# Representations with Thresholds & Representation of Choice Probabilities

Chapters 16 and 17

# The Basic Problem

- $n$ : a standard cup of coffee containing $n$ granules of sugar
- Given any two cups, $m$ & $n$, the subject expresses preference for one over the other or an indifference relation between them.
- The subject cannot distinguish between $n$ and $n+1$ by taste for any $n$. So ($n \sim n+1$).
- But for some $k$, the subject isn't indifferent between $n$ and $n + k$.
- Therefore, $\sim$ cannot be transitive.
- But $\succ$ is expected to be.
- How do we represent this?

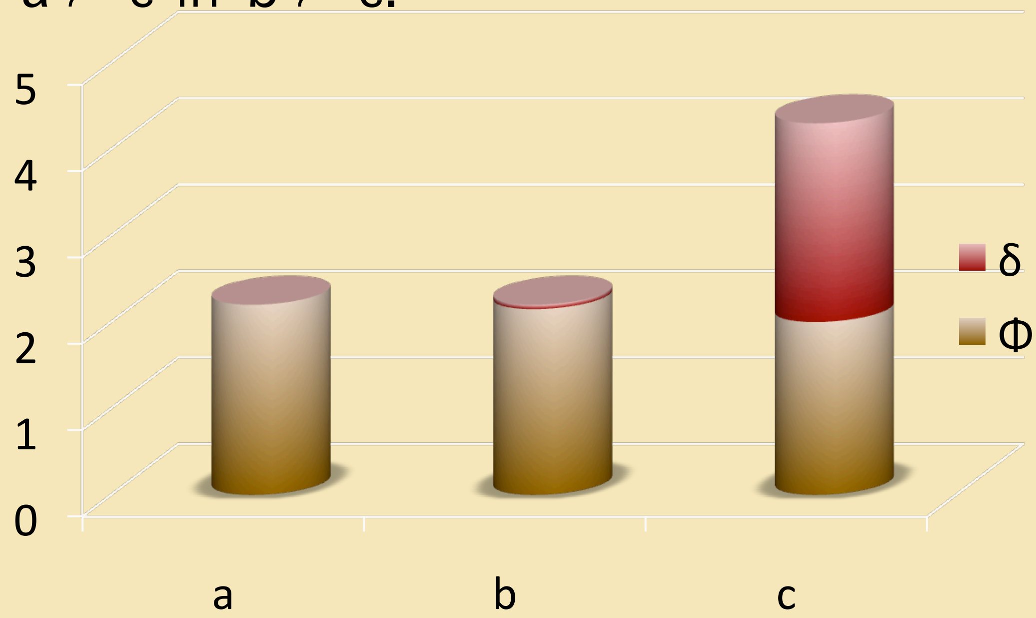
# 16: Representations with Thresholds

**<u>Definition 1</u>** (p. 303): **Suppose $\succ$ and ~ are binary relations on A, where A is non-empty**

1. $< A, \succ >$ is a Strict Partial Order iff $\succ$ is asymmetric and transitive
2. $< A, \succ >$ is a Graph iff ~ is reflexive and symmetric
3. ~ is the Symmetric Complement of $\succ$ iff "a ~ b" is equivalent to "not(a $\succ$ b) and not(b $\succ$ a)"

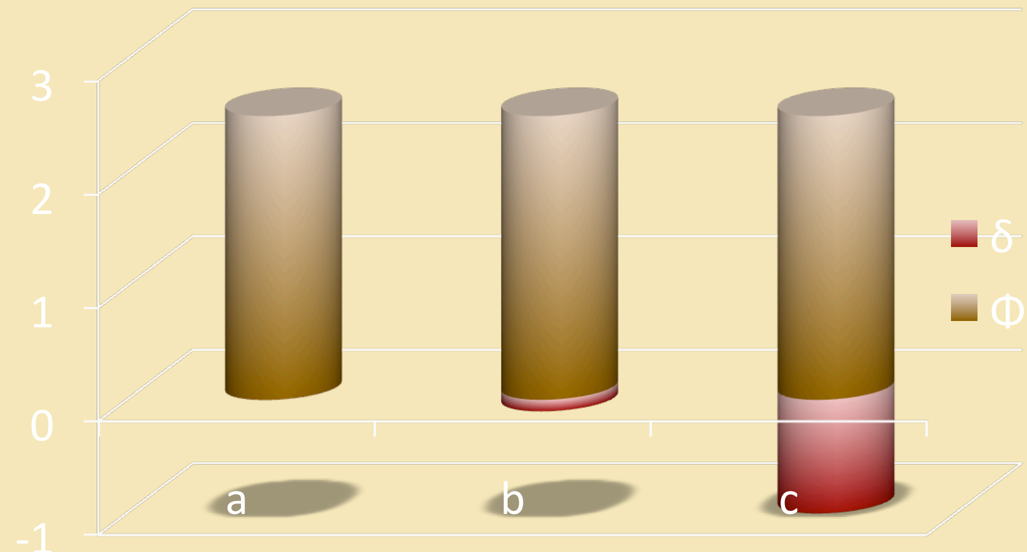
**<u>Definition 2</u>** (p. 305): **Let $\succ$ be an asymmetric binary relation on *A*. A pair of real-valued functions $<\varphi^{--},\delta^{--}>$ on A is an Upper-Threshold Representation iff**

