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Minimal Conditions for Additive Conjoint Measurement and Qualitative Probability[1]

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Axioms for additive conjoint measurement and qualitative probability are given. Representation theorems and uniqueness theorems are proved for structures that satisfy these axioms. Both Archimedean and nonarchimedean cases are considered. Approximations of infinite structures by sequences of finite structures are also considered.

INTRODUCTION

At the present time, there is one set of techniques for proving representation theorems for finite measurement structures an another set for infinite structures. Techniques for finite structures were developed in Scott (1964) and basically consist of solving finite sets of inequalities; techniques for infinite structures in one way or another resemble those used in Hölder (1901) and consist of the construction of fundamental sequences. Although finite structures often admit good axiomatizations in the sense that necessary and sufficient conditions for their representations can be given, they do not admit good uniqueness results. Infinite structures, however, often have uniqueness results for their representations but assume structural (nonnecessary) conditions in their axiomatizations. In this paper, new techniques are developed which allow infinite structures to be represented in terms of their finite substructures and thus simultaneously achieve good axiomatizations and representation theorems. These new techniques use the compactness theorem of mathematical logic in a way similar to Abraham Robinson's use in his *Nonstandard Analysis* (Robinson, 1966). However, to avoid the introduction of a large amount of mathematical logic into this paper, algebraic constructions are given for the various uses of the compactness theorem. This makes the paper relatively self-contained. These new techniques also allow a bridge to be built from finite to infinite structures. Thus, in Section 7 it is shown that certain infinite structures with unique representations are limits of sequences of finite structures. In terms of representations this means that as more elements are included into the qualitative structure the more "unique" the representation becomes. These new techniques also avoid the use of Archimedean axioms.

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1. Preliminaries

The following definitions, notations, and conventions will be observed throughout this paper.

$A = \mathsf{X}_{i=1}^{n} A_i$ will mean that A is the Cartesian product of the sets $A_1, \ldots, A_n$. $\mathsf{X}_{i=1}^{n} A_i$ is sometimes written as $A_1 \times \cdots \times A_n$. By convention it is assumed that $n \geqslant 2$ and for each $i \leqslant n$, A_i is nonempty. Members of $\mathsf{X}_{i=1}^{n} A_i$ are often written as $a_1 \cdots a_n$ where it is understood that for each $i \leqslant n$, $a_i \in A_i$. a_i is called the ith *coordinate* of $a_1 \cdots a_n$.

Let $\gtrsim$ be a binary relation on A. By convention, it is assumed that $\gtrsim$ is nonempty. (Thus $A \neq \varnothing$.) By definition,

(1) $x \sim y$ if and only if $x \gtrsim y$ and $y \gtrsim x$,
(2) $x \succ y$ if and only if $x \gtrsim y$ and not $x \sim y$,
(3) $y \lesssim x$ if and only if $x \gtrsim y$,
(4) $y \prec x$ if and only if $x \succ y$.

A statement of the form $u \gtrsim v$, $u \succ v$, or $u \sim v$ is called an *inequality*, and u is called *the left side* (of the inequality) and v is called *the right side*. If the inequality is of the form $u \succ v$ it is called a *strict inequality*. If the inequality is of the form $u \sim v$, it is called an *equivalence*.

By convention, *if* $A = \mathsf{X}_{i=1}^{n} A_i$ *and* $\gtrsim$ *is a binary relation on* A, *then it is assumed that for each* $i \leqslant n$ *there are* $a_1 \cdots a_n$ *in* A *and* $x, y \in A_i$ *such* that $a_1 \cdots a_{i-1} x a_{i+1} \cdots a_n \succ a_1 \cdots a_{i-1} y a_{i+1} \cdots a_n$.

Let $\gtrsim$ be a binary relation on A. By definition,

(1) $\gtrsim$ is *reflexive* if and only if for each $x \in A$, $x \sim x$,

(2) $\gtrsim$ is a *weak order* if and only if (i) $\gtrsim$ is reflexive, (ii) (*transitivity*) for each x, y, z in A, if $x \gtrsim y$ and $y \gtrsim z$, then $x \gtrsim z$, and (iii) (*connectivity*) for each $x, y \in A$, either $x \gtrsim y$ or $y \gtrsim x$.

(x, y), $\langle x, y \rangle$ will denote ordered pairs, (x, y, z), $\langle x, y, z \rangle$ will denote ordered triplets, etc. If $R(x_1, \ldots, x_n)$ is a relation on A and B is a set, then, by definition, the restriction of R to B, $R \upharpoonright B$ is $\{(b_1, \ldots, b_n) \mid b_1, \ldots, b_n \in B \text{ and } R(b_1, \ldots, b_n)\}$.

Re will denote the set of real numbers, I the set of integers, Re^+ the set of positive real numbers, and I^+ the set of positive integers. *The real number field* is the ordered 4-tuple $\langle \mathrm{Re}, +, \cdot, \geqslant \rangle$. $\langle \mathrm{Re}, +, \cdot, \geqslant \rangle$ is sometimes called *the reals*. $\langle F, \oplus, \odot, \gtrsim \rangle$ is said to be a *weakly ordered field* if and only if the following conditions hold:

(1) the elements 0 and 1 are in F and not $0 \sim 1$,
(2) $\gtrsim$ is a weak order on F,
(3) for all $x, y, z, w \in F$, $x \oplus y \sim y \oplus x$, $x \oplus (y \oplus z) \sim (x \oplus y) \oplus z$, $x \oplus 0 \sim x$, and if $x \gtrsim y$ and $z \gtrsim w$, then $x \oplus z \gtrsim y \oplus w$,

(4) for each $x \in F$ there is a $y \in F$ such that $x \oplus y \sim 0$,

(5) for all $x, y, z, w \in F$, $x \odot y \sim y \odot x$, $x \odot (y \odot z) \sim (x \odot y) \odot z$, $x \odot 1 = x$, if $x \succsim 0$ and $y \succsim 0$, then $x \odot y \succsim 0$,

(6) for each $x \in F$, if not $x \sim 0$ then there is a $y \in F$ such that $x \odot y \sim 1$,

(7) for each x, y, z in F, $x \odot (y \oplus z) \sim (x \odot y) \oplus (x \odot z)$.

By definition, $\langle F, \oplus, \odot, \succsim\rangle$ is said to be a *weakly ordered field extension of the reals* if and only if $\langle F, \oplus, \odot, \succsim\rangle$ is a weakly ordered field such that $F \supseteq \mathrm{Re}$, $\oplus \supseteq +$, $\odot \supseteq \cdot$, and $\succsim \supseteq \geqslant$. For notational simplicity, weakly ordered field extensions of the reals will often be written as $\langle {}^*\mathrm{Re}, +, \cdot, \succsim\rangle$ where it is understood that $+, \cdot$ are extensions of the addition and multiplication operations of the reals, etc. $^*\mathrm{Re}^+$ will denote $\{x \in {}^*\mathrm{Re} \mid x \succ 0\}$.

Let X be a set. If $A \subseteq X$ then $X - A$ is said to be *the complement of A (relative to X)*. We often write $A^\sim$ for the complement of A when it is clear from the context that this complement is relative to X. $\mathscr{E}$ is said to be an algebra of subsets of X if and only if (i) X is a nonempty set and each member of $\mathscr{E}$ is a subset of X, (ii) $X \in \mathscr{E}$ and $\phi \in \mathscr{E}$, and (iii) if $x, y \in \mathscr{E}$ then $x^\sim \in \mathscr{E}$ and $x \cup y \in \mathscr{E}$. Let $\mathscr{F}$ be a nonempty family of subsets of X. A is said to be a *maximal element of $\mathscr{F}$* if and only if for each $B \in \mathscr{F}$, if $B \supseteq A$ then $B = A$. $\mathscr{C}$ is said to be a *chain* in $\mathscr{F}$ if and only if $\mathscr{C} \subseteq \mathscr{F}$ and for each $A, B \in \mathscr{C}$, either $A \subseteq B$ or $B \subseteq A$. By a fundamental theorem of set theory miscalled Zorn's lemma, if $\mathscr{F}$ is such that for each chain $\mathscr{C}$ in $\mathscr{F}$, $\bigcup\mathscr{C} \in \mathscr{F}$, then for some $A \in \mathscr{F}$, A is a maximal element of $\mathscr{F}$.

2. Additive Conjoint Structures

Definition 2.1. $\langle A, \succsim\rangle$ is said to be an *additive conjoint structure* if and only if the following three conditions hold:

(1) for some $n \in \mathrm{I}^+$, $A = \mathsf{X}_{i=1}^n A_n$;

(2) $\succsim$ is a reflexive relation on A; and

(3) for $i = 1,..., n$, there are functions Φ_i from A_i into Re such that for each $a_1 \cdots a_n$ and each $b_1 \cdots b_n$ in A the following two properties hold:

(i) if $a_1 \cdots a_n \succ b_1 \cdots b_n$ then $\Phi_1(a_1) + \cdots + \Phi_n(a_n) > \Phi_1(b_1) + \cdots + \Phi_n(b_n)$,

and

(ii) if $a_1 \cdots a_n \sim b_1 \cdots b_n$ then $\Phi_1(a_1) + \cdots + \Phi_n(a_n) = \Phi_1(b_1) + \cdots + \Phi_n(b_n)$.

The functions $\Phi_1, ..., \Phi_n$ that satisfy condition (3) are called a *set of strict representation functions for $\langle A, \succsim\rangle$*.

Definition 2.2. Let $A = \mathsf{X}_{i=1}^n A_i$ and $\succsim$ be a binary relation on A. Then $\langle A, \succsim\rangle$ is said to be *independent* if and only if for each $i \leqslant n$ and each $x_i, y_i \in A_i$, if for some a_j, $j \neq i$, $a_1 \cdots a_{i-1}x_ia_{i+1} \cdots a_n \succsim a_1 \cdots a_{i-1}y_ia_{i+1} \cdots a_n$ then for each $b_j \in A_j$, $j \neq i$, $b_1 \cdots b_{i-1}x_ib_{i+1} \cdots b_n \succsim b_1 \cdots b_{i-1}y_ib_{i+1} \cdots b_n$.

THEOREM 2.1. *If $\langle A, \succsim \rangle$ is an additive conjoint structure and $\succsim$ is a weak order then $\langle A, \succsim \rangle$ is independent.*

Proof. Left to reader.

DEFINITION 2.3. Let $A = \times_{i=1}^n A_i$ and $\succsim$ be a reflexive relation on A. Then for each i, let $\succsim_i$ be the relation defined on A_i as follows: $x \succsim_i y$ if and only if for some $a_j \in A_j$, $j \neq i$, $a_1 \cdots a_{i-1}xa_{i+1} \cdots a_n \succsim a_1 \cdots a_{i-1}ya_{i+1} \cdots a_n$.

DEFINITION 2.4. Let $A = \times_{i=1}^n A_i$ and $\succsim$ be a weak ordering on A and $\langle A, \succsim \rangle$ be independent. Then $\langle A, \succsim \rangle$ is said to be *Archimedean* if and only if it is not the case that there are

(1) $i, j \leqslant n$ such that $i \neq j$ and
(2) $a, b \in A_j$ such that $b \succ_j a$ and
(3) for $p \neq i, j$, $c_p \in A_p$ and
(4) $x, x^1, x^2, \ldots$ in A_i such that either (i) for each positive integer k, $x \succ_i x^{k+1} \succ_i x^k$ and $c_1 \cdots x^{k+1} \cdots a \cdots c_n \succsim c_1 \cdots x^k \cdots b \cdots c_n$, or (ii) for each positive integer k, $x^k \succ_i x^{k+1} \succ_i x$ and $c_1 \cdots x^k \cdots a \cdots c_n \succsim c_1 \cdots x^{k+1} \cdots b \cdots c_n$.

