Chapter 5:
Probability Representations

Definition

Suppose that X is a nonempty set (sample space) and that $\mathcal{E}$ is a nonempty family of subsets of X. Then $\mathcal{E}$ is an algebra of sets on X iff, for every $A, B \in \mathcal{E}$:

1. $-A \in \mathcal{E}$.
2. $A \cup B \in \mathcal{E}$

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1. $-A \in \mathcal{E}$.
2. $A \cup B \in \mathcal{E}$

Furthermore, if $\mathcal{E}$ is closed under countable unions, the $\mathcal{E}$ is called a σ-algebra on X.

Kolmogorov Axioms

Definition

Suppose that X is a nonempty set, that that $\mathcal{E}$ is an algebra of sets on X, and that P is a function from $\mathcal{E}$ into the real numbers. The triple $\langle X, \mathcal{E}, P \rangle$ is a (finitely additive) probability space iff, for every $A, B \in \mathcal{E}$:

1. $P(A) \geq 0$.
2. $P(X) = 1$.
3. If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

Kolmogorov Axioms

Definition

It is a probability space $\langle X, \mathcal{E}, P\rangle$ is countably additive if in addition:

1. $\mathcal{E}$ is a σ-algebra on X.
2. If $A_i \in \mathcal{E}$ and $A_i \cap A_j = \emptyset$, $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

finite X + algebra	$\Rightarrow$	σ-algebra
finite X + probability space	$\Rightarrow$	countably additive probability space
$\langle X, \mathcal{E}, P \rangle$ measure space + $P(X) = 1$	$\Leftrightarrow$	$\langle X, \mathcal{E}, P \rangle$ countably additive probability space
Non-countably-additive probability space	$\Leftrightarrow$	infinite X + (non-σ) algebra

Necessary Conditions

Definition

Suppose that X is a nonempty set, that $\mathcal{E}$ is an algebra of sets on X, and that $\succsim$ is a relation on $\mathcal{E}$. The triple $\langle X, \mathcal{E}, \succsim \rangle$ is a structure of qualitative probability iff for every $A, B, C \in \mathcal{E}$:

1. $\langle \mathcal{E}, \succsim \rangle$ is a weak ordering.
2. $X \succ \emptyset$ and $A \succsim \emptyset$.
3. Suppose that $A \cap B = A \cap C = \emptyset$. Then $B \succsim C$ iff $A \cup B \succsim A \cup C$.

Necessary Conditions

Definition

Suppose $\mathcal{E}$ is an algebra of sets an that $\sim$ is an equivalence relation on $\mathcal{E}$. A sequence $A_1, \ldots, A_i, \ldots$, where $A_i \in \mathcal{E}$, is a standard sequence relative to $A \in \mathcal{E}$ iff there exist $B_i, C_i \in \mathcal{E}$ such that:

(i) $A_1 = B_1$ and $B1 \sim A$;

(ii) $B_i \cap C_i = \emptyset$;

(iii) $B_i \sim A_i$;

(iv) $C_i \sim A$;

(v) $A_{i+1} = B_i \cup C_i$.

Necessary Conditions

Definition

A structure of qualitative probability is Archimedean iff, for every $A \succ \emptyset$, any standard sequence relative to A is finite.

Nonsufficiency of Qualitative Probability

Let $X = \{a, b, c, d, e\}$ and let $\mathcal{E}$ be all subsets of X. Consider any order for which

$$\{a\} \succ \{b, c\}, \quad \{c, d\} \succ \{a, b\} \quad \text{and} \quad \{b, e\} \succ \{a, c\} \,. \tag{1}$$

Nonsufficiency of Qualitative Probability

Let $X = \{a, b, c, d, e\}$ and let $\mathcal{E}$ be all subsets of X. Consider any order for which

(1) $\{a\} \succ \{b, c\}, \quad \{c, d\} \succ \{a, b\} \quad \text{and} \quad \{b, e\} \succ \{a, c\}$.