- $\delta^{--}$ is nonnegative for all a, b, c, in *A*
- **If $a \succ b$, then $\varphi^{--}(a) \geq \varphi^{--}(b) + \delta^{--}(b)$**
- **If $\varphi^{--}(a) > \varphi^{--}(b) + \delta^{--}(b)$, then $a \succ b$.**
- If $\varphi^{--}(a) = \varphi^{--}(b)$, then $a \succ c$ iff $b \succ c$.



**Definition 2** (continued): $<\varphi_{-},\delta_{-}>$ on A is a **Lower-Threshold Representation** iff

- $\delta_{-}$ is nonpositive
- **If $a \prec b$, then $\varphi_{-}(a) \leq \varphi_{-}(b) + \delta_{-}(b)$**
- **If $\varphi_{-}(a) < \varphi_{-}(b) + \delta_{-}(b)$, then $a \prec b$.**
- If $\varphi_{-}(a) = \varphi_{-}(b)$, then $a \prec c$ iff $b \prec c$.



**Definition 2** (continued): **$<\varphi, \delta^{-}, \delta_{-}>$ on A is a Two-Sided Threshold Representation iff**

- $< \varphi, \delta^{-}>$ is an upper-threshold representation
- $< \varphi, \delta_{-}>$ is a lower-threshold representation
- $a \sim b$ then $\varphi(a)$ lies in the interval $[\varphi(b) + \delta_{-}(b), \varphi(b) + \delta^{-}(b)]$

**Definition 2** (continued): **$<\varphi^{-}, \delta^{-}>$ is said to be Strong iff iv holds. $<\varphi^{-}, \delta^{-}>$ is said to be Strong* iff iv* holds.**

(iv) $a \succ b$ iff $\varphi^{-}(a) > \varphi^{-}(b) + \delta^{-}(b)$

(iv*) $a \succ b$ iff $\varphi^{-}(a) \geq \varphi^{-}(b) + \delta^{-}(b)$

(same idea for lower and two-sided representations)

**<u>Definition 3</u>** (p.307): **Let $\succ$ be an asymmetric binary relation on *A*. The Upper Quasiorder induced by $\succ$, $Q^-$ and the Lower Quasiorder induced by $\succ$, $Q_-$ are defined as follows:**

- ($a\ Q^-\ b$) iff for all c in A**, if b $\succ$ c then a $\succ$ c.**
- ($a\ Q_-\ b$) iff for all c in A, **if c $\succ$ a then c $\succ$ b.**

We define the I relation in terms of Quasiorders…

1. ($a\ I^-\ b$) iff ($a\ Q^-\ b$) and ($b\ Q^-\ a$)

   In other words, **(b $\succ$ c iff a $\succ$ c)**

2. Same thing for $I_-$

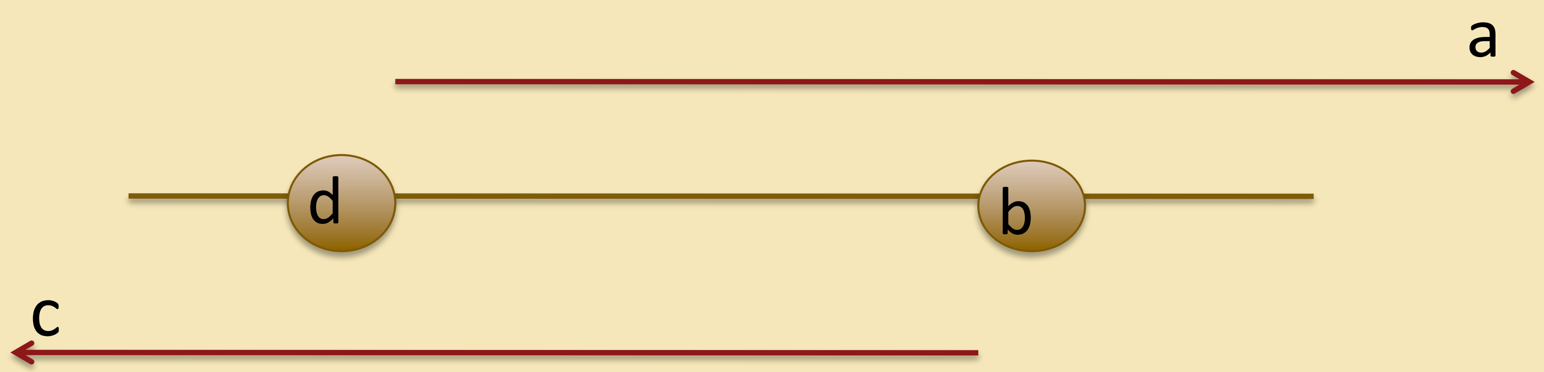
# Theorem 1 (p. 307)