THEOREM 2.2. *If $\langle A, \succsim \rangle$ is an additive conjoint structure and $\succsim$ is a weak order then $\langle A, \succsim \rangle$ is Archimedean.*

Proof. Suppose not. Let $A = \times_{i=1}^n A_i$. Since by Theorem 2.1 $\langle A, \succsim \rangle$ is independent, $\succsim_i$ is defined on A_i for each $i \leqslant n$. Let $i, j \leqslant n$, $i \neq j$, $a, b \in A_j$, $b \succ_j a$, $c_k \in A_k$ for $k \neq i, j$, and $x, x^1, x^2, \ldots$ be in A_i, and $x \succ_i x^{p+1} \succ_i x^p$ (the case where $x^p \succ_i x^{p+1} \succ_i x$ will follow similarly), and for each positive integer p,

$$c_1 \cdots a \cdots c_k \cdots x^{p+1} \cdots c_n \succsim c_1 \cdots b \cdots c_k \cdots x^p \cdots c_n .$$

Let $\Phi_1, \ldots, \Phi_n$ be a set of strict representation functions for $\langle A, \succsim \rangle$. Then for each positive integer p,

$$(1) \quad \begin{aligned} &\Phi_1(c_1) + \cdots + \Phi_j(a) + \cdots + \Phi_k(c_k) + \cdots + \Phi_i(x^{p+1}) + \cdots + \Phi_n(c_n) \\ &\geqslant \Phi_1(c_1) + \cdots + \Phi_j(b) + \cdots + \Phi_k(c_k) + \cdots + \Phi_i(x^p) + \cdots + \Phi_n(c_n). \end{aligned}$$

By subtraction we get

$$(2) \quad \Phi_j(a) + \Phi_i(x^{p+1}) \geqslant \Phi_j(b) + \Phi_i(x^p).$$

Let $r_p = \Phi_i(x^p)$ and $s = \Phi_j(b) - \Phi_j(a)$. Since $b \succ_j a$, $s > 0$. From (2) it follows that $r_{p+1} - r_p \geqslant s$ for each positive integer p. Since $x \succ_i x^p$, $\Phi_i(x) > r_p$ for each positive integer p. This violates the Archimedean axiom for the real number system.

DEFINITION 2.5. Let $A = \mathsf{X}_{i=1}^{n} A_i$ and $\gtrsim$ be a binary relation on A. $\langle B, \gtrsim' \rangle$ is said to be a *finite substructure of* $\langle A, \gtrsim \rangle$ if and only if for $i = 1,..., n$, B_i is a finite subset of A_i, $B = \mathsf{X}_{i=1}^{n} B_i$, and $\gtrsim'$ is the restriction of $\gtrsim$ to B. $\langle A, \gtrsim \rangle$ is said to have *the finiteness property for additive conjoint structures* if and only if each finite substructure of $\langle A, \gtrsim \rangle$ is an additive conjoint structure.

THEOREM 2.3. *If* $\langle A, \gtrsim \rangle$ *is an additive conjoint structure then* $\langle A, \gtrsim \rangle$ *has the finiteness property for additive conjoint structures.*

Proof. Obvious.

DEFINITION 2.6. Let $A = \mathsf{X}_{i=1}^{n} A_i$ and $\gtrsim$ be a reflexive relation on A. Let Γ be a set of equivalences of members of A, or strict inequalities of members of A. That is, each $\gamma \in \Gamma$ has the form $a_1 \cdots a_n \sim b_1 \cdots b_n$, or the form $a_1 \cdots a_n \succ b_1 \cdots b_n$. For each $i \leqslant n$ and each $x \in A_i$ define Γ_x^{1i} and Γ_x^{ri} as follows: Γ_x^{1i} is the number of γ in Γ such that x occurs in the ith coordinate of the left side of γ, and Γ_x^{ri} is the number of γ in Γ such that x occurs in the ith coordinate of the right side of γ. $\langle A, \gtrsim \rangle$ is said *to satisfy the* kth *cancellation axiom* if and only if for each set Γ which consists of equivalences of members of A or strict inequalities of members of A, if Γ has at most k elements and for each $i \leqslant n$ and each $x \in A_i$, $\Gamma_x^{1i} = \Gamma_x^{ri}$, then each member of Γ is an equivalence of members of A. $\langle A, \gtrsim \rangle$ is said *to satisfy the finite cancellation axioms* if and only if $\langle A, \gtrsim \rangle$ satisfies the kth cancellation axiom for each $k \in \mathrm{I}^+$.

THEOREM 2.4. *If* $\langle A, \gtrsim \rangle$ *is an additive conjoint structure then* $\langle A, \gtrsim \rangle$ *satisfies the* kth *cancellation axiom for each* $k \in \mathrm{I}^+$.

Proof. Let $A = \mathsf{X}_{i=1}^{n} A_i$, $\Phi_1, ..., \Phi_n$ be a set of strict representation functions for $\langle A, \gtrsim \rangle$ and Γ be a set of equivalences and strict inequalities on A. Suppose that Γ has at most k elements where $k \in \mathrm{I}^+$ and for each $i \leqslant n$ and each $x \in A_i$, $\Gamma_x^{1i} = \Gamma_x^{ri}$. Let γ be in Γ. Then γ is $a_1 \cdots a_n \succ b_1 \cdots b_n$ or γ is $a_1 \cdots a_n \sim b_1 \cdots b_n$. In either case let $1(\gamma) = \Phi_1(a_1) + \cdots + \Phi_n(a_n)$ and $r(\gamma) = \Phi_1(b_1) + \cdots + \Phi_n(b_n)$. Then if γ is a strict inequality, $1(\gamma) > r(\gamma)$ and if γ is an equivalence $1(\gamma) = r(\gamma)$. Therefore,

$$\sum_{\gamma \in \Gamma} 1(\gamma) = \sum_{i \leqslant n} \sum_{x \in A_i} \Gamma_x^{1i} \Phi(x) \qquad \text{and} \qquad \sum_{\gamma \in \Gamma} r(\gamma) = \sum_{i \leqslant n} \sum_{x \in A_i} \Gamma_x^{ri} \Phi_i(x).$$

Since $\Gamma_x^{1i} = \Gamma_x^{ri}$, $\sum_{\gamma \in \Gamma} 1(\gamma) = \sum_{\gamma \in \Gamma} r(\gamma)$. This can only happen if for all $\gamma \in \Gamma$, $1(\gamma) = r(\gamma)$. That is, each γ is an equivalence. Thus $\langle A, \gtrsim \rangle$ satisfies the cancellation axiom.

THEOREM 2.5. *Let* A *be a finite set and* $\gtrsim$ *be a reflexive relation on* A. *If* $\langle A, \gtrsim \rangle$ *satisfies the finite cancellation axioms then* $\langle A, \gtrsim \rangle$ *is an additive conjoint structure.*

Proof. Chapter 9 of Krantz, Luce, Suppes, and Tversky [1971].

THEOREM 2.6. *Let $\succsim$ be a reflexive relation on A. If $\langle A, \succsim\rangle$ satisfies the finite cancellation axioms then $\langle A, \succsim\rangle$ satisfies the finiteness condition for additive conjoint structures.*

Proof. Immediate from Theorem 2.5.

THEOREM 2.7. *Let $\succsim$ be a weak ordering on A. If $\langle A, \succsim\rangle$ satisfies the second cancellation axiom then $\langle A, \succsim\rangle$ is independent.*

Proof. Left to reader.

3. QUALITATIVE PROBABILITY

DEFINITION 3.1. $\langle X, \mathscr{E}, \succsim\rangle$ is said to be a *qualitative probability structure* if and only if X is a nonempty set, $\mathscr{E}$ is an algebra of subsets of X, $\succsim$ is a reflexive relation on $\mathscr{E}$, and there is a function P from $\mathscr{E}$ into [0, 1] such that the following four conditions hold for all A, B in $\mathscr{E}$:

(1) $P(X) = 1$, $P(\varnothing) = 0$;
(2) if $A \succ B$ then $P(A) > P(B)$;
(3) if $A \sim B$ then $P(A) = P(B)$; and
(4) if $A \cap B = \varnothing$ then $P(A \cup B) = P(A) + P(B)$.

The above function P is called a *probability representation for* $\langle X, \mathscr{E}, \succsim\rangle$.

DEFINITION 3.2. Let X be a nonempty set, $\mathscr{E}$ an algebra of subsets of X, and $\succsim$ a binary relation on A. Then $\langle X, \mathscr{E}, \succsim\rangle$ is said to be *Archimedean* if and only if for each A, A_1, A_2,... in $\mathscr{E}$, it is not the case that

(1) $A \succ \varnothing$,
(2) $A_i \succ A$ for each $i \in \mathrm{I}^+$, and
(3) $A_i \cap A_j = \varnothing$ for each $i, j \in \mathrm{I}^+$ such that

THEOREM 3.1. *If $\langle X, \mathscr{E}, \succsim\rangle$ is a qualitative probability structure, then $\langle X, \mathscr{E}, \succsim\rangle$ is Archimedean.*

Proof. Left to reader.

DEFINITION 3.3. $\langle X, \mathscr{E}, \succsim\rangle$ is said *to satisfy the finiteness property for qualitative probability* if and only if X is a nonempty set, $\mathscr{E}$ is an algebra of subsets of X, $\succsim$ is a binary relation on $\mathscr{E}$. and for each finite subalgebra $\mathscr{E}'$ of $\mathscr{E}$, $\langle X, \mathscr{E}', \succsim'\rangle$ is a qualitative probability structure where $\succsim'$ is the restriction of $\succsim$ to $\mathscr{E}'$.

THEOREM 3.2. *If $\langle X, \mathscr{E}, \succsim\rangle$ is a qualitative probability structure then $\langle X, \mathscr{E}, \succsim\rangle$ satisfies the finiteness property for qualitative probability.*

Proof. Immediate from Definition 3.3.

DEFINITION 3.4. Let X be a nonempty set, $\mathscr{E}$ an algebra of subsets of X, and $\succsim$ a reflexive relation on $\mathscr{E}$. Let Γ be a set of equivalences or strict inequalities of members of $\mathscr{E}$. That is, each $\gamma \in \Gamma$ is of the form $A \succ B$ or $A \sim B$. For each $x \in X$, define Γ_x^{1} and Γ_x^{r} as follows: Γ_x^{1} is the number of γ in Γ such that x is a member of the left side of γ, and Γ_x^{r} is the number of γ in Γ such that x is a member of the right side of γ. $\langle X, \mathscr{E}, \succsim\rangle$ is said *to satisfy the* kth *cancellation axiom* if and only if for each Γ which is a set of equivalences or strict inequalities of members of $\mathscr{E}$, if Γ has at most k elements and for each $x \in X$, $\Gamma_x^{1} = \Gamma_x^{r}$, then each member of Γ is an equivalence of members of $\mathscr{E}$. $\langle X, \mathscr{E}, \succsim\rangle$ is said *to satisfy the finite cancellation axioms* if and only if $\langle X, \mathscr{E}, \succsim\rangle$ satisfies the kth cancelaltion axiom for each $k \in \mathrm{I}^{+}$.

THEOREM 3.3. *If $\langle X, \mathscr{E}, \succsim\rangle$ is a qualitative probability structure then $\langle X, \mathscr{E}, \succsim\rangle$ satisfies the* kth *cancellation axiom for each $k \in \mathrm{I}^{+}$.*

Proof. Left to reader.

