξ^a is a Killing field of (M, g'_{ab}), the second by Proposition 1.7.4 of Malament (2012), and the third since ξ^n is constant. Since p and ξ^a were arbitrary, this implies that $\nabla_n g'_{ab} = \mathbf{0}$ everywhere on M. This means that ∇ is the derivative operator associated with g'_{ab}, so the two metrics are affine equivalent and, therefore, projectively equivalent as well.

We now claim that, at all points in M, a vector is null with respect to g_{ab} if and only if it is null with respect to g'_{ab}. Let $p \in M$ with α^a and β^a vectors at p. As above, we can extend them to constant fields α^a and β^a on some open set O containing p. By choosing O to be a small enough neighborhood about p, we may assume that there is a position field χ^a on O that vanishes at p and satisfies $\nabla_a \chi^b = \delta^b_a$ on O (Malament, 2012, Proposition 1.7.12). We now consider the constant anti-symmetric field $F_{ab} = \alpha_a \beta_b - \alpha_b \beta_a$. (Note that all indices throughout are lowered using g_{ab}.) One now easily shows that

$$\lambda^a = g^{ac} \chi^b F_{bc}$$

is a Killing field on O with respect to g_{ab}. Note that this implies that $\nabla_n \lambda^a = \alpha_n \beta^a - \alpha^a \beta_n$. Lemma 7 implies that it is a Killing field with respect to g'_{ab}. We then compute the following:

$$\mathbf{0} = \mathcal{L}_\lambda g'_{ab} = \lambda^n \nabla_n g'_{ab} + g'_{mb} \nabla_a \lambda^m + g'_{an} \nabla_b \lambda^n$$

$$= g'_{an}\alpha_b\beta^n - g'_{an}\alpha^n\beta_b + g'_{mb}\alpha_a\beta^m - g'_{mb}\alpha^m\beta_a$$

The first equality follows since ξ^a is a Killing field of g'_{ab}, the second from Proposition 1.7.4 of Malament (2012), and the third from the fact (mentioned above) that $\nabla_n g'_{ab} = \mathbf{0}$ and the easily computable fact that $\nabla_n\lambda^a = \alpha_n\beta^a - \alpha^a\beta_n$. We now contract the result with $\alpha^a\alpha^b$ to see that

$$\begin{aligned} 0 &= (g'_{an}\alpha^a\beta^n)(\alpha_b\alpha^b) - (g'_{an}\alpha^a\alpha^n)(\beta_b\alpha^b) \\ &\quad + (g'_{mb}\alpha^b\beta^m)(\alpha_a\alpha^a) - (g'_{mb}\alpha^m\alpha^b)(\alpha^a\beta_a) \\ &= 2(g'_{an}\alpha^a\beta^n)(\alpha_b\alpha^b) - 2(g'_{an}\alpha^a\alpha^n)(\beta_b\alpha^b) \end{aligned}$$

Now assume that α^a is null with respect to g_{ab} and that β^a is not orthogonal to α^a with respect to g_{ab}. It follows from the preceding equation that $g'_{an}\alpha^a\alpha^n = 0$. So, at all points $p \in M$, if a vector α^a at p is null with respect to g_{ab}, it must be null with respect to g'_{ab} too. One establishes the converse via an analogous argument.

With this claim in hand, we can finish the argument. Since g_{ab} and g'_{ab} agree on which vectors are null, they must be conformally equivalent (Malament, 2012, Proposition 2.1.1). Since they are both projectively and conformally equivalent, Proposition 1.9.6 of Malament (2012) implies that the conformal factor connecting them is constant. And hence there is a constant $c\mathbb{R}$ such that $g'_{ab} = cg_{ab}$. (Note that Proposition 2.1.1 only holds for spacetimes of dimension 3 or greater. But if the dimension is 2, agreement on null vectors will entail that either g'_{ab} is conformally equivalent to g_{ab} or it is conformally equivalent to $-g_{ab}$. We can then use Proposition 1.9.6 to show that either $g'_{ab} = cg_{ab}$ with c positive or $g'_{ab} = cg_{ab}$ with c negative.) □

Definition. Let (S, C) and (S', C') be locally G- and G'-structured spaces, respectively. An **isomorphism** $f : (S, C) \to (S', C')$ is a bijection $f : S \to S'$ such that

1. f is a diffeomorphism between (S, C^+) and (S', C'^+) and
2. the map $s \mapsto f \circ s \circ f^{-1}$ is a bijection between Γ and Γ', the pseudogroups associated with (S, C) and (S', C').