If < A, $\succ$ > has an upper threshold representation, then the upper quasiorder induced by $\succ$ is connected.  So it is a weak order.

- In layman's terms, if, in our structure, preference implies the sort of function we described, then the way in which the elements of preference pairs relate to a third element is connected. So we get a weak order. Bam!

**Definition 4** (p.309): **Suppose $\succ$ is an irreflexive binary relation on *A*. < A, $\succ$ > is an Interval Order iff**

- For all a, b, c, d, in *A*, **If a $\succ$ c and b $\succ$ d, then either a $\succ$ d or b $\succ$ c**

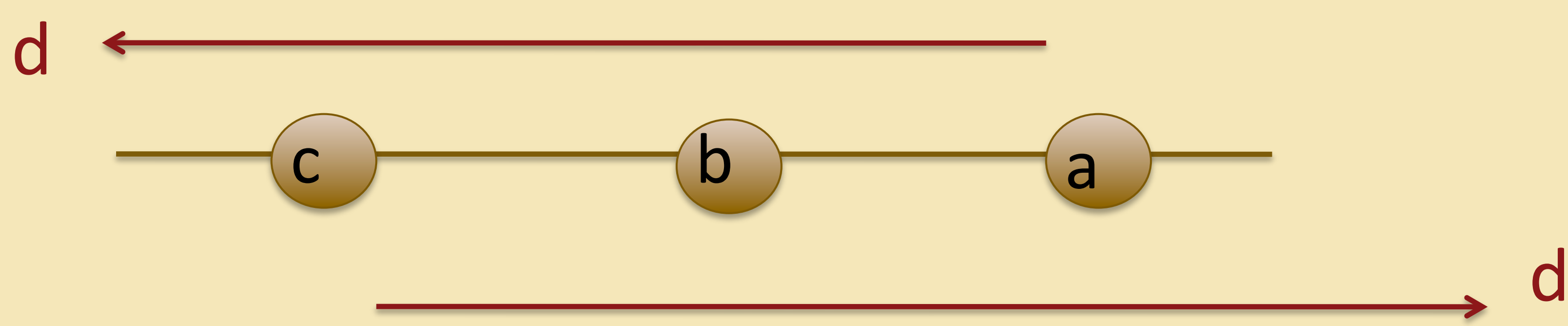


# Theorem 2 (p. 310)

- If $\succ$ is binary on A and is asymmetric, then the upper quasiorder and the lower quasiorder are connected and equivalent.
- Furthermore, having one of these connected quasiorders plus asymmetry is equivalent to something being an interval order.
- What this means is that transitivity of $\succ$ falls out of the definition of interval order.
- Wait for it…

**Definition 5** (P. 310): **Suppose <A, $\succ$ > is an interval order. Then it is a Semiorder iff**

- **For all a, b, c, d in A, If a $\succ$ b and b $\succ$ c, then either a $\succ$ d or d $\succ$ c**
- This may look trivial (because it looks like we're only saying that d falls somewhere on this line), but it is not.
- Suppose you're standing at c trying to judge the ordering: obviously a $\succ$ c and not: d $\succ$ c
- Now hang out at a. a $\succ$ b but not: a $\succ$ d



# What we've learned so far…

- A necessary condition for the construction of a one-sided threshold representation of $\succ$ is that $< A, \succ >$ is an interval order.
- A necessary condition for the construction of a two-sided threshold representation of $\succ$ is that $< A, \succ >$ is an semiorder.

And now we come to…

- A necessary condition for the construction of the set of equivalence classes for a two-sided threshold representation is that they contain a finite or countable order-dense subset.
- But is this enough? Can we construct a threshold representation for *any* interval order or semiorder whose equivalence classes have a countable order-dense subset?
- Yes we can!