Proposition

If the relation $\succsim$ on $\mathcal{E}$ satisfies (1) and has an order-preserving (finitely additive) probability representation, then

$$\{d, e\} \succ \{a, b, c\} \; .$$

Proposition

There is a relation $\succsim$ such that $\langle X, \mathcal{E}, \succsim \rangle$ is a structure of qualitative probability and $\{a, b, c\} \succ \{d, e\}$.

Nonsufficiency of Qualitative Probability

Lesson?

A probability representation has metrical structure that a (Archimedean) structure of qualitative probability does not.

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Recall that, to solve this sort of problem wrt extensive measurement, we had axiom (4) in Definition 3 of Chapter 3 (p. 84). Why not impose a similar axiom here?

Nonsufficiency of Qualitative Probability

Lesson?

A probability representation has metrical structure that a (Archimedean) structure of qualitative probability does not.

Recall that, to solve this sort of problem wrt extensive measurement, we had axiom (4) in Definition 3 of Chapter 3 (p. 84). Why not impose a similar axiom here?

What a great idea! Let's call it 'Axiom 5'.

Sufficient Conditions

Axiom 5

Suppose $\langle X, \mathcal{E}, \succsim \rangle$ is a structure of qualitative probability. If $A, B, C, D \in \mathcal{E}$ are such that $A \cap B = \emptyset$, $A \succ C$, and $B \succsim D$, then there exist $C', D', E \in \mathcal{E}$ such that:

(i) $E \sim A \cup B$;

(ii) $C' \cap D' = \emptyset$;

(iii) $E \supset C' \cup D'$;

(iv) $C' \sim C$ and $D' \sim D$.

Sufficient Condition

Proposition

If a finite structure of qualitative probability satisfies Axiom 5, then its equivalence classes form a single standard sequence.

Sufficient Condition

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If a finite structure of qualitative probability satisfies Axiom 5, then its equivalence classes form a single standard sequence.

Similar to “Lego blocks” in the case of extensive measurement.

Representation Theorem

Theorem 2

Suppose that $\langle X, \mathcal{E}, \succsim \rangle$ is an Archimedean structure of qualitative probability for which Axiom 5 holds, then there exists a unique order-preserving function P such that $\langle X, \mathcal{E}, P \rangle$ is a finitely additive probability space.

Countably Additive Representation

Countably Additive Representation

Definition

Suppose that $\langle X, \mathcal{E}, \succsim \rangle$ is a structure of qualitative probability and that $\mathcal{E}$ is a σ-algebra. We say that $\succsim$ is monotonically continuous on $\mathcal{E}$ iff for any sequence $A_1, A_2, \ldots$ in $\mathcal{E}$ and any $B \in \mathcal{E}$, if $A_i \subset A_{i+1}$ and $B \succsim A_i$, for all i, then $B \succsim \bigcup_{i=1}^{\infty} A_i$.

Countably Additive Representation

Definition

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Theorem 4

A finitely additive probability representation of a structure of qualitative probability, on a σ-algebra, is countably additive iff the structure is monotonically continuous.

Countably Additive Representation

Definition

Let $\succsim$ be a weak ordering of an algebra of sets $\mathcal{E}$. An even $A \in \mathcal{E}$ is an atom iff $A \succ \mathcal{E}$ and for any $B \in \mathcal{E}$, if $A \supset B$, then $A \sim B$ or $B \sim \emptyset$.

Countably Additive Representation

Definition

Let $\succsim$ be a weak ordering of an algebra of sets $\mathcal{E}$. An even $A \in \mathcal{E}$ is an atom iff $A \succ \mathcal{E}$ and for any $B \in \mathcal{E}$, if $A \supset B$, then $A \sim B$ or $B \sim \emptyset$.

Theorem 5

Suppose that $\langle X, \mathcal{E}, \succsim \rangle$ is a structure of qualitative probability, $\mathcal{E}$ is a σ-algebra, and there are no atoms. Then there is a unique order preserving probability representation, and it is countably additive.