Theorem 4. *Let (M, g_{ab}) be a relativistic spacetime. Then (M, g_{ab}) is locally presentable (up to homothety) if and only if it is determined (up to homothety) by local isometry.*

Proof. We exactly follow the contours of the proof of Theorem 1 given by Barrett and Manchak (2024, Theorem 4.2.1). Suppose first that (M, g_{ab}) is locally presentable (up to homothety). Let (M, g'_{ab}) be a relativistic spacetime that has the same isometry pseudogroup as (M, g_{ab}). Let (S', C') be the locally G'-structured space determined by (M, g'_{ab}) and (S, C) the locally G-structured space determined by (M, g_{ab}). Theorem 1 implies that the identity maps $1_M : S' \to M$ and $1_M : S \to M$ (which make sense since $S = M = S'$) are diffeomorphisms from (S', C'^+) to M and from (S, C^+) to M, and that Γ' and Γ are the isometry pseudogroups of (M, g'_{ab}) and (M, g_{ab}). This implies that 1_M is an isomorphism between (S, C) and (S', C'). Since (M, g_{ab}) is locally presentable (up to homothety), (M, g_{ab}) and (M, g'_{ab}) are related by homothety, and hence (M, g_{ab}) is determined (up to homothety) by isometry.

Now suppose that (M, g_{ab}) is determined (up to homothety) by local isometry and let (M', g'_{ab}) be a relativistic spacetime. Suppose that $f : S \to S'$ is an isomorphism between (S, C) and (S', C'), the locally G- and G'-structured spaces determined by (M, g_{ab}) and (M, g'_{ab}), respectively. We show that f is a homothety between (M, g_{ab}) and (M', g'_{ab}). We first show that $f : M \to M'$ is a diffeomorphism. (Note that f is a function $M \to M'$ since $S = M$ and $S' = M'$.) Since f is an isomorphism, we know that it is a diffeomorphism between the manifolds (S, C^+) and (S', C'^+). Theorem 1 implies that $1_M : M \to S$ is a diffeomorphism between M and (S, C^+) and $1_{M'} : S' \to M'$ is a diffeomorphism between (S', C'^+) and M', so the composition $f = 1_{M'} \circ f \circ 1_M : M \to M'$ is a diffeomorphism.

Consider the metric $f^*(g'_{ab})$ on M. We show that (M, g_{ab}) and $(M, f^*(g'_{ab}))$ have the same isometry pseudogroup. First, suppose that $h : U \to V$ is in the isometry pseudogroup of $(M, f^*(g'_{ab}))$, so $h^*(f^*(g'_{ab})) = f^*(g'_{ab})|_U$. This immediately implies that $f_* \circ h^* \circ f^*(g'_{ab}) = g'_{ab}$, so $(f \circ h \circ f^{-1})^*(g'_{ab}) = g'_{ab}$. Since Γ' is by Theorem 1 the isometry pseudogroup of (M', g'_{ab}), this means that $f \circ h \circ f^{-1} \in \Gamma'$. Clause 2 of the definition of isomorphism implies that $f^{-1} \circ f \circ h \circ f^{-1} \circ f \in \Gamma$, so $h \in \Gamma$. Theorem 1 then implies that $h : U \to V$ is in the isometry pseudogroup of (M, g_{ab}). Second, suppose that $h : U \to V$ is in the isometry pseudogroup of (M, g_{ab}), so $h^*(g_{ab}) = g_{ab}|_U$. Theorem 1 implies that $h \in \Gamma$. Clause 2 of the definition of isomorphism implies that $f \circ h \circ f^{-1} \in \Gamma'$. Theorem 1 then implies that $(f \circ h \circ f^{-1})^*(g'_{ab}) = g'_{ab}$, which means that $h^*(f^*(g'_{ab})) = f^*(g'_{ab})$, so h is in the isometry pseudogroup of $(M, f^*(g_{ab}))$. Since (M, g_{ab}) and $(M, f^*(g'_{ab}))$ have the same isometry pseudogroup, the fact that (M, g_{ab}) is determined (up to homothety) by isometry implies that (M, g_{ab}) and $(M, f^*(g'_{ab}))$ are related by a homothety. So there is some scalar $c \in \mathbb{R}$ and a diffeomorphism $h : M \to M$ such that $h^*(cf^*(g'_{ab})) = g_{ab}$. This implies that $(f \circ h)^*(cg'_{ab}) = g_{ab}$, and hence $f \circ h$ is a homothety between (M, g_{ab}) and (M', g'_{ab}). This means that (M, g_{ab}) is locally presentable (up to homothety). □