**Definition 7** (P. 315): **Suppose $<\varphi, \delta^{--}, \delta_{--}>$ is a two-sided threshold representation of $<A, \succ>$. It is said to be Tight iff**

- For all a, b, c in *A*,

1. If *a I b then* $\varphi(a) = \varphi(b)$
2. $\delta^{--}(a) = \sup \{\varphi(a') - \varphi(a) \mid a' \in A \text{ and } a' \sim a\}$
3. $\delta_{--}(a) = \inf \{\varphi(a') - \varphi(a) \mid a' \in A \text{ and } a' \sim a\}$

Key points about Tight Representations

-The order induced by $\varphi$ is the maximal ordering compatible with $\succ$. (coarsest ordering)

-The threshold of delta is as small in absolute value as it can be.

# **Theorem 11** (p. 320)

- **Theorem 11 officially gives us uniqueness and existence!**
- **To get this, we have to assume…**
- $\varphi$ is dense on an interval
- $\delta^{--}(\varphi)$ is continuous and bounded away from 0 on that interval
- Monotonicity (i.e. $\varphi + \delta$ is a strictly increasing function of $\varphi$
- Wait for it…

**Definition 11** (P. 337): **Let P be a binary probability function on A x A. For all a, b, c, d in A:**

- P has Weak Stochastic Transitivity iff:

if $P$ (a, b) ≥ ½ and $P$ (b, c) ≥ 1/2 , then $P$ (a, c) ≥ ½

- P has Weak Independence iff:

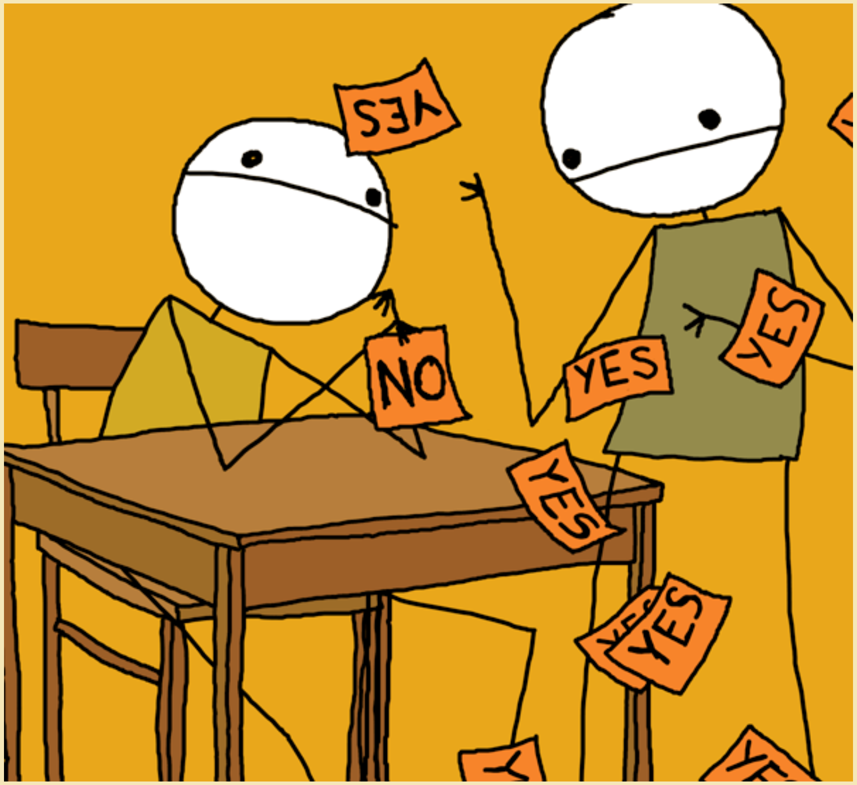
if $P$ (a, c) > $P$ (b, c), then $P$ (a, d) ≥ $P$ (b, d)

- P has Strong Stochastic transitivity iff

If $P$ (a, b) ≥ ½ and $P$ (b, c) ≥ ½, then $P$ (a, c) ≥ max [$P$ (a, b), $P$ (b, c)

# 17: Representation of Choice Probabilities

- Choose the option that maximizes.
- **Problem:** If we choose to maximize, how do we explain our inconsistent choices?
- Choice probabilities are the function of two arguments, an option and a set of options.
- Given option set B and option a ∈ B, the probability of choosing a if B is the set of feasible options is ***P* (a, B).**



**Definition 1** (p.384):

**< A, M, P > is a Structure of Choice Probabilities iff:**

- A is a set.
  - It comprises all objects in the domain that we're looking at.
- M is also a set.
  - It is nonempty and finite
  - It has $2^A$ members (the set of characteristic functions of subsets of A).
- P is a real-valued function with the following features…