QM-Algebra

QM-Algebra

Definition

Suppose that X is a nonempty set and that $\mathcal{E}$ is a nonempty family of subsets of X. Then $\mathcal{E}$ is a QM-algebra of sets on X iff, for every $A, B \in \mathcal{E}$

1. $-A \in \mathcal{E}$;
2. If $A \cap B = \emptyset$, then $A \cup B \in \mathcal{E}$.

Furthermore, if $\mathcal{E}$ is closed under countable unions of mutually disjoint sets, then $\mathcal{E}$ is called a QM σ-algebra.

QM-Algebra

Axiom 3′

Suppose that $A \cap B = C \cap D = \emptyset$. If $A \succsim C$ and $B \succsim D$, then $A \cup B \succsim C \cup D$; moreover, if either hypothesis is $\succ$, then the conclusion is $\succ$.

QM-Algebra

Axiom 3′

Suppose that $A \cap B = C \cap D = \emptyset$. If $A \succsim C$ and $B \succsim D$, then $A \cup B \succsim C \cup D$; moreover, if either hypothesis is $\succ$, then the conclusion is $\succ$.

Theorem 3

If $\mathcal{E}$ is a QM-algebra and if $\langle X, \mathcal{E}, \succsim \rangle$ satisfies Axioms 1, 2, 3′, 4, and 5, then there is a unique order-preserving (finitely additive) probability representation on $\mathcal{E}$.

Independent Events

Necessary Conditions

Definition

Suppose $\mathcal{E}$ is an algebra of sets on X and $\perp$ is a binary relation on $\mathcal{E}$. Then $\perp$ is an independence relation iff

1. $\perp$ is symmetric.
2. For $A \in \mathcal{E}$, $\{B \mid A \perp B\} \subset \mathcal{E}$ is a QM-algebra.

Independent Events

Necessary Conditions

Definition

Suppose $\mathcal{E}$ is an algebra of sets on X and $\perp$ is a binary relation on $\mathcal{E}$. Then $\perp$ is an independence relation iff

1. $\perp$ is symmetric.
2. For $A \in \mathcal{E}$, $\{B \,|\, A \perp B\} \subset \mathcal{E}$ is a QM-algebra.

Definition

Let $\mathcal{E}$ be an algebra of sets and $\perp$ an independence relation on $\mathcal{E}$. For $m \geq 2$, $A_1, \ldots, A_m \in \mathcal{E}$ are $\perp$-independent iff, for every $M \subset \{1, \ldots, m\}$, every B in the smallest subalgebra containing $\{A_i | i \in M\}$, and every C in the smallest subalgebra containing $\{A_i | i \notin M\}$, we have $B \perp C$.

Independent Events

Necessary Conditions

Definition

Suppose that $\langle X, \mathcal{E}, \succsim \rangle$ is a structure of qualitative probability and $\perp$ is an independence relation on $\mathcal{E}$. The quadruple $\langle X, \mathcal{E}, \succsim, \perp \rangle$ is a structure of qualitative probability with independence iff

3. Suppose that $A, B, C, D \in \mathcal{E}$, $A \perp B$, and $C \perp D$. If $A \succsim C$ and $B \succsim D$, then $A \cap B \succsim C \cap D$; moreover, if $A \succ C$, $B \succ D$, and $B \succ \emptyset$, then $A \cap B \succ C \cap D$.

Structural Condition

Definition

The structure $\langle X, \mathcal{E}, \succsim, \perp \rangle$ is complete iff the following additional axiom holds:

4. For any $A_1, \ldots, A_m, A \in \mathcal{E}$, there exists $A' \in \mathcal{E}$ with $A' \sim A$ and $A' \perp A_i$. Moreover, if $A_1, \ldots, A_m$ are $\perp$-independent, then A' can be chosen so that $A_1, \ldots, A_m, A'$ are also $\perp$-independent.