THEOREM 3.4. *Let X be a nonempty set, $\mathscr{E}$ a finite algebra of subsets of X, and $\succsim$ a reflexive relation on $\mathscr{E}$. If $\langle X, \mathscr{E}, \succsim\rangle$ satisfies the finite cancellation axioms then $\langle X, \mathscr{E}, \succsim\rangle$ is a qualitative probability structure.*

Proof. Chapter 9 of Krantz *et al.* [1971].

THEOREM 3.5. *Let X be a nonempty set, $\mathscr{E}$ an algebra of subsets of X, and $\succsim$ a reflexive relation on $\mathscr{E}$. If $\langle X, \mathscr{E}, \succsim\rangle$ satisfies the finite cancellation axioms then $\langle X, \mathscr{E}, \succsim\rangle$ satisfies the infiniteness condition for qualitative probability structures.*

Proof. Immediate from Theorem 3.4.

4. ULTRAPRODUCTS

DEFINITION 4.1. Let $\mathscr{F}$ be a nonempty collection of sets. $\mathscr{F}$ is said to have *the finite intersection property* if and only if for each finite nonempty subset $\mathscr{G}$ of $\mathscr{F}$, $\bigcap \mathscr{G} \neq \varnothing$.

DEFINITION 4.2. $\mathscr{U}$ is said to be an *ultrafilter on* X if and only if the following six conditions hold for all subsets A, B of X:

(1) X is a nonempty set;
(2) $\mathscr{U}$ is a collection of subsets of X;
(3) $X \in \mathscr{U}$ and $\varnothing \notin \mathscr{U}$;
(4) if $A \in \mathscr{U}$ and $B \supseteq A$ then $B \in \mathscr{U}$;
(5) if $A \in \mathscr{U}$ and $B \in \mathscr{U}$ then $A \cap B \in \mathscr{U}$; and
(6) either $A \in \mathscr{U}$ or $A^{\sim} \in \mathscr{U}$.

THEOREM 4.1. *If X is a nonempty set and $\mathscr{F}$ is a collection of subsets of X that has the finite intersection property, then there is an ultrafilter $\mathscr{U}$ on X such that $\mathscr{U} \supseteq \mathscr{F}$.*

Proof. Let Γ be defined as follows: $\mathscr{G} \in \Gamma$ if and only if (i) $\mathscr{G}$ is a family of subsets of X, (ii) $\mathscr{G} \supseteq \mathscr{F}$, and (iii) $\mathscr{G}$ has the finite intersection property. We will show by Zorn's lemma that Γ has a maximal element. Let Δ be a chain in Γ. Let $\mathscr{W} = \bigcup \Delta$. Then it is easy to show that $\mathscr{W}$ is in Γ. Thus by Zorn's lemma, Γ has a maximal element $\mathscr{U}$. Suppose that $A \subseteq X$ and $A \notin \mathscr{U}$. Then, since $\mathscr{U}$ is a maximal element of Γ, $\mathscr{U} \cup \{A\}$ does not have the finite intersection property. Thus for some $A_1, \ldots, A_n$ in $\mathscr{U}$, $A_1 \cap \cdots \cap A_n \cap A = \varnothing$. Therefore $A^{\sim} \supseteq A_1 \cap \cdots \cap A_n$. $\mathscr{U} \cup \{A^{\sim}\}$ has the finite intersection property since if for some $B_1, \ldots, B_m$ in $\mathscr{U}$, $B_1 \cap \cdots \cap B_m \cap A^{\sim} = \varnothing$, then $B_1 \cap \cdots \cap B_m \cap A_1 \cap \cdots \cap A_n = \varnothing$, which would contradict that $\mathscr{U}$ has the finite intersection property. Therefore $\mathscr{U} \cup \{A^{\sim}\}$ is in Γ. Since $\mathscr{U}$ is a maximal element of Γ, $A^{\sim} \in \mathscr{U}$. In other words, for each subset A of X, either $A \in \mathscr{U}$ or $A^{\sim} \in \mathscr{U}$. Since $\mathscr{U}$ has the finite intersection property, $\varnothing \notin \mathscr{U}$. Therefore, $\varnothing^{\sim} = X \in \mathscr{U}$. Suppose $D \in \mathscr{U}$ and $E \in \mathscr{U}$. Since $\mathscr{U}$ has the finite intersection property and $D \cap E \cap (D \cap E)^{\sim} = \varnothing$, $(D \cap E)^{\sim}$ is not in $\mathscr{U}$. Therefore, $D \cap E$ is in $\mathscr{U}$. Suppose that $F \in \mathscr{U}$ and $X \supseteq G \supseteq F$. Since $\mathscr{U}$ has the finite intersection property and $F \cap G^{\sim} = \varnothing$, $G^{\sim}$ is not in $\mathscr{U}$. Therefore, G is in $\mathscr{U}$. Therefore, by Definition 4.2, $\mathscr{U}$ is an ultrafilter.

DEFINITION 4.3. Let $\mathscr{U}$ be an ultrafilter on J and for each $j \in J$, let $\succsim_j$ be a binary relation on the nonempty set A_j. Then, by definition, *the $\mathscr{U}$ ultraproduct of* $\{\langle A_j, \succsim_j\rangle \mid j \in J\}$ is $\langle A, \succsim\rangle$ where $A = \times_{j\in J} A_j$ and $\succsim$ is the binary relation on A defined by: $f \succsim g$ if and only if $\{j \mid f(j) \succsim_j g(j)\} \in \mathscr{U}$.

THEOREM 4.2. *Let $\mathscr{U}$ be an ultrafilter on J and for each $j \in J$, let $\succsim_j$ be a weak ordering on the nonempty set A_j. Let $\langle A, \succsim\rangle$ be the $\mathscr{U}$ ultraproduct of $\{\langle A_j, \succsim_j\rangle \mid j \in J\}$. Then $\succsim$ is a weak ordering on A.*

Proof. Let f, g, h be in A. Since $J = \{j \mid f(j) \succsim_j f(j)\}$, by Definition 4.3, $f \succsim f$. Thus $\succsim$ is reflexive. Suppose that $f \succsim g$ and $g \succsim h$. Then by Definition 4.3, $J_1 = \{j \mid f(j) \succsim_j g(j)\} \in \mathscr{U}$ and $J_2 = \{j \mid g(j) \succsim_j h(j)\} \in \mathscr{U}$. Since $\mathscr{U}$ is an ultrafilter, $J_1 \cap J_2 \in \mathscr{U}$. Since $\succsim_j$ is a weak order, $J_3 = \{j \mid f(j) \succsim_j h(j)\} \supseteq \{j \mid f(j) \succsim_j g(j)$ and $g(j) \succsim_j h(j)\} = J_1 \cap J_2$. Since $\mathscr{U}$ is an ultrafilter, $J_3 \in \mathscr{U}$. Therefore, by Definition 4.3, $f \succsim h$. Thus, $\succsim$ is transitive. To show that $\succsim$ is connected, suppose that d and e are in A. Let $J_1 = \{j \mid d(j) \succsim_j e(j)\}$ and $J_2 = \{j \mid e(j) \succsim_j d(j)\}$. Since for each $j \in J$ $\succsim_j$ is a weak order, $J_1 \cup J_2 = J$. Since $\mathscr{U}$ is an ultrafilter on J, either $J_1 \in \mathscr{U}$ or $J_2 \in \mathscr{U}$. Thus by Definition 4.3 we have either $d \succsim e$ or $e \succsim d$.

DEFINITION 4.4. Let $\mathscr{U}$ be an ultrafilter on J. Then $\langle {}^*\mathrm{Re}, \oplus, \odot, \succsim\rangle$ is said to be *an $\mathscr{U}$ ultrapower of* $\langle \mathrm{Re}, +, \cdot, \geqslant\rangle$ if and only if for all f, g, h in ${}^*\mathrm{Re}$ the following four conditions hold:

(1) $^*\text{Re} = \{d \mid d$ is a function from J into Re$\}$;
(2) $f \gtrsim g$ if and only if $\{j \mid f(j) \geqslant g(j)\} \in \mathscr{U}$;
(3) $f \oplus g \sim h$ if and only if $\{j \mid f(j) + g(j) = h(j)\} \in \mathscr{U}$; and
(4) $f \odot g \sim h$ if and only if $\{j \mid f(j) \cdot g(j) = h(j)\} \in \mathscr{U}$.

Let $\mathscr{U}$ be an ultrafilter on J, $\mathscr{R}_j = \langle \text{Re}, +, \cdot, \geqslant \rangle$ for each $j \in J$, and $^*\mathscr{R} = \langle ^*\text{Re}, \oplus, \odot, \gtrsim \rangle$ be an $\mathscr{U}$ ultrapower of $\langle \text{Re}, +, \cdot, \geqslant \rangle$. Then $\langle ^*\text{Re}, \gtrsim \rangle$ is the $\mathscr{U}$ ultraproduct of $\mathscr{R}_j$. Also note that by Theorem 4.2 $\gtrsim$ is a weak ordering in *Re. Also note that for each $f, g \in {}^*\text{Re}$, $f \sim g$ if and only if $\{j \in J \mid f(j) = g(j)\} \in \mathscr{U}$.

THEOREM 4.3. *Let $\mathscr{U}$ be an ultrafilter on J. If $^*\mathscr{R} = \langle ^*\text{Re}, +, \cdot, \gtrsim \rangle$ is an $\mathscr{U}$ ultrapower of $\mathscr{R} = \langle \text{Re}, +, \cdot, \geqslant \rangle$, then $^*\mathscr{R}$ is a weakly ordered field extension of $\mathscr{R}$.*

Proof. By Theorem 4.2 $\gtrsim$ is a weak ordering on *Re. Suppose that f, g, h are in *Re. Then

$$\begin{aligned} J &= \{j \mid f(j) + g(j) = g(j) + f(j)\} \\ &= \{j \mid f(j) + [g(j) + h(j)] = [f(j) + g(j)] + h(j)\} = \{j \mid f(j) \cdot g(j) = g(j) \cdot f(j)\} \\ &= \{j \mid f(j) \cdot [g(j) \cdot h(j)] = [f(j) \cdot g(j)] \cdot h(j)\} \\ &= \{j \mid f(j) \cdot [g(j) + h(j)] = f(j) \cdot g(j) + f(j) \cdot h(j)\}. \end{aligned}$$