**<u>Definition 1</u>** (continued):

**< A, M, P > is a Structure of Choice Probabilities iff**

- Dom(P)= {(a, B) | a $\in$ B $\in$ M}
- P(a, B) $\geq$ 0
- $\sum_{b \in B} P(b, B) = 1$
- Furthermore…
- < A, M, P> is **<u>finite</u>** iff A is finite
- <A, M, P> is **<u>closed</u>** iff
  - A is finite
  - M={B $\subset$ A | B $\neq \varnothing$ }

**<u>Definition 2</u>** (P. 388):

**<A, M, P> is a Pair Comparison Structure iff:**

- <A, M, P> is a structure of choice probability
- M is a reflexive binary relation on A

  e.g. if our parameters are *P* (a, b), then, since M is reflexive, *P* (a, a) = ½.

*Notation note: instead of writing (a, b) $\in$ M, we write aMb. They will all be in pairs

- M is a symmetric binary relation of A
  - aMb implies bMa
- <A, M, P> is **<u>complete</u>** iff M = A x A

**<u>Definition 3</u>** (p. 389)**:**

**Let <A, M, P> be a complete structure of pair comparison (so, M = A x A), and *P*(a, b) ≥ ½ and *P*(b, c) ≥ ½:**

Weak Stochastic Transitivity (WST)

holds iff *P* (a, c) ≥ ½

This means that if a ≻/~ b, b ≻/~ c, then a ≻/~ c

Moderate Stochastic Transitivity (MST)

holds iff *P* (a, c) ≥ *min* [*P* (a, b), *P* (b, c)]

Strong Stochastic Transitivity (SST)

holds iff P(a, c) ≥ *max* [*P* (a, b), *P* (b, c)]

Strict Stochastic Transitivity (ST)

holds iff SST holds and a strict inequality in the hypotheses implies a strict inequality in the conclusion.

**<u>Definition 4</u> (390):**

**A complete structure of pair comparison satisfies the Strong-Utility Model iff:**

- There exists a real-valued function $\varphi$ on A such that for all a, b, c, d $\in A$:

$$\varphi(a) - \varphi(b) \geq \varphi(c) - \varphi(d) \textit{ iff } P(a, b) \geq P(c, d)$$

$$* \; P(a, b) \geq P(c, d) \text{ iff } ab \succ/\sim cd$$

**Definition 5** (p. 390):

**A structure of pair comparison <A, M, P> is a Complete Difference Structure iff:**

- $M$ = A x A (same as completeness for pair comparison structures)
- The Monotonicity and the Solvability axioms hold.
- The Monotonicity Axiom:
  - If $P$ (a, b) ≥ $P$ (a′, b′) and $P$ (b, c) ≥ $P$ (b′, c′), then $P$ (a, c) ≥ $P$ (a′, c′).
  - if either antecedent inequality is strict, the conclusion is also strict.
- The Solvability Axiom:
  - For any t ∈ (0, 1) that satisfies $P$ (a, b) ≥ t ≥ $P$ (a, d), there exists c ∈ $A$, such that $P$ (a, c) = t.

# Theorems 1 & 2 p. 391-2)

## Theorem 1

- If < A, M, P > is a COMPLETE difference structure, then we can get a function that takes us from A onto some real interval.
- This function is unique up to a positive linear transformation.

## Theorem 2

- If < A, M, P > is a LOCAL difference structure, then we can get a function that takes us from A onto some real interval.
- This function is unique up to a positive linear transformation.

Definition 6 (p. 392): a $\succ/\sim$ b iff aMb and *P* (a, b) $\geq$ ½. A pair comparison structure <A, M, P> is a Local Difference Structure iff, for all a, a′, b, b′, c, c′ $\in$ *A:*

- The following Axioms hold:

1. Comparability: Any two elements that are bounded from above or below by the same third element are comparable
2. Monotonicity: same as before, except adding M's (thereby restricting the domain of P)
3. Solvability: same as before, except adding M's
4. Connectedness: Any two nonequivalent elements of A are connected either by an increasing or a decreasing sequence, but not both.

**Definition 7** (p. 394): **A complete pair comparison structure <A, M, P> with $A = A_1 \times \ldots \times A_n$ is an Additive-Difference Structure iff, for all a, a′, b, b′, c, c′, d, d′ $\in A$:**

- The following axioms hold:

1. Independence: Primes and not primes agree on one component. And the pairs: (a, c), (a′, c′), (b, d), (b′, c′) agree on all others.