Conditional Probability

Definition

Suppose $\langle X, \mathcal{E}, \succsim, \perp \rangle$ is a structure of qualitative probability with independence. Let $\mathcal{N} = \{A \mid A \sim \emptyset\} \subset \mathcal{E}$. If $A, C \in \mathcal{E}$ and $B, D \in \mathcal{E} - \mathcal{N}$, define

$$A|B \succsim' C|D$$

iff there exist $A', B', C', D' \in \mathcal{E}$ with

$$A' \sim A \cap B, \quad B' \sim B, \quad C' \sim C \cap D, \quad D' \sim D;$$

$$A' \perp D' \quad \text{and} \quad C' \perp B';$$

and

$$A' \cap D' \succsim C' \cap B.'$$

Conditional Probability

Definition

The structure $\langle X, \mathcal{E}, \succsim, \perp \rangle$ is Archimedean iff every standard sequence is finite, where $\{A_i\}$ is a standard sequence iff for all i, $A_i \in \mathcal{E} - \mathcal{N}$, $A_{i+1} \supset A_i$, and

$$X|X \succ' A_i|A_{i+1} \sim' A_1|A_2 \ .$$

Conditional Probability

Axiom 8

If $A|B \succsim' C|D$, then there exists $C' \in \mathcal{E}$ such that $C \cap D \subset C'$ and $A|B \sim' C'|D$.

Conditional Probability

Axiom 8

If $A|B \succsim' C|D$, then there exists $C' \in \mathcal{E}$ such that $C \cap D \subset C'$ and $A|B \sim' C'|D$.

* Axiom 8 is somewhere in strength between Axiom 5 and Axiom 5$'$. In particular, it requires an infinite sample space.

Conditional Probability

Theorem 7

Suppose that $\langle X, \mathcal{E}, \succsim, \perp \rangle$ is an Archimedean and complete structure of qualitative probability with independence such that Axiom 8 is satisfied. Then there is a unique probability representation in which conditional probabilities preserve $\succsim'$.

Conditional Probability

Theorem 7

Suppose that $\langle X, \mathcal{E}, \succsim, \perp \rangle$ is an Archimedean and complete structure of qualitative probability with independence such that Axiom 8 is satisfied. Then there is a unique probability representation in which conditional probabilities preserve $\succsim'$.

* Axiom 8 is somewhere in strength between Axiom 5 and Axiom 5$'$. In particular, it requires an infinite sample space.

Chapter 6:
Additive Conjoint Measurement

Decomposable Structures

Definition

Let A_1, A_2 be nonempty sets, and let $\succsim$ be a weak ordering on $A_1 \times A_2$. The triple $\langle A_1, A_2, \succsim \rangle$ is decomposable if there are real valued functions $\phi_1 : A_1 \to \mathbb{R}$, $\phi_2 : A_2 \to \mathbb{R}$, and $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where F is 1-1 in each variable, such that, for all $a, b \in A_1$ and $p, q \in A_2$,

$$ap \succsim bq \quad \text{iff} \quad F[\phi_1(a), \phi_2(p)] \geq F[\phi_1(b), \phi_2(q)] \; .$$

Additive Independence

Definition

A decomposable structure $\langle A_1, A_2, \succsim \rangle$ is additively independent if, for all $a, b \in A_1$ and $p, q \in A_2$,

$$ap \succsim bq \quad \text{iff} \quad \phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q) \ .$$

Examples

Proposition

Suppose $\langle A_1, A_2, \succsim \rangle$ is a decomposable structure such that

$$ap \succsim bq \quad \text{iff} \quad \psi_1(a)\psi_2(p) \geq \psi_1(b)\psi_2(q) \ ,$$

for positive real-valued functions ψ_1, ψ_2. Then $\langle A_1, A_2, \succsim \rangle$ is additively independent.