Thus by Definition 4.4, $f \oplus g \sim g \oplus f$, $f \oplus (g \oplus h) \sim (f \oplus g) \oplus h$, $f \odot g \sim g \odot h$, $f \odot (g \odot h) \sim (f \odot g) \odot h$, and $f \odot (g \oplus h) \sim (f \odot g \oplus (f \odot h)$. Let *0 and *1 be the following functions on J: For each $j \in J$, $0^*(j) = 0$ and $^*1(j) = 1$. Then *0 and *1 are in *Re. Since $J = \{j \mid {}^*0(i) \neq {}^*1(i)\}$, it is not the case that $^*0 \sim {}^*1$. Since for each $f \in {}^*\text{Re}$, $J = \{j \mid f(j) + {}^*0(j) = f(j)\} = \{j \mid f(j) \cdot {}^*1(j) = f(j)\}$, it follows from Definition 4.4 that $f \oplus {}^*0 \sim f$ and $f \odot {}^*1 \sim f$. Suppose that $d \in {}^*\text{Re}$. Let d' be the following function on J: for each $j \in J$, $d'(j) = -d(j)$. Then $d' \in {}^*\text{Re}$. Since $J = \{j \mid d(j) + d'(j) = {}^*0(j)\}$, by Definition 4.4, $d \oplus d' \sim {}^*0$. Suppose that $e \in {}^*\text{Re}$ and not $e \sim {}^*0$. Let e'' be the following function on J: for each $j \in J$, if $e(j) \neq 0$ then $e''(j) = 1/e(j)$, and if $e(j) = 0$ then $e''(j) = 0$. Let $J_1 = \{j \mid e(j) \neq 0\}$. Since not $e \sim {}^*0$, $J_1 \in \mathscr{U}$. But then $J_1 = \{j \mid e(j) \cdot e''(j) = {}^*1(j)\}$. Thus by Definition 4.4, $e \odot e'' \sim {}^*1$. Suppose $p \gtrsim q$ and $r \gtrsim s$. Let $J_2 = \{j \mid p(j) \geqslant q(j)\}$ and $J_3 = \{j \mid r(j) \geqslant s(j)\}$. Then by Definition 4.4, $J_2 \in \mathscr{U}$ and $J_3 \in \mathscr{U}$. Since $\mathscr{U}$ is an ultrafilter, $J_2 \cap J_3 \in \mathscr{U}$. Since $J_4 = \{j \mid p(j) + r(j) \geqslant q(j) + s(j)\} \supseteq \{j \mid p(j) \geqslant q(j)$ and $r(j) \geqslant s(j)\} = J_2 \cap J_3$, $J_4 \in \mathscr{U}$. Therefore, by Definition 4.4, $p \oplus r \gtrsim s \oplus q$. Suppose that $a, b \in {}^*\text{Re}$ and $a \gtrsim {}^*0$ and $b \gtrsim {}^*0$. Then by Definition 4.4, $J_5 = \{j \mid a(j) \gtrsim {}^*0(j)\} \in \mathscr{U}$ and $J_6 = \{j \mid b(j) \gtrsim {}^*0(j)\} \in \mathscr{U}$. Since $\mathscr{U}$ is an ultrafilter, $J_5 \cap J_6 \in \mathscr{U}$. Since $J_7 = \{j \mid a(j) \cdot b(j) \gtrsim {}^*0(j)\} \supseteq \{j \mid a(j) \gtrsim {}^*0(j)$ and $b(j) \gtrsim {}^*0(j)\} = J_5 \cap J_6$, $J_7 \in \mathscr{U}$. Thus by Definition 4.4, $a \odot b \gtrsim {}^*0$. For each $x \in \text{Re}$ let $^*x$ be the following function on J: For $j \in J$, $^*x(j) = x$. For each $x \in \text{Re}$ let $F(x) = {}^*x$. Then it is easy to show that F is an isomorphic imbedding of Re into *Re. (We may therefore consider Re as a subset of *Re.) Therefore, $^*\mathscr{R}$ is a weakly orderend field extension of $\mathscr{R}$.

NOTATION. To simplify notation we will from here on write "$\oplus$" as "$+$" and "$\odot$" as "$\cdot$". Although this introduces some ambiguity, it makes the text much more readable.

DEFINITION 4.5. Let $\langle {}^*\mathrm{Re}, +, \cdot, \gtrsim \rangle$ be a weakly ordered field extension of the reals. An element $f \in {}^*\mathrm{Re}$ is said to be *finite* if and only if there are r and s in Re such that $r \succ f \succ s$.

DEFINITION 4.6. Let $\langle {}^*\mathrm{Re}, +, \cdot, \gtrsim \rangle$ be a weakly ordered field extension of the reals, $f \in {}^*\mathrm{Re}$, and f be finite. Let $A_1 = \{x \in \mathrm{Re} \mid x \gtrsim f\}$ and $A_2 = \{x \in \mathrm{Re} \mid f \succ x\}$. Then since f is finite, the ordered pair (A_1, A_2) is a Dedekind cut of Re. Let b be the cut number of (A_1, A_2). Then, by definition, ${}^\circ f = b$.

THEOREM 4.4. *Let* $\langle {}^*\mathrm{Re}, +, \cdot, \gtrsim \rangle$ *be a weakly ordered field extension of the reals and f be a finite element of* ${}^*\mathrm{Re}$. *Then for each positive real number r,* $| {}^\circ f - f | \lesssim r$.

Proof. Let r be a postive real number. Let $A_1 = \{x \in \mathrm{Re} \mid x \prec f\}$ and $A_2 = \{x \in \mathrm{Re} \mid x \gtrsim f\}$. Since ${}^\circ f$ is the cut number of the Dedekind cut (A_1, A_2), ${}^\circ f - r/2$ is in A_1 and ${}^\circ f + r/2$ is in A_2. But then ${}^\circ f - r/2 \lesssim f \lesssim {}^\circ f + r/2$. In other words, $| {}^\circ f - f | \lesssim r/2$.

THEOREM 4.5. *Let* $\langle {}^*\mathrm{Re}, +, \cdot, \gtrsim \rangle$ *be a weakly ordered field extension of the reals and e, f, g, h be finite elements of* ${}^*\mathrm{Re}$ *such that $e \succ f$ and $g \sim h$. Then ${}^\circ e \geqslant {}^\circ f$ and ${}^\circ g = {}^\circ h$.*

Proof. Since for each positive real number r, $| e - {}^\circ e | \prec r$ and $| f - {}^\circ f | \lesssim r$, it follows that for each $r \in \mathrm{Re}^+$, ${}^\circ e - {}^\circ f + 2r = {}^\circ e + r - ({}^\circ f - r) \gtrsim e - f$. Thus, ${}^\circ e - {}^\circ f \gtrsim 0$. Similarly it can be shown that ${}^\circ g = {}^\circ h$.

THEOREM 4.6. *Let* $\langle {}^*\mathrm{Re}, +, \cdot, \gtrsim \rangle$ *be a weakly ordered field extension of the reals and f, g be finite elements of* ${}^*\mathrm{Re}$. *Then $f + g$ is a finite element of* ${}^*\mathrm{Re}$ *and* ${}^\circ(f + g) = {}^\circ f + {}^\circ g$.

Proof. It is immediate from Definition 4.5 that $f + g$ is finite. By Theorem 4.4, for each positive real number r, $| {}^\circ f - f | \prec r$, $| {}^\circ g - g | \lesssim r$, and $| {}^\circ(f + g) - (f + g)| \prec r$. Thus $| {}^\circ(f + g) - ({}^\circ f + {}^\circ g)| \prec 3r$ for each $r \in \mathrm{Re}^+$. Therefore, ${}^\circ(f + g) = {}^\circ f + {}^\circ g$.

THEOREM 4.7. *Let* $\langle {}^*\mathrm{Re}, +, \cdot, \gtrsim \rangle$ *be a weakly ordered field extension of the reals and $f_1, f_2, \ldots, f_n$ be finite elements of* ${}^*\mathrm{Re}$. *Then $f_1 + f_2 + \cdots + f_n$ is a finite element of* ${}^*\mathrm{Re}$ *and* ${}^\circ(f_1 + f_2 + \cdots + f_n) = {}^\circ f_1 + {}^\circ f_2 + \cdots + {}^\circ f_n$.

Proof. Left to reader.

5. Minimal Conditions for Qualitative Probability

Theorem 5.1. *Let X be a nonempty set, $\mathscr{E}$ an algebra of subsets of X, and $\gtrsim$ a reflexive relation on $\mathscr{E}$. Suppose that $\langle X, \mathscr{E}, \gtrsim\rangle$ satisfies the finite cancellation axioms (Definition 3.4). Then there is a weakly ordered field extension of the reals $\langle {}^*\mathrm{Re}, +, \cdot, \gtrsim\rangle$ and a function P from $\mathscr{E}$ into ${}^*[0, 1] = \{x \in {}^*\mathrm{Re} \mid 0 \lesssim x \lesssim 1\}$ such that the following four conditions hold:*

(1) $P(X) \sim 1$ *and* $P(\varnothing) \sim 0$;

(2) *if* $x \succ y$ *then* $P(x) \succ P(y)$;

(3) *if* $x \sim y$ *then* $P(x) \sim P(y)$; *and*

(4) *if* $x, y \in \mathscr{E}$ *and* $x \cap y = \varnothing$ *then* $P(x \cup y) \sim P(x) + P(y)$.

Proof. Let $S = \{\alpha \mid \alpha$ is a finite subalgebra of $\mathscr{E}\}$. Let $\mathbf{Y} = \{\Delta \mid \Delta$ is a nonempty finite subset of $S\}$. For each $\alpha \in S$ let $\hat{\alpha} = \{\Delta \mid \Delta \in \mathbf{Y}$ and $\alpha \in \Delta\}$. Let $\mathscr{F} = \{\hat{\alpha} \mid \alpha \in S\}$. If $\hat{\alpha}_1, \ldots, \hat{\alpha}_n$ are in $\mathscr{F}$ then $\hat{\alpha}_1 \cap \cdots \cap \hat{\alpha}_n \neq \varnothing$ since $\{\alpha_1, \ldots, \alpha_n\} \in \hat{\alpha}_i$ for $i = 1, \ldots, n$. Thus, $\mathscr{F}$ has the finite intersection property. Therefore, by Theorem 4.1, let $\mathscr{U}$ be an ultrafilter on $\mathbf{Y}$ such that $\mathscr{U} \supseteq \mathscr{F}$.

For each $\alpha \in S$ let $\gtrsim_\alpha$ be the restriction of $\gtrsim$ to α. Since $\langle X, \mathscr{E}, \gtrsim\rangle$ satisfies the finite cancellation axioms, by Theorem 3.5, $\langle X, \mathscr{E}, \gtrsim\rangle$ satisfies the finiteness condition for qualitative probability structures. Therefore, for each $\alpha \in S$ let P_α be a probability representation for $\langle X, \alpha, \gtrsim_\alpha\rangle$. For each $x \in \mathscr{E}$ define the function F_x from $\mathbf{Y}$ into Re as follows: Let β be the finite subalgebra of $\mathscr{E}$ generated by $\bigcup \Delta$, and:

$$\text{if } x \in \beta \text{ then } F_x(\Delta) = P_\beta(x);$$

$$\text{if } x \notin \beta \text{ then } F_x(\Delta) = 0.$$

Let $\langle {}^*\mathrm{Re}, +, \cdot, \gtrsim\rangle$ be an $\mathscr{U}$ ultrapower of $\langle \mathrm{Re}, +, \cdot, \geqslant\rangle$. Define P on $\mathscr{E}$ as follows: for each $x \in \mathscr{E}$ let $P(x) = F_x$. Then P is a function from $\mathscr{E}$ into *Re.

Let $\alpha \in S$. If $\Delta \in \hat{\alpha}$ and β is the finite subalgebra generated by $\bigcup \Delta$ then $F_X(\Delta) = P_\beta(X) = 1$ and $F_\varnothing(\Delta) = P_\beta(\varnothing) = 0$. In other words $\{\Delta \mid F_X(\Delta) = 1\} \supseteq \hat{\alpha}$. Since $\hat{\alpha} \in \mathscr{U}$, we can conclude that $F_X \sim 1$. Therefore, $P(X) \sim 1$. Similarly, $P(\varnothing) \sim 0$.

Suppose that $x, y \in \mathscr{E}$ and $x \succ y$. Let $\alpha \in S$ be such that $x, y \in \alpha$. If $\Delta \in \hat{\alpha}$ and β is the finite subalgebra generated by $\cup \Delta$ then $F_x(\Delta) = P_\beta(x) > P_\beta(y) = F_y(\Delta)$. In other words, $\{\Delta \mid F_x(\Delta) > F_y(\Delta)\} \supseteq \hat{\alpha}$. Since $\hat{\alpha} \in \mathscr{U}$, we can conclude that $F_x \succ F_y$. Therefore, $P(x) \succ P(y)$. Similarly it can be shown that if $x, y \in \mathscr{E}$ and $x \sim y$ then $P(x) \sim P(y)$.