**$P$ (a, b) ≥ $P$ (a\`, b\`) iff $P$ (c, d) ≥ $P$ (c\`, d\`)**

1. Monotonicity: Same as before, except now we are supposing that a, a\`, b, b\`, c, c\` coincide on all but one factor.
2. Solvability: Same as before, except that c coincides with b and d on any factor on which they coincide
3. The Thomsen Condition: cancellation stuff

# Theorems 3 & 4 (p. 395-7)

- Theorem 3 just tells us that we get a representation theorem for additive difference structures.
- But intransitive preferences can survive Theorem 3 (see P. 398-9)
- So we introduce Theorem 4, which gets rid of intransitive preferences.
- And everyone is happy

**<u>Definition 8</u>** (p. 410):

**A closed structure of choice probabilities <A, M, P> satisfies Simple Scalability iff:**

- There exists a real-valued $\varphi$ on *A* & a family of real-valued functions $\{F_\beta\}$
- $2 \leq \beta \leq \alpha$, (the cardinality of A is at least as big as the cardinality of B, which is at least as big as 2)
- For any B = {a, b, …, h} $\subseteq$ *A*, with *P* (a, B) $\neq$ 1, the following holds:

$$P\,(a, B) = F_\beta\,[\varphi(a), \varphi(b), \ldots, \varphi(h)]$$

**Definition 9** (p. 411):

**A closed structure of choice probabilities satisfies Order-independence iff:**

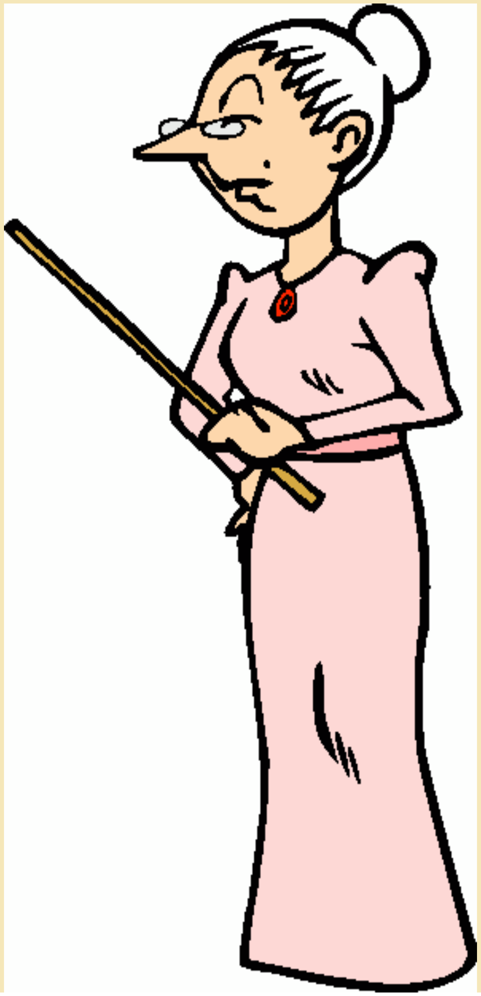
- For all a, b $\in$ B – C and c $\in$ C:

  **$P$ (a, B) ≥ $P$ (b, B) iff $P$ (c, C ∪ {a}) ≤ $P$ (c, C ∪ {b})**
- (So long as the choice probabilities on either side of the inequality are not both 0 and 1.)
- Let's say a = red ball, b = black ball, and B = urn of red, black, and yellow balls. C = the set of yellow balls
- **$P$ (a, B) ≥ $P$ (b, B)** means that there are at least as many red balls in the urn as there are black (maybe more).
- **$P$ (c, C ∪ {a}) ≤ $P$ (c, C ∪ {b})** says that the probability we choose a yellow ball given all the black balls and the yellow balls is at least as great as the probability of choosing a yellow given all the reds and yellows.
- This makes sense, since there are at least as many red balls than black ones.

**Definition 10** (p. 414): **A closed structure of choice probabilities satisfies the Strict-Utility Model iff:**

- There exists a positive real-valued function φ
- on $A$ such that for all $a \in B \subseteq A$:

$$P(a, B) = \varphi(a) \div \sum_{b \in B} \varphi(b)$$



**Definition 11** (p. 415):

**A closed structure of choice probabilities satisfies the Constant-Ratio Rule iff:**

- For all a, b $\in$ B $\subseteq$ *A*, the following holds:

  $P$ (a, b) ÷ $P$ (b, a) = $P$ (a, B) ÷ $P$ (b, B)

(assuming that the denominators don't vanish)

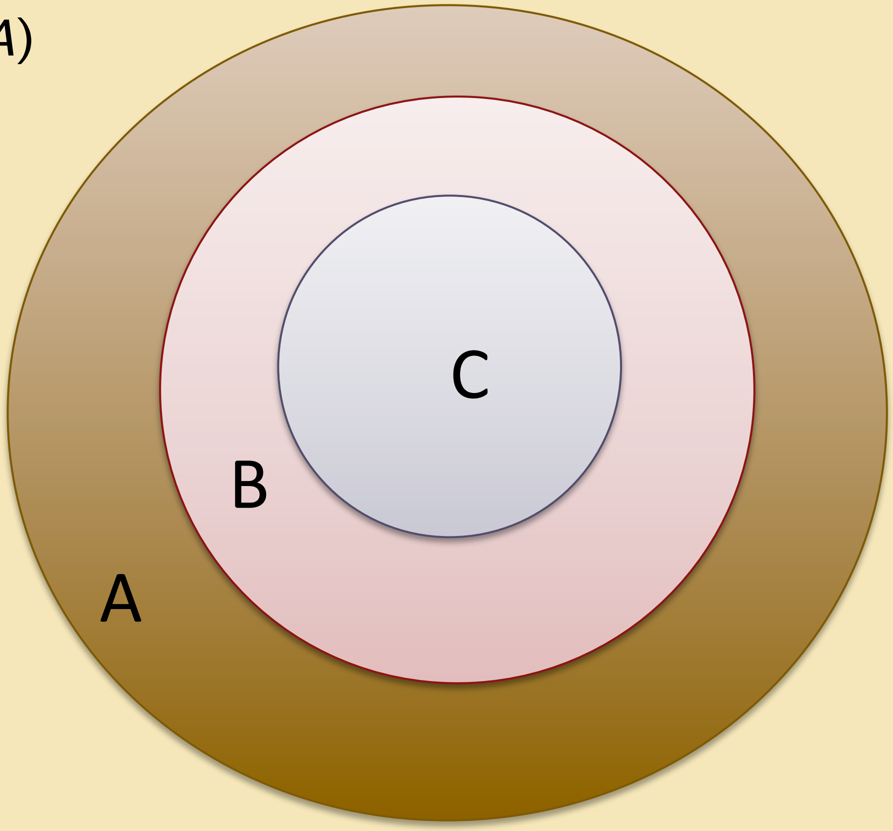
- Tells us that the strength of preference for a over b (the ratio) is unaffected by the other alternatives.
- This is a notch up from the independence of irrelevant alternatives, which only tells us that the ordering of probabilities is unaffected by the alternatives.

**Definition 12** (p. 416):

**A closed structure of choice probabilities satisfies the Choice Axiom iff, for all C ⊆ B ⊆ *A*:**

- $P$ (C, A) = $P$ (C, B) x $P$ (B, A)

(where $P$ (B, A) ≠ 0 for all B ⊆ *A*)



# Theorem 6 & 8 (p. 416 & 418)

- If we have a closed structure of choice probabilities where the probability of a single event is neither zero or one, and some other stuff is true,  then the strict utility model is satisfied if the constant ratio rule or the choice axiom holds.
- 8 gives necessary and sufficient conditions for getting a strict utility model of binary form.
- Gives us **the product rule**: a $\succ$ b $\succ$ c $\succ$ a/ a $\succ$ c $\succ$ b $\succ$ a

**Definition 13** (p. 421):

A closed structure of choice probabilities satisfies a Random Utility Model iff there exists a collection $U = \{U_a \mid a \in A\}$ of jointly distributed random variables, such that for all $a \in B$

$$P(a, B) = Pr\,(\mathbf{U}_a = \max\,\{\mathbf{U}_b \mid b \in B\})$$

**Definition 14** (p. 423): **A closed structure of choice probabilities satisfies Nonnegativity iff for any a ∈ A and $B_0$, $B_1$, …, $B_n$ ⊂ *A***

- The probability of any event, a, given a subset of B, $B_0$, subtracted by the probability of a, given all even combinations of subsets of B, plus the probability of a, given all odd combinations of subsets of B, is greater than or equal to zero.

## Cool Stuff about Nonnegativity…

- Nonnegativity is equivalent to the random-utility model whenever A contains 4 or fewer elements.
- Regularity says that the choice probability can't be increased by enlarging the offered set.
- When n=1, nonnegativity reduces to $P(a, B) \geq P(a, B \cup C)$
- Nonegativity is both necessary and sufficient for the representation of choice probabilities by a random-utility model.

Definition 19 (p. 436): An Elimination Structure is quadruple < A, M, P, Q > where < A, M, P > is a closed structure of choice probabilities, and $Q = \{Q_B \mid B \subseteq A\}$ is the **corresponding family of transition probability functions.** $Q_B \in \boldsymbol{Q}$ is a mapping from $2^B$ onto [0, 1] satisfying *i-iii* (in book).