Examples

Proposition

Suppose $\langle A_1, A_2, \succsim \rangle$ is a decomposable structure such that

$$ap \succsim bq \quad \text{iff} \quad \psi_1(a)\psi_2(p) \geq \psi_1(b)\psi_2(q) \ ,$$

for positive real-valued functions ψ_1, ψ_2. Then $\langle A_1, A_2, \succsim \rangle$ is additively independent.

$$ap \succsim bq \quad \text{iff} \quad \log \psi_1(a) + \log \psi_2(p) \geq \log \psi_1(b) + \log \psi_2(q)$$

Examples

Momentum

$$p = mv$$

$$m_1 v_1 \geq m_2 v_2 \quad \text{iff} \quad \log m_1 + \log v_1 \geq \log m_2 + \log v_2$$

Examples

Independent Random Variables

Suppose Y_1, Y_2 are random variables on the same probability space, and let $\sigma(Y_i)$ be the smallest σ-algebra for which Y_i is continuous. Define $\succsim$ on $\sigma(Y_1) \times \sigma(Y_2)$ by

$$ap \succsim bq \quad \text{iff} \quad Pr(a \cap p) \geq Pr(b \cap q) \ ,$$

for all $a, b \in \sigma(Y_1)$ and $p, q \in \sigma(Y_2)$.

Examples

Independent Random Variables

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$$ap \succsim bq \quad \text{iff} \quad Pr(a \cap p) \geq Pr(b \cap q) \ ,$$

for all $a, b \in \sigma(Y_1)$ and $p, q \in \sigma(Y_2)$.

Proposition

$\langle \sigma(Y_1), \sigma(Y_2), \succsim \rangle$ is additively independent if and only if X_1 and X_2 are independent.

Examples

Expected Utility

Suppose $\langle X, \mathcal{E}, Pr \rangle$ is a probability space and $\mathcal{A}$ is a set of commodities with associated utility function U. Define $\succsim$ on $\mathcal{E} \times \mathcal{A}$ by

$$ap \succsim bq \quad \text{iff} \quad Pr(a)U(p) \geq Pr(b)U(q) \; ,$$

for all $a, b \in \mathcal{E}$ and $p, q \in \mathcal{A}$.

Necessary Conditions

Independence (a.k.a. single cancelation)

Definition

A relation $\succsim$ on $A_1 \times A_2$ is independent iff, for all $a, b \in A_1$, $ap \succsim bp$ for some $p \in A_2$ implies that $aq \succsim bq$ for every $q \in A_2$; and, for all $p, q \in A_2$, $ap \succsim aq$ for some $a \in A_1$ implies that $bq \succsim bp$ for every $b \in A_1$.

Necessary Conditions

Independence (a.k.a. single cancelation)

Definition

A relation $\succsim$ on $A_1 \times A_2$ is independent iff, for all $a, b \in A_1$, $ap \succsim bp$ for some $p \in A_2$ implies that $aq \succsim bq$ for every $q \in A_2$; and, for all $p, q \in A_2$, $ap \succsim aq$ for some $a \in A_1$ implies that $bq \succsim bp$ for every $b \in A_1$.

* $\succsim$ is an independent relation if $\langle A_1, A_2, \succsim \rangle$ is additively independent.

Necessary Conditions

Independence (a.k.a. single cancelation)

Definition

Suppose that $\succsim$ is an independent relation on $A_1 \times A_2$.

(i) Define $\succsim_1$ on A_1: for $a, b \in A_1$, $a \succsim_1 b$ iff $ap \succsim bp$ for some $p \in A_2$; and

(ii) define $\succsim_2$ on A_2 similarly.

Necessary Conditions

Independence (a.k.a. single cancelation)

Lemma 1

If $\succsim$ is an independent weak ordering of $A_1 \times A_2$, then

(i) $\succsim_i$ is a weak ordering of A_i.

(ii) For $a, b \in A_1$ and $p, q \in A_2$, if $a \succsim_1 b$ and $p \succsim_2 q$, then $ap \succsim bq$.

(iii) If either antecedent inequality of (ii) is strict, so is the conclusion.