Suppose that $x, y \in \mathscr{E}$ and $x \cap y = \varnothing$. Let $\alpha \in S$ be such that $x, y \in \alpha$. If $\Delta \in \hat{\alpha}$ and β is the finite subalgebra generated by $\bigcup \Delta$ then $F_{x \cup y}(\Delta) = P_\beta(x \cup y) = P_\beta(x) + P_\beta(y) = F_x(\Delta) + F_y(\Delta)$. In other words, $\{\Delta \mid F_{x \cup y}(\Delta) = F_x(\Delta) + F_y(\Delta)\} \supseteq \hat{\alpha}$. Since $\hat{\alpha} \in \mathscr{U}$ we can conclude that $P(x \cup y) \sim P(x) + P(y)$.

DEFINITION 5.1. Let $\mathscr{E}$ be an algebra of subsets of X and $\gtrsim$ be a reflexive relation on $\mathscr{E}$. Then P is said to be a *weak probability representation for* $\langle X, \mathscr{E}, \gtrsim\rangle$ if and only if P is a function from $\mathscr{E}$ into [0, 1] such that the following four conditions hold for all $x, y \in \mathscr{E}$:

(1) $P(X) = 1$, $P(\varnothing) = 0$;
(2) if $x \succ y$ then $P(x) \geqslant P(y)$;
(3) if $x \sim y$ then $P(x) = P(y)$; and
(4) if $x \cap y = \varnothing$ then $P(x \cup y) = P(x) + P(y)$.

THEOREM 5.2. *Let $\mathscr{E}$ be an algebra of subsets of X, $\gtrsim$ a reflexive relation on $\mathscr{E}$, and $\langle X, \mathscr{E}, \gtrsim\rangle$ satisfy the finite cancellation axioms (Definition 3.4). Then there is a weak probability representation for $\langle X, \mathscr{E}, \gtrsim\rangle$.*

Proof. By Theorem 5.1 let $\langle {}^*\mathrm{Re}, +, \cdot, \gtrsim\rangle$ be a weakly ordered field extension of the reals and P a function from $\mathscr{E}$ into *Re such that the following four conditions hold for each $x, y \in \mathscr{E}$:

(1) $P(X) \sim 1$, $P(\varnothing) \sim 0$;
(2) if $x \succ y$ then $P(x) \succ P(y)$;
(3) if $x \sim y$ then $P(x) \sim P(y)$; and
(4) if $x \cap y = \varnothing$ then $P(x \cup y) \sim P(x) + P(y)$.

Since for each $x \in \mathscr{E}$ $X \gtrsim x \gtrsim \varnothing$, by (1), (2), and (3), $1 \sim P(X) \gtrsim P(x) \gtrsim P(\phi) \sim 0$. Therefore, $P(x)$ is finite for each $x \in \mathscr{E}$. Thus, by Definition 4.6 for each $x \in \mathscr{E}$, let $P'(x) = {}^\circ P(x)$. Then P' is a function from $\mathscr{E}$ into [0, 1]. By Theorem 4.5, $P'(X) = 1$ and $P'(\varnothing) = 0$. Let x, y be elements of $\mathscr{E}$. If $x \succ y$ then by (2) $P(x) \succ P(y)$, and thus by Theorem 4.5, $P'(x) \geqslant P'(y)$. If $x \sim y$ then by (3) $P(x) \sim P(y)$, and thus by Theorem 4.5, $P'(x) = P'(y)$. If $x \cap y = \varnothing$ then by (4) $P(x \cup y) \sim P(x) + P(y)$, and thus by Theorem 4.5, $P'(x \cup y) = P'(x) + P'(y)$. Therefore, by Definition 5.1 P' is a weak probability representation for $\langle X, \mathscr{E}, \gtrsim\rangle$.

DEFINITION 5.2. Let $\mathscr{E}$ be an algebra of subsets of X, $\gtrsim$ a weak order on $\mathscr{E}$, and $a, b \in \mathscr{E}$ such that $a \subseteq b$. Then (b, c, d, a) is said to be a *trisplit* of b, a if and only if the following four conditions hold:

(1) $b \supseteq c \supseteq d \supseteq a$;
(2) $b - c \gtrsim c - d$;
(3) $b - d \gtrsim d - a$; and
(4) $c - a \gtrsim b - c$.

$\langle X, \mathscr{E}, \gtrsim\rangle$ is said to be *trisplittable* if and only if for each $a, b \in \mathscr{E}$, if $a \subseteq b$ then there are $c, d \in \mathscr{E}$ such that (b, c, d, a) is a trisplit of b, a.

LEMMA 5.1. *Let $\mathscr{E}$ be an algebra of subsets of X, $\succsim$ a weak order on $\mathscr{E}$, $a, b \in \mathscr{E}$ such that $a \subseteq b$, (b, c, d, a) a trisplit of b, a, and P a weak probability representation for $\langle X, \mathscr{E}, \succsim\rangle$. Then*

$$\tfrac{1}{2}[P(b) - P(a)] \geqslant P(b) - P(c) \geqslant \tfrac{1}{4}[P(b) - P(a)].$$

Proof. Since $b \supseteq c \supseteq d$, $(b - c) \cap (c - d) = \varnothing$ and $(b - c) \cup (c - d) = b - d$. Thus,

(1) $P(b - d) = P(b - c) + P(c - d)$.

Since $b - c \succsim c - d$, $P(b - c) \geqslant P(c - d)$. Thus, by (1) we can conclude

(2) $2P(b - c) \geqslant P(b - d)$.

Since $b - d \succsim d - a$, $P(b - d) \geqslant P(d - a)$. Thus, by (2) we can conclude that

(3) $4P(b - c) \geqslant P(b - d) + P(d - a)$.

Since $b \supseteq d \supseteq a$, $(b - d) \cap (d - a) = \varnothing$ and $b - a = (b - d) \cup (d - a)$. Thus, $P(b - a) = P(b - d) + P(d - a)$. Therefore, by (3)

(4) $4P(b - c) \geqslant P(b - a)$.

Since $b \supseteq c \supseteq a$, $(b - c) \cap (c - a) = \varnothing$ and $(b - c) \cup (c - a) = b - a$. Therefore

(5) $P(b - c) + P(c - a) = P(b - a)$.

Since $c - a \succsim b - c$, $P(c - a) \geqslant P(b - c)$. Thus, by (5)

(6) $P(b - a) \geqslant 2P(b - c)$.

Since for each $x, y \in \mathscr{E}$ such that $x \supseteq y$, $y \cap (x - y) = \varnothing$ and $x = y \cup (x - y)$, we can conclude that $P(y) + P(x - y) = P(x)$, i.e., $P(x - y) = P(x) - P(y)$. Therefore, by (6) and (4)

(7) $\tfrac{1}{2}[P(b) - P(a)] \geqslant P(b) - P(c) \geqslant \tfrac{1}{4}[P(b) - P(a)]$.

LEMMA 5.2. *Let $\mathscr{E}$ be an algebra of subsets of X, $\succsim$ a weak ordering on $\mathscr{E}$, P a weak probability representation for $\mathscr{E}$, $\langle X, \mathscr{E}, \succsim\rangle$ be trisplittable, $r, s \in [0, 1]$ and $s > r$. Then for some $s \in \mathscr{E}$, $s > P(x) > r$.*

Proof. Suppose not. A contradiction will be shown. Let $\mathscr{D} = \{x \in \mathscr{E} \mid P(x) \geqslant s\}$. $\mathscr{D} \neq \varnothing$ since $X \in \mathscr{D}$. Let $t = \inf\{P(x) \mid x \in \mathscr{D}\}$ and $\epsilon = t - r$. Then for each $x \in \mathscr{E}$ it is not the case that $t > P(x) > r$. Then by the definition of *inf* let $y \in \mathscr{D}$ be such that $P(y) - t < \epsilon/100$. Let $a_1 = \varnothing$ and for each $i \in I^+$, a_i, b_i be such that (y, a_{i+1}, b_i, a_i) is a trisplit of y, a_i. By repeated applications of Lemma 5.1, it is easy to show that $(1/2^i)P(y) \geqslant (1/2^i)[P(y) - P(a_1)] \geqslant P(y) - P(a_{i+1})$. Note that since $a_1 = \varnothing$, $P(y) - P(a_1) \geqslant \epsilon$. Therefore let $n \in I^+$ be such that $P(y) - P(a_n) \geqslant \epsilon$ and

$P(y) - P(a_{n+1}) < \epsilon$. Then by Lemma 5.1 we can conclude that $\epsilon > P(y) - P(a_{n+1}) \geqslant \frac{1}{4}[P(y) - P(a_n)] \geqslant \frac{1}{4}\epsilon$. Then $t > P(a_{n+1}) > r$, a contradiction.

LEMMA 5.3. *Let $\mathscr{E}$ be an algebra of subsets of X, $\succsim$ a weak ordering on $\mathscr{E}$, P, Q weak probability representations for $\langle X, \mathscr{E}, \succsim\rangle$, and $\langle X, \mathscr{E}, \succsim\rangle$ be trisplittable. Then for each $a, b \in \mathscr{E}$, if $b \succ a$ and $P(b) = P(a)$ then $Q(b) = Q(a)$.*

Proof. Let $a, b \in \mathscr{E}$ be such that $b \succ a$ and $P(b) = P(a)$. Let $c = b - a$. Since $c \cap a = \varnothing$ and $P(b) = P(a \cup c) = P(a) + P(c)$, $P(c) = 0$. Let $d_1 = \varnothing$ and for each $i \in \mathrm{I}^+$, let d_i, e_i be such that $(c^\sim, d_{i+1}, e_i, d_i)$ is a trisplit of $c^\sim, d_i$. By repeated applications of Lemma 5.1 one can show that for each $i \in \mathrm{I}^+$, $P(c^\sim - d_i) > 0$, $Q(c^\sim - d_i) > 0$, and $\lim_{i\to\infty} Q(c^\sim - d_i) = 0$. Since $\succsim$ is a weak order and for each $i \in \mathrm{I}^+$, $P(c^\sim - d_i) > P(c) = 0$, we can conclude that $c^\sim - d_i \succ c$ for each $i \in \mathrm{I}^+$. Thus $Q(c^\sim - d_i) \geqslant Q(c)$ for each $i \in \mathrm{I}^+$. Since $\lim_{i\to\infty} Q(c^\sim - d_i) = 0$, $Q(c) = 0$. Therefore, $Q(b) = Q(a \cup c) = Q(a) + Q(c) = Q(a)$.

THEOREM 5.3. *Let $\mathscr{E}$ be an algebra of subsets of X, $\succsim$ be a weak ordering on $\mathscr{E}$, $\langle X, \mathscr{E}, \succsim\rangle$ be trisplittable, and P, Q be weak probability representations for $\langle X, \mathscr{E}, \succsim\rangle$. Then $P = Q$.*

Proof. Define Q' on $[0, 1]$ as follows: For each $t \in [0, 1]$ let $Q'(t) = \sup_{a\in A} Q(a)$ where $A = \{x \in \mathscr{E} \mid P(x) \leqslant t\}$. Note that if $P(z) = P(y)$ then by Lemma 5.3 $Q(y) = Q(z)$ and thus

(1) for each $x \in \mathscr{E}$, $Q'[P(x)] = Q(x)$.