- Given some set B, one selects a nonempty subset of B. Call this C. The probability with which one chooses C is $Q_B(C)$.
- We select a subset of C, call this D. The probability of choosing D is $Q_C(D)$, and we keep doing this over and over again until the subset eventually consists of a single alternative.
- *i* just says that will $Q_B(B)$ only ever equal one when we've gotten to a single alternative.
- *ii* says that when we sum up all of the C's and multiply them by the probability their probabilities given B, this equals one.
- *iii* says that *P* (a, B) is the absorbing probability of the Markov Chain. *P* (a, B) equals one times the probability of a, given $C_i$.

**<u>Definition 20</u> (p. 437): An Elimination Structure < A, M, P, Q > satisfies a Random-Elimination Model iff there exists a random vector U defined on *A* which satisfies that satisfies i and ii.**

- i says that the probability that utility of a = the utility of b ≠ 1, for any two elements in A.
- ii says that where c and d are elements of C and b is an element of B that is not in C, the probability of choosing subset C from B equals the probability that the utility of c equals the utility of d and that the utility of d and c are each greater than the utility of b.

*A random elimination model is **Boolean** iff the components of U are all 0 or 1.

**Definition 21** (p. 437-8): A closed structure of choice probabilities satisfies Proportionality iff there exists a family $Q = \{Q_B \mid B \subseteq A\}$ of functions such that (i) and (ii).

- (i) just says that our structure is an elimination structure.
- (ii) says that for all D, C ⊆ B ⊆ A, the ratio of the probability of choosing C to choosing D, given B is equal to the ratio of the probability of choosing C to D, given A, when we multiply the probability of C, given A, by the sum of the C's that intersect with B, and likewise, we multiply the probability of D given A by the sum of all the Ds that intersect with B.
- The only other conditions are that the denominations are positive and that if one denominator goes away, so do the other. This is an equality after all…

**Definition 22** (p. 440): **A closed structure of choice probabilities satisfies the Model of Elimination-by-Aspects iff there exists a positive-valued function, *f*, defined on A` - $A^0$, such that for all a ∈ B ⊆ A…. (see book)**

- The idea is that each alternative consists of a collection of aspects.
- There is a utility scale defined over all of these aspects.
- At each stage in the process, one selects an aspect with a probability proportional to its utility.
- Selecting this aspect eliminates all the alternatives that don't include it.
- This process continues until there is only a single alternative left.

The EBA model, the Boolean random-elimination model, and the proportionality condition are all equivalent! (Theorem 16).

# The End

**<u>Definition 6</u>** (P. 312): **Suppose $\succ$ and R are binary relations on A and $\succ$ is asymmetric. R is Upper Compatible with $\succ$ iff**

- For every a, b, c in *A*, aRb and b $\succ$ c imply a $\succ$ c

**<u>Definition 6</u>** (continued): **Suppose $\succ$ and R are binary relations on A and $\succ$ is asymmetric. R is Lower Compatible with $\succ$ iff**

- For every a, b, c in *A*, aRb and c $\succ$ a imply c $\succ$ b

* *R* is fully compatible with $\succ$ iff it is both upper and lower compatible with $\succ$

**<u>Definition 10</u>** (P. 333): **Suppose $\succ_1$ and $\succ_2$ are asymmetric relations on *A*. They satisfy upper- (and lower-) interval homogeneity iff**

- For all a, b, c, d in *A*, whenever a $\succ_1$ c and b $\succ_2$ d, then either **a $\succ_2$ d or b $\succ_1$ c** (a $\succ_1$ d or b $\succ_2$ c)

Definition 8 (P. 317): Suppose $< A, \succ >$ is a one-sided threshold representation. It is said to be Monotonic iff

- $\varphi + \delta$ is a strictly increasing function of $\varphi$ .

Definition 9 (P. 332): Suppose $\succ_1$, $\succ_2$ are asymmetric relations on $A$, and $\varphi$, $\delta_1$, $\delta_2$ are real-valued functions on $A$. $< \varphi, \delta_1, \delta_2 >$ is a Homogeneous, Upper Representation of $< A, \succ_1, \succ_2 >$ iff

- $< \varphi, \delta_i >$ is an upper representation of $< A, \succ_i >$, $i = 1, 2$.

- (Same deal for homogeneous lower and homogeneous two-sided representations.)

**Definition 12** (P. 338): **P satisfies Interval Stochastic Transitivity iff**

- max [*P* (a, d), *P* (b, c)] ≥ min [*P* (a, c), *P* (b, d)]