(iv) For $a, b \in A_1$ and $p, q \in A_2$, if $ap \sim bq$, then $a \succsim_1 b$ iff $q \succsim_2 p$.

Necessary Conditions

Double Cancelation

Definition

A relation $\succsim$ on $A_1 \times A_2$ satisfies double cancelation provided that, for every $a, b, f \in A_1$ and $p, q, x \in A_2$, if $ax \succsim fq$ and $fp \succsim bx$, then $ap \succsim bq$. The weaker condition in which $\succsim$ is replaced by $\sim$ is the Thomsen condition.

Necessary Conditions

Archimedean Axiom

Definition

Suppose $\succsim$ is an independent weak ordering of $A_1 \times A_2$. For any set N of consecutive integers (positive or negative, finite or infinite), a set $\{a_i \mid a_i \in A_1, i \in N\}$ is a standard sequence on component 1 iff there exists $p, q \in A_2$ such that not($p \sim_2 q$) and, for all $i, i+1 \in N$, $a_i p \sim a_{i+1} q$. A parallel definition holds for the second component.

Necessary Conditions

Archimedean Axiom

Definition

Suppose $\succsim$ is an independent weak ordering of $A_1 \times A_2$. For any set N of consecutive integers (positive or negative, finite or infinite), a set $\{a_i \,|\, a_i \in A_1, i \in N\}$ is a standard sequence on component 1 iff there exists $p, q \in A_2$ such that not$(p \sim_2 q)$ and, for all $i, i+1 \in N$, $a_i p \sim a_{i+1} q$. A parallel definition holds for the second component.

Definition

A standard sequence on component 1 $\{a_i \,|\, i \in N\}$ is strictly bounded iff there exist $a, b \in A_2$ such that, for all $i \in N$, $c \succ_1 a_i \succ_1 b$. A parallel definition holds for the second component.

Necessary Conditions

Archimedean Axiom

Definition

Suppose $\succsim$ is an independent weak ordering of $A_1 \times A_2$. $\langle A_1, A_2, \succsim \rangle$ is Archimedean iff every strictly bounded standard sequence (on either component) is finite.

Sufficient Condition

Solvability

Definition

A relation $\succsim$ on $A_1 \times A_2$ satisfies unrestricted solvabillity provided that, given three of $a, b \in A_1$ and $p, q \in A_2$, the fourth exists so that $ap \sim bq$.

Sufficient Condition

Solvability

Definition

A relation $\succsim$ on $A_1 \times A_2$ satisfies restricted solvabillity provided that:

(i) whenever there exist $a, \bar{b}, \underline{b} \in A_1$ and $p, q \in A_2$ for which $\bar{b}q \succsim ap \succsim \underline{b}q$, then there exists $b \in A_1$ such that $bq \sim ap$;

(ii) a similar condition holds on the second component.

Definition

Suppose that $\succsim$ is a relation on $A_1 \times A_2$. Component A_1 is essential iff there exist $a, b \in A_1$ and $p \in A_2$ such that not($ap \sim bp$). A similar definition holds for A_2.

Sufficient Condition

Definition

Suppose that $\succsim$ is a relation on $A_1 \times A_2$. Component A_1 is essential iff there exist $a, b \in A_1$ and $p \in A_2$ such that not($ap \sim bp$). A similar definition holds for A_2.

Lemma 2

Suppose that $\succsim$ is an independent relation on $A_1 \times A_2$. Then component A_1 is essential iff there exist $a, b \in A_1$ such that $a \succ_1 b$.

Additive Conjoint Structure

Definition

Suppose that A_1 and A_2 are nonempty sets and $\succsim$ is a binary relation on $A_1 \times A_2$. The triple $\langle A_1, A_2, \succsim \rangle$ is an additive conjoint structure iff $\succsim$ satisfies the following six axioms:

1. Weak ordering
2. Independence
3. Thomsen condition
4. Restricted solvability
5. Archemedean property
6. Each component is essential

   The structure is symmetric iff, in addition,

7. For $a, b \in A_1$, there exist $p, q \in A_2$ such that $ap \sim bq$, and for $p', q' \in A_2$, there exist $a', b' \in A_1$ such that $a'p' \sim b'q'$.