Suppose that $r, s \in [0, 1]$ and $s > r$. By Lemma 5.2, let $x, y \in \mathscr{E}$ be such that $s > P(x) > P(y) > r$. Since $\succsim$ is a weak order, $x \succ y$. Thus, $Q(x) > Q(y)$. Then by the definition of Q', $Q'(s) \geqslant Q'[P(x)] = Q(x) > Q(y) \geqslant Q'(r)$. (Note that by Lemma 5.3, $Q(x) > Q(y)$ since $P(x) > P(y)$.) In other words,

(2) for each $r, s \in [0, 1]$, if $s > r$ then $Q'(s) > Q'(r)$.

We will now show the following:

(3) if for each $i \in \mathrm{I}^+$, $P(z_i) \leqslant w$ and $\lim_{i\to\infty} P(z_i) = w$, then $\sup_{i\in\mathrm{I}^+} Q(z_i) = Q'(w)$.

To show (3) there are two cases to consider: *Case* 1. for each $z \in \mathscr{E}$, $P(z) \neq w$. Let $A = \{z \in \mathscr{E} \mid P(z) \leqslant w\}$. Then for each $z \in A$ there is an $i \in \mathrm{I}^+$ such that $z_i \succ z$. Thus $Q'(w) = \sup_{i\in\mathrm{I}^+} Q(z_i)$. *Case* 2. z' is such that $P(z') = w$. Since $\succsim$ is a weak order and $P(z') \geqslant P(z_i)$, $z' \succsim z_i$ for each $i \in \mathrm{I}^+$. Thus, $Q(z') \geqslant Q(z_i)$ for each $i \in \mathrm{I}^+$. Therefore, $Q(z') \geqslant \sup_{i\in\mathrm{I}^+} Q(z_i)$. Assume that $Q(z') > \sup_{i\in\mathrm{I}^+} Q(z_i)$. We will show a contradiction thus establishing that $Q(z') = \sup_{i\in\mathrm{I}^+} Q(z_i)$. By Lemma 5.2, let $x \in \mathscr{E}$ be such that $Q(z') > Q(x) > \sup_{i\in\mathrm{I}^+} Q(z_i)$. Then since $\succsim$ is a weak ordering,

$z' \succ x \succ z_i$. Since $\lim_{i \in I^+} P(z_i) = w = P(z')$, $P(x) = P(z')$. Therefore, by Lemma 5.3 $Q(x) = Q(z')$, a contradiction.

We will now show the following:

(4) For all $u, v \in [0, 1]$ such that $u + v \in [0, 1]$, $Q'(u + v) = Q'(u) + Q'(v)$.

There are two cases to consider: *Case* 1. $u = 0$ or $v = 0$. Without loss of generality suppose that $u = 0$. Let $A = \{x \in \mathscr{E} \mid P(x) = 0\}$. Then by Lemma 5.3, for each $x \in A$, $Q(x) = 0$. Therefore, $Q'(u) = 0$. Thus, $Q'(u + v) = Q'(v) = Q'(u) + Q'(v)$. *Case* 2. $u \neq 0$ and $v \neq 0$. Then since $0 < u + v \leqslant 1$, $u \neq 1$ and $v \neq 1$. By Lemma 5.2 let $\{x_i \mid i \in I^+\}$ be such that for each $i \in I^+$, $0 < P(x_i) < u$ and $\lim_{i \to \infty} P(x_i) = u$. Since $u + v \leqslant 1$ and for each $i \in I^+$, $P(x_i) \leqslant u$, we can conclude that $v \leqslant P(x_i{}^\sim)$. For each $i \in I^+$, let $\mathscr{E}_i = \{y \mid y = x_i{}^\sim \cap z \text{ for some } z \in \mathscr{E}\}$, $\succsim_i$ be the restriction of $\succsim$ to $\mathscr{E}_i$, and $P_i = P/P(x_i{}^\sim)$. Then for each $i \in I^+$ it is easy to show that $\mathscr{E}_i$ is an algebra of subsets of $x_i{}^\sim$, $\succsim_i$ is a weak ordering on $x_i{}^\sim$, $\langle x_i{}^\sim, \mathscr{E}_i, \succsim_i\rangle$ is trisplittable, and P_i is a weak probability representation for $\langle x_i{}^\sim, \mathscr{E}_i, \succsim_i\rangle$. By Lemma 5.3, for each $i \in I^+$ let $\{y_{ij} \mid j \in I^+\}$ be such that $0 < P_i(y_{ij}) \leqslant v/P(x_i{}^\sim)$ and $\lim_{j \to \infty} P_i(y_{ij}) = v/P(x_i{}^\sim)$. Then it follows that $0 < P(y_{ij}) \leqslant v$ and $\lim_{j \to \infty} P(y_{ij}) = v$ and for each $i, j \in I^+$, $x_i \cap y_{ij} = \varnothing$. Thus

$$\lim_{i \to \infty} \lim_{j \to \infty} P(x_i \cup y_{ij}) = \lim_{i \to \infty} \lim_{j \to \infty} [P(x_i) + P(y_{ij})] = \lim_{i \to \infty} [P(x_i) + \lim_{j \to \infty} P(y_{ij})]$$
$$= \lim_{i \to \infty} [P(x_i) + v] = u + v.$$

Therefore, by (3) $\sup_{i,j \in I^+} Q(x_i \cup y_{ij}) = Q'(u + v)$. Since

$$\sup_{i,j \in I^+} Q(x_i \cup y_{ij}) = \sup_{i,j \in I^+} [Q(x_i) + Q(y_{ij})] = \sup_{i,j \in I^+} Q(x_i) + \sup_{i,j \in I^+} Q(y_{ij})$$

and by (3) $\sup_{i,j \in I^+} Q(x_i) = Q'(u)$ and $\sup_{i,j \in I^+} Q(y_{ij}) = Q'(v)$, we can conclude that $Q'(u + v) = Q'(u) + Q'(v)$.

Note that

(5) $Q'(0) = 0$ and $Q'(1) = 1$,

since if $A = \{x \in \mathscr{E} \mid P(x) = 0\}$ and $B = \{y \in \mathscr{E} \mid P(y) = 1\}$ then from Lemma 5.3 it follows that for each $x \in A$ and each $y \in B$, $P(\varnothing) = P(x) = Q(\varnothing) = Q(x)$ and $P(X) = P(y) = Q(X) = Q(y)$.

It is a well-known theorem of analysis that the only function Q' that satisfies (2), (4), and (5) is the identity function. (This can also easily be shown by using the representation and uniqueness theorem for Archimedean, regular, positive, ordered local semigroups that is given in Chapter 2 Krantz *et al.* [1971].) Thus, for each $r \in [0, 1]$, of $Q'(r) = r$. By (1) this means that for each $x \in \mathscr{E}$, $Q(x) = Q'[P(x)] = P(x)$.

THEOREM 5.4. *Suppose that $\mathscr{E}$ is an algebra of subsets of X, $\succsim$ is a weak ordering on $\mathscr{E}$, $\langle X, \mathscr{E}, \succsim\rangle$ is trisplittable and Archimedean (Definition* 3.2), *$\langle X, \mathscr{E}, \succsim\rangle$ satisfies the finite cancellation axioms (Definition* 3.4), *and P is a weak probability representation for $\langle X, \mathscr{E}, \succsim\rangle$. Then P is a probability representation for $\langle X, \mathscr{E}, \succsim\rangle$.*

Proof. Suppose that P is not a probability representation. A condtradiction will be shown. Let $x, y \in \mathscr{E}$ be such that $x \succ y$ and $P(x) = P(y)$. Let $d_1 = \varnothing$ and for each $i \in \mathrm{I}^+$, let c_i d_i be such that (X, d_{i+1}, c_i, d_i) is a trisplit of X, d_i. Then by Lemma 5.1, one can easily show that for each $i \in \mathrm{I}^+$, $P(d_{i+1} - d_i) > 0$. Let $e_i = d_{i+1} - d_i$. Then if $i, j \in \mathrm{I}^+$ and $i \neq j$, then $e_i \cap e_j = \varnothing$. Since $\succsim$ is a weak ordering, by Theorem 5.1, let $\langle {}^*\mathrm{Re}, +, \cdot, \succsim\rangle$ be a weakly ordered field extension of the reals and Q a function from $\mathscr{E}$ into $^*[0, 1]$ such that for all u, v, w, t in $\mathscr{E}$, (i) $u \succ v$ if and only if $Q(u) \succ Q(v)$, and (ii) if $w \cap t = \varnothing$ then $Q(w \cup t) = Q(w) + Q(t)$. Let $z = x - y$. Since $x \succ y$ and $z \cap y = \varnothing$ and $y \cup z = x$, $Q(x) = Q(y \cup z) = Q(y) + Q(z) \succ Q(y)$. Thus, $Q(z) \succ 0$. Thus, $z \succ 0$. Since $P(x) = P(y)$ and $P(x) = P(y \cup z) = P(y) + P(z)$, $P(z) = 0$. Since for each $i \in \mathrm{I}^+$, $P(e_i) > 0 = P(z)$, $e_i \succ z$. Therefore, for each $i, j \in \mathrm{I}^+$, $e_i \cap e_j = \varnothing$ and $e_i \succ z \succ \varnothing$. This contradicts the Archimedean axiom (Definition 3.2).

THEOREM 5.5. *Suppose that $\mathscr{E}$ is an algebra of subsets of X, $\succsim$ is a weak ordering on $\mathscr{E}$, $\langle X, \mathscr{E}, \succsim\rangle$ is trisplittable and satisfies the finite cancellation axioms (Definition* 3.4). *Then there is an unique weak probability representation P for $\langle X, \mathscr{E}, \succsim\rangle$. Furthermore, if in addition $\langle X, \mathscr{E}, \succsim\rangle$ is Archimedean, then P is a probability representation for $\langle X, \mathscr{E}, \succsim\rangle$.*

Proof. Theorem 5.2, 5.3, and 5.4.

6. Minimal Conditions for Additive Conjoint Measurement

THEOREM 6.1. *Let $A = \times_{i=1}^{n} A_i$ and $\succsim$ be a reflexive relation on A. Suppose that $\langle A, \succsim\rangle$ satisfies the finite cancellation axioms (Definition* 2.6). *Then there is a weakly ordered field extension of the reals $\langle {}^*\mathrm{Re}, +, \cdot, \succsim\rangle$ and functions Φ_i on A_i for $i = 1,..., n$, such that the following two conditions hold*:

(1) *if* $a_1 \cdots a_n \succ b_1 \cdots b_n$ *then* $\Phi_1(a_1) + \cdots + \Phi_n(a_n) \succ \Phi_1(b_1) + \cdots + \Phi_n(b_n)$;

and

(2) *if* $a_1 \cdots a_n \sim b_1 \cdots b_n$ *then* $\Phi_1(a_1) + \cdots + \Phi_n(a_n) \sim \Phi_1(b_1) + \cdots + \Phi_n(b_n)$.

Proof. To simplify notation, we will assume that $A = A_1 \times A_2$.

Let $S = \{\alpha \mid \alpha$ is a finite substructure of $\langle A, \succsim\rangle\}$. Let $X = \{\Delta \mid \Delta$ is a nonempty finite subset of $S\}$. For each $\alpha \in S$ let $\hat{\alpha} = \{\Delta \mid \Delta \in X$ and $\alpha \in \Delta\}$. Let $\mathscr{F} = \{\hat{\alpha} \mid \alpha \in S\}$. If $\hat{\alpha}_1, ..., \hat{\alpha}_m$ are in $\mathscr{F}$ then $\hat{\alpha}_1 \cap \cdots \cap \hat{\alpha}_m \neq \varnothing$ since $\{\alpha_1, ..., \alpha_m\} \in \hat{\alpha}_i$ for $i = 1,..., m$.

Thus, $\mathscr{F}$ has the finite intersection property. Therefore, by Theorem 4.1 let $\mathscr{U}$ be an ultrafilter on X such that $\mathscr{U} \supseteq \mathscr{F}$.

Suppose that Δ is in X, $\Delta = \{\alpha_1, \ldots, \alpha_m\}$, and $\alpha_i = \langle B_1^i \times B_2^i, \succsim^i \rangle$ for $i = 1, \ldots, m$. Let $B_1 = B_1^1 \cup \cdots \cup B_1^m$, $B_2 = B_2^1 \cup \cdots \cup B_2^m$, and $\succsim^1$ be the restriction of $\succsim$ to $B_1 \times B_2$. Then $\alpha = \langle B_1 \times B_2, \succsim^1 \rangle \in S$ and α_i is a finite substructure of α for $i = 1, \ldots, m$. Let β be the function from X into S defined by $\beta(\Delta) = \alpha$.

Since $\langle A, \succsim \rangle$ satisfies the kth cancellation axiom for each $k \in \mathrm{I}^+$, by Theorem 2.6 $\langle A, \succsim \rangle$ satisfies the finiteness condition for additive conjoint structures. Therefore, for each $\alpha \in S$, let Ψ_1^α and Ψ_2^α be a set of representation functions for α. For each $a \in A_1$, $b \in A_2$, and $\Delta \in X$, the functions F_a and G_b on X are defined as follows:

(1) $F_a(\Delta) = \Psi_1^{\beta(\Delta)}(a)$ if for some $\alpha \in \Delta$, if $\alpha = \langle B_1 \times B_2, \succsim^1 \rangle$ then $a \in B_1$, otherwise $F_a(\Delta) = 0$;

(2) $G_b(\Delta) = \Psi_2^{\beta(\Delta)}(a)$ if for some $\alpha \in \Delta$, if $\alpha = \langle B_1 \times B_2, \succsim^1 \rangle$ then $b \in B_2$, otherwise $G_b(\Delta) = 0$.

Let $\langle {}^*\mathrm{Re}, +, \cdot, \succsim \rangle$ be the $\mathscr{U}$ ultrapower of $\langle \mathrm{Re}, +, \cdot, \geqslant \rangle$. Define Φ_1 on A_1 and Φ_2 on A_2 as follows: For each $a \in A_1$, let $\Phi_1(a) = F_a$, and for each $b \in A_2$, let $\Phi_2(b) = G_b$. Since F_a and G_b are functions from X into Re, F_a and G_b are in *Re. Thus, Φ_1 and Φ_2 are functions from A_1 and A_2 into *Re.

Suppose that $x, u \in A_1$ and $y, v \in A_2$. Let $A^1 = \{x, u\} \times \{y, v\}$ and $\succsim^1$ be the restriction of $\succsim$ on A^1. Let $\alpha = \langle A^1, \succsim^1 \rangle$. Then $\alpha \in X$. By definition of $\mathscr{U}$, $\hat{\alpha} \in \mathscr{U}$. Let $\Delta \in \hat{\alpha}$. Then $\alpha \in \Delta$. Since $\alpha = \langle \{x, y\} \times \{u, v\}, \succsim^1 \rangle$, $F_x(\Delta) = \Psi_1^{\beta(\Delta)}(x)$, $F_u(\Delta) = \Psi_1^{\beta(\Delta)}(u)$, $G_y(\Delta) = \Psi_2^{\beta(\Delta)}(y)$, and $G_v(\Delta) = \Psi_2^{\beta(\Delta)}(v)$.

Case 1. $xy \succ uv$. Since $\alpha = \langle \{x, u\} \times \{y, v\}, \succsim^1 \rangle$ and $\alpha \in \Delta$, $\Psi_1^{\beta(\Delta)}(x) + \Psi_2^{\beta(\Delta)}(y) > \Psi_1^{\beta(\Delta)}(u) + \Psi_2^{\beta(\Delta)}(v)$. Thus, $F_x(\Delta) + G_y(\Delta) > F_u(\Delta) + G_v(\Delta)$. In other words, $\{\Delta \mid F_x(\Delta) + G_y(\Delta) > F_u(\Delta) + G_v(\Delta)\} \supseteq \hat{\alpha}$. Since $\hat{\alpha} \in \mathscr{U}$, $F_x + G_y \succ F_u + G_v$. That is, $\Phi_1(x) + \Phi_2(y) \succ \Phi_1(u) + \Phi_2(v)$.

Case 2. $xy \sim uv$. By a proof similar to Case 1 we show that $\Phi_1(x) + \Phi_2(y) \sim \Phi_1(u) + \Phi_2(v)$.

DEFINITION 6.1. Let $A = \mathsf{X}_{i=1}^n A_i$ and $\succsim$ be a binary relation on A. Then $\Phi_1, \ldots, \Phi_n$ is said to be a *set of representation functions for* $\langle A, \succsim \rangle$ if and only if the following three conditions hold for all $a_1 \cdots a_n$, $b_1 \cdots b_n$ in A:

(1) for each $i \leqslant n$, Φ_i is a function from A_i into Re;

(2) if $a_1 \cdots a_n \succ b_1 \cdots b_n$ then $\Phi_1(a_1) + \cdots + \Phi_n(a_n) \geqslant \Phi_1(b_1) + \cdots + \Phi_n(b_n)$;

and

(3) if $a_1 \cdots a_n \sim b_1 \cdots b_n$ then $\Phi_1(a_1) + \cdots + \Phi_n(a_n) = \Phi_1(b_1) + \cdots + \Phi_n(b_n)$.

DEFINITION 6.2. Let $A = \times_{i=1}^{n} A_i$, $\succsim$ be a binary relation on A, and $\langle A, \succsim \rangle$ be independent. Then $\langle A, \succsim \rangle$ is said to be *bounded by* $a_1, b_1, a_2, \ldots, a_n$ if and only if the following three conditions hold for each $x_1 \cdots x_n$ in A and each i, $2 \leqslant i \leqslant n$:

(1) $b_1 \succsim_1 x_1 \succsim_1 a_1$;

(2) $x_i \succsim_i a_i$; and

(3) $b_1 a_2 \cdots a_i \cdots a_n \succsim a_1 \cdots a_{i-1} x_i a_{i+1} \cdots a_n$.

THEOREM 6.2. *Let $A = \times_{i=1}^{n} A_i$, $\succsim$ be a reflexive relation on A, $\langle A, \succsim \rangle$ be bounded by $a_1, b_1, a_2, \ldots, a_n$, and $\langle A, \succsim \rangle$ satisfy the finite cancellation axioms (Definition 2.6). Then there is a set of representation functions for $\langle A, \succsim \rangle$, $\Phi_1, \ldots, \Phi_n$, such that $\Phi_1(a_1) = \Phi_2(a_2) = \cdots = \Phi_n(a_n) = 0$ and $\Phi_1(b_1) = 1$.*

Proof. To simplify notation, we will assume that $A = A_1 \times A_2$. By Theorem 6.1, let $\langle {}^*\mathrm{Re}, +, \cdot, \succsim \rangle$ be a weakly ordered field extension of the reals and Ψ_1', Ψ_2' functions from A_1, A_2 into *Re such that for all uv, xy in A, (i′) if $uv \succ xy$ then $\Psi_1'(u) + \Psi_2'(v) \succ \Psi_1'(x) + \Psi_2'(y)$, and (ii′) if $uv \sim xy$ then $\Psi_1'(u) + \Psi_2'(v) \sim \Psi_1'(x) + \Psi_2'(y)$. Let $\Psi_1'' = \Psi_1' - \Psi_1'(a_1)$ and $\Psi_2'' = \Psi_2' - \Psi_2'(a_2)$. Then $\Psi_1''(a_1) = \Psi_2''(a_2) = 0$. Since $b \succ a$, $\Psi_1''(b) \succ 0$. Let $\Psi_1 = \Psi_1''/\Psi_1''(b)$ and $\Psi_2 = \Psi_2''/\Psi_2''(b)$. Then it is easy to show that $\Psi_1(a_1) = \Psi_2(a_2) = 0$, $\Psi_1(b_1) = 1$, and for each uv, xy in A, (i) if $uv \succ xy$ then $\Psi_1(u) + \Psi_2(v) \succ \Psi_1(x) + \Psi_2(y)$, and (ii) if $uv \sim xy$ then $\Psi_1(u) + \Psi_2(v) \sim \Psi_1(x) + \Psi_2(y)$. Since for each $x \in A_1$, $b_1 \succsim_1 x \succsim_1 a_1$, it follows that $1 = \Psi_1(b_1) \succsim \Psi_1(x) \succsim \Psi_1(a_1) = 0$. Since for each $y \in A_2$, $b_1 a_2 \succsim a_1 y$, it follows that $\Psi_1(b_1) + \Psi_2(a_2) \succsim \Psi_1(a_1) + \Psi_2(y)$ and thus $1 = \Psi_1(b_1) \succsim \Psi_2(x) \succsim 0$. Therefore, for each $x \in A_1$ and each $y \in A_2$, $\Psi_1(x)$ and $\Psi_2(y)$ are finite (Definition 4.5). Therefore, for each $xy \in A$, let $\Phi_1(x) = {}^\circ\Psi_1(x)$ and $\Phi_2(y) = {}^\circ\Psi_2(y)$. Then by Theorem 4.4, $\Phi_1(a_1) = \Phi_2(a_2) = 0$ and $\Phi_1(b_1) = 1$. Suppose that uv, xy are in A and $uv \succ xy$. Then $\Psi_1(u) + \Psi_2(v) \succ \Psi_1(x) + \Psi_2(y)$. Thus, by Theorem 4.5, ${}^\circ[\Psi_1(u) + \Psi_2(v)] \geqslant {}^\circ[\Psi_1(x) + \Psi_2(y)]$, and by Theorem 4.7, ${}^\circ\Psi_1(u) + {}^\circ\Psi_2(v) \geqslant {}^\circ\Psi_1(x) + {}^\circ\Psi_2(y)$. Therefore, $\Phi_1(u) + \Phi_2(v) \geqslant \Phi_1(x) + \Phi_2(y)$. In a similar manner it can be shown that if ef, gh are in A and $ef \sim gh$, then $\Phi_1(e) + \Phi_2(f) = \Phi_1(g) + \Phi_2(h)$.

The first interesting thing to not about Theorem 6.2 is that no type of Archimedean axiom is assumed. In general, Archimedean axioms are used to guarantee that no distinguished pair of measured objects (i.e., a pair x, y where $x \succ y$ or $y \succ x$) are "too far" or "too close" with respect to a fixed distinguished pair. While in most areas of physical sciences there are reasonable grounds for making such an assumption, it seems to me to be a highly dubious assumption to make in the social sciences, especially when one is measuring quantities like utility or subjective probability. In Theorem 6.2, all distinguished pairs of elements of A that are "too close" with respect to the distinguished pair $a_1 \cdots a_n$, $b_1 a_2 \cdots a_n$, are assigned the same numerical value.

DEFINITION 6.3. Let $A = \bigtimes_{i=1}^{n} A_i$, $\gtrsim$ be a weak ordering on A, $\langle A, \gtrsim \rangle$ be independent, $i \leqslant n$, and $u, v, x, y \in A_i$. Then, by definition, $u - v \succ_i x - y$ if and only if (1) $u \succ_i v$ and $x \succ_i y$, and (2) for some $a_1 \cdots a_n$ in A and for some $j \neq i$ and some b, d in A_j,

$$a_1 \cdots a_{i-1} u a_{i+1} \cdots a_{j-1} d a_{j+1} \cdots a_n \succ a_1 \cdots a_{i-1} v a_{i+1} \cdots a_{j-1} b a_{j+1} \cdots a_n$$

and

$$a_1 \cdots a_{i-1} y a_{i+1} \cdots a_{j-1} b a_{j+1} \cdots a_n \succ a_1 \cdots a_{i-1} x a_{i+1} \cdots a_{j-1} d a_{j+1} \cdots a_n .$$

DEFINITION 6.4. Let $A = \bigtimes_{i=1}^{n} A_i$, $\gtrsim$ be a weak ordering on A, $\langle A, \gtrsim \rangle$ be independent, and $i \leqslant n$. Then (b, c, d, a) is said to be an *i-trisplit* of b, a if and only if the following four conditions hold:

(1) $b \succ_i c \succ_i d \succ_i a$;
(2) $b - d \succ_i d - a$;
(3) $c - a \succ_i b - c$; and
(4) $b - c \succ_i c - d$, and $d - a \succ_i c - d$.

DEFINITION 6.5. Let $A = \bigtimes_{i=1}^{n} A_i$, $\gtrsim$ be a weak ordering on A, and $\langle A, \gtrsim \rangle$ be independent. Then $\langle A, \gtrsim \rangle$ is said to be *trisplittable* if and only if for each $i \leqslant n$, if $b \succ_i a$ then there is an i-trisplit (b, c, d, a) of b, a.

Let $A = \bigtimes_{i=1}^{n} A_i$, $\gtrsim$ be a weak ordering on A, $\langle A, \gtrsim \rangle$ be independent and trisplittable, and $\Phi_1, \ldots, \Phi_n$ a set of representation functions for $\langle A, \gtrsim \rangle$. Then for each $i \leqslant n$ the following five lemmas hold:

LEMMA 6.1. *If $b - a \succ_i e - f$ then $\Phi_i(b) - \Phi_i(a) \geqslant \Phi_i(e) - \Phi_i(f)$.*

Proof. For notational simplicity, assume that $A = A_1 \times A_2$ and $i = 1$. By hypothesis, $b - a \succ_1 e - f$. Therefore, by Definition 5.3 let $c, d \in A_2$ be such that $c \succ_2 d$ and $bd \succ ac$ and $fc \succ ed$. Then $\Phi_1(b) + \Phi_2(d) \geqslant \Phi_1(a) + \Phi_2(c)$, i.e.,

$$\Phi_1(b) - \Phi_1(a) \geqslant \Phi_2(c) - \Phi_2(d), \quad \text{and} \quad \Phi_1(f) + \Phi_2(c) \geqslant \Phi_1(e) + \Phi_2(d),$$

i.e., $\Phi_2(c) - \Phi_2(d) \geqslant \Phi_1(e) - \Phi_1(f)$. Thus, $\Phi_1(b) - \Phi_1(a) \geqslant \Phi_1(e) - \Phi_1(f)$.

LEMMA 6.2. *Suppose that $\Phi_i(b) > \Phi_i(a)$ and (b, c, d, a) is an i-trisplit of b, a. Then $\Phi_i(b) > \Phi_i(c)$.*

Proof. For notational simplicity, assume that $A = A_1 \times A_2$ and $i = 1$. Since (b, c, d, a) is an 1-trisplit of b, a, $b \succ_1 c$. Thus $\Phi_1(b) \geqslant \Phi_1(c)$. Suppose that $\Phi_1(b) = \Phi_1(c)$. A contradiction will be shown. Since (b, c, d, a) is an i-trisplit of b, a, $b - c \succ_1 c - d$ and $b - d \succ_1 d - a$. Thus, by Lemma 6.1 (i) $\Phi_1(b) - \Phi_1(c) \geqslant \Phi_1(c) - \Phi_1(d)$ and (ii) $\Phi_1(b) - \Phi_1(d) \geqslant \Phi_1(d) - \Phi_1(a)$. Since $\Phi_1(b) = \Phi_1(c)$ and $\Phi_1(c) \geqslant \Phi_1(d)$, by (i) we conclude that $\Phi_1(c) = \Phi_1(d)$. Thus $\Phi_1(b) = \Phi_1(d)$. Since $\Phi_1(b) = \Phi_1(d)$ and

$\Phi_1(d) \geqslant \Phi_1(a)$, by (ii) we conclude that $\Phi_1(d) = \Phi_1(a)$. Thus $\Phi_1(b) = \Phi_1(a)$, a contradiction.

LEMMA 6.3. *Suppose that* $\Phi_i(b) > \Phi_i(a)$ *and* (b, c, d, a) *is an i-trisplit of* b, a. *Then* $\frac{1}{2}[\Phi_i(b) - \Phi_i(a)] \geqslant \Phi_i(b) - \Phi_i(c)$ *and* $\frac{1}{2}[\Phi_i(b) - \Phi_i(a)] \geqslant \Phi_i(d) - \Phi_i(a)$.

Proof. For notational simplicity, assume that $A = A_1 \times A_2$ and $i = 1$. Since (b, c, d, a) is an i-trisplit of b, a, $c - a >_1 b - c$. Therefore, let $e, f \in A_2$ be such that $e >_2 f$, $cf \succ ae$, and $ce \succ bf$. Thus,

(1) $\Phi_1(c) + \Phi_2(f) \geqslant \Phi_1(a) + \Phi_2(e)$, and
(2) $\Phi_1(c) + \Phi_2(e) \geqslant \Phi_1(b) + \Phi_2(f)$.

Adding (1) and (2) and subtracting the common term, $\Phi_2(e) + \Phi_2(f)$, of both sides of the resulting inequality, we get

(3) $2\Phi_1(c) \geqslant \Phi_1(a) + \Phi_1(b)$.

Thus,

(4) $-\Phi_1(a) - \Phi_1(b) \geqslant -2\Phi_1(c)$.
(5) $2\Phi_1(b) - \Phi_1(a) - \Phi_1(b) \geqslant 2\Phi_1(b) - 2\Phi_1(c)$,
(6) $\Phi_1(b) - \Phi_1(a) \geqslant 2[\Phi_1(b) - \Phi_1(c)]$, and
(7) $\frac{1}{2}[\Phi_1(b) - \Phi_1(a)] \geqslant \Phi_1(b) - \Phi_1(c)$.

Similarly it can be shown that $\frac{1}{2}[\Phi_1(b) - \Phi_1(a)] \geqslant \Phi_1(d) - \Phi_1(a)$.

LEMMA 6.4. *Suppose that* $\Phi_i(b) > \Phi_i(b_1)$, *and* $b_1, b_2, b_3, \ldots$ *are such that for each* $j \in \mathrm{I}^+$ *there is a* c_j *such that* (b, b_{j+1}, c_j, b_j) *is an i-trisplit of* b, b_j. *Also suppose that* $x >_i y$ *and* $\Phi_i(x) = \Phi_i(y)$. *Then for each* $j \in \mathrm{I}^+$, $b - b_j >_i x - y$.

Proof. For notational simplicity, suppose that $A = A_1 \times A_2$ and $i = 1$. By repeated applications of Lemma 6.2 one can easily show that for each $j \in \mathrm{I}^+$, $\Phi_1(b) > \Phi_1(b_j)$. Since (b, b_{j+1}, c_j, b_j) is a 1-trisplit of b, b_j, $b - b_j >_1 b - b_{j+1}$. Thus let e, f in A_2 be such that $e >_2 f$ and $bf \succ b_je$ and $b_{j+1}e \succ bf$. Since, by hypothesis, $\succsim$ is a weak order, either $ye \succ xf$ or $xf \succsim ye$. If $xf \succsim ye$, then

(1) $\Phi_1(b_{j+1}) + \Phi_2(e) \geqslant \Phi_1(b) + \Phi_2(f)$, and
(2) $\Phi_1(x) + \Phi_2(f) \geqslant \Phi_1(y) + \Phi_2(e)$,

which by adding (1) and (2) and then subtracting the common term, $\Phi_2(e) + \Phi_2(f)$, from the resulting inequality yields

(3) $\Phi_1(b_{j+1}) + \Phi_1(x) \geqslant \Phi_1(b) + \Phi_1(y)$,

and, thus,

(4) $\Phi_1(x) - \Phi_1(y) \geqslant \Phi_1(b) - \Phi_1(b_{j+1})$,

which—since $\Phi_1(x) = \Phi_1(y)$—yields

(5) $\Phi_1(b) = \Phi_1(b_{j+1})$,

which contradicts $\Phi_1(b) > \Phi_1(b_{j+1})$. Thus $ye \succ xf$. Since $bf \succ b_j e$, we conclude that $b - b_j \succ_1 x - y$.

LEMMA 6.5. *Suppose that $\Phi_i(b) > \Phi_i(a)$, $x \succ_i y$, $\Phi_i(x) = \Phi_i(y)$, and $\Psi_1, \ldots, \Psi_n$ is a set of representation functions for $\langle A, \succsim \rangle$ such that $\Psi_i(b) > \Psi_i(a)$. Then $\Psi_i(x) = \Psi_i(y)$.*

Proof. Since $\Phi_i(b) > \Phi_i(a)$ and $\langle A, \succsim \rangle$ is trisplittable, we can find $b_1 = a$, b_2, $b_3, \ldots, c_1, c_2, \ldots,$ such that for each $j \in \mathrm{I}^+$, (b, b_{j+1}, c_j, b_j) is a trisplit of b, b_j. By Lemma 6.4, for each $j \in \mathrm{I}^+$, $b - b_j \succ_i x - y$. Therefore, by Lemma 6.1 for each $j \in \mathrm{I}^+$,

(1) $\Psi_i(b) - \Psi_i(b_j) \geqslant \Psi_i(x) - \Psi_i(y)$.

Thus, by Lemma 6.3 for each $j \in \mathrm{I}^+$,

(2) $\frac{1}{2}[\Psi_i(b) - \Psi_i(b_j)] \geqslant \Psi_i(b) - \Psi_i(b_{j+1})$.

Therefore, for each $j \in \mathrm{I}^+$

(3) $\frac{1}{2}^j[\Psi_i(b) - \Psi_i(b_1)] \geqslant \Psi_i(b) - \Psi_i(b_{j+1}) \geqslant \Psi_i(x) - \Psi_i(y)$.

Thus, $\Psi_i(x) = \Psi_i(y)$.

THEOREM 6.3. *Let $A = \mathsf{X}_{i=1}^n A_i$, $\succsim$ be a weak ordering on A, $\langle A, \succsim \rangle$ satisfy the finite cancellation axioms, $\langle A, \succsim \rangle$ be bounded by $b_1, a_1, \ldots, a_n$, and $\langle A, \succsim \rangle$ be trisplittable. By Theorem 6.2 let $\Phi_1, \ldots, \Phi_n$ and $\Psi_1, \ldots, \Psi_n$ be sets of representation functions for $\langle A, \succsim \rangle$ such that $\Phi_1(b_1) = \Psi_1(b_1) = 1$, $\Phi_1(a_1) = \Psi_1(a_1) = \cdots = \Phi_n(a_n) = \Psi_n(a_n) = 0$. Then $\Phi_1 = \Psi_1$, $\Phi_2 = \Psi_2, \ldots, \Phi_n = \Psi_n$.*

Proof. To simplify notation, we will assume that $A = A_1 \times A_2$. Let $B_1 = B_2 = [0, 1]$ and $B = B_1 \times B_2$. Define $\succsim'$ on B as follows: $rs \succsim' ut$ if and only if $r + s \geqslant u + t$.

Suppose that $xy \succ' uv$. We will show that for some $a, b \in A$, $xy \succ' \Phi_1(a)\Phi_2(b) \succ' uv$. Let $x + y - (u + v) = \epsilon > 0$. For simplicity we will