Additive Conjoint Structure

Theorem 1

Suppose $\langle A_1, A_2, \succsim \rangle$ is a structure for which the weak ordering, double cancellation, unrestricted solvability, and the Archimedean axioms hold. If at least one component is essential, then $\langle A_1, A_2, \succsim \rangle$ is a symmetric, additive conjoint structure.

Representation Theorem

Theorem 2

Suppose $\langle A_1, A_2, \succsim \rangle$ is an additive conjoint structure. Then there exist functions $\phi_i : A_i \to \mathbb{R}$ such that, for all $a, b \in A_1$ and $p, q \in A_2$,

$$ap \succsim bq \quad \text{iff} \quad \phi_1(a) + \phi_2(p) \geq \phi_1(b) + \phi_2(q) \ .$$

If ϕ_i' are two other functions with the same property, then there exists constants $\alpha > 0$, β_1 and β_2 such that

$$\phi_1' = \alpha\phi_1 + \beta_1 \quad \text{and} \quad \phi_2' = \alpha\phi_2 + \beta_2 \ .$$

Representation Theorem

Uniqueness of multiplicative representation

Proposition

Suppose $\langle A_1, A_2, \succsim \rangle$ is an additive conjoint structure. Then there exist functions $\psi_i : A_i \to \mathbb{R}^+$ such that, for all $a, b \in A_1$ and $p, q \in A_2$,

$$ap \succsim bq \quad \text{iff} \quad \psi_1(a)\psi_2(p) \geq \psi_1(b)\psi_2(q) \ .$$

If ψ_i' are two other functions with the same property, then there exists constants $\alpha > 0$, β_1 and β_2 such that

$$\phi_1' = \beta_1 \psi_1^\alpha \quad \text{and} \quad \psi_2' = \beta_2 \psi_2^\alpha \ .$$

Extensive Structure

Definition

Suppose $\langle A_1, A_2, \succsim \rangle$ is a symmetric, additive conjoint structure. It is bounded iff there are $\underline{a}, \bar{a} \in A_1$, $\underline{p}, \bar{p} \in A_2$ such that

$$\underline{a}\bar{p} \sim \bar{a}\underline{p}$$

and, for $a \in A_1$ and $p \in A_2$,

$$\bar{a} \succsim_1 a \succsim_1 \underline{a} \quad \text{and} \quad \bar{p} \succsim_2 p \succsim_2 \underline{p} \, .$$

Extensive Structures

Extensive Structures

Moreover, for $a, b \in A_1$, we define: $\pi(a) \in A_2$ as the (unique up to $\sim_2$) solution to $\underline{a}\pi(a) \sim a\underline{p}$;
$B_1 = \{ab \,|\, a, b \succ_1 \underline{a}$ and $\bar{a}\underline{p} \succsim a\pi(b)\}$; for $ab \in B_1$, $a \circ b$ is the (unique up to $\sim_1$) solution to $(a \circ b)\underline{p} \sim a\pi(b)$. Similar definitions hold for A_2 with $\alpha(p)$ playing the role of $\pi(a)$.

Lemma 5

If $\langle A_1, A_2, \succsim\rangle$ is a bounded, symmetric, additive conjoint structure, and if B_1 is nonempty, then $\langle A_1, \succsim_1, B_1, \circ\rangle$ is an extensive structure with no essential maximum.

Subtractive Structures

Subtractive Structures

Define the dual relations $\succsim'$ and $\succsim'$ as follows:

$$ap \succsim bq \quad \text{iff} \quad aq \succsim' bp \,.$$

Theorem 5

If two relations are dual, then transitivity and double cancellation are dual properties, and independence, restricted and unrestricted solvability, and the Archemedean property are self-dual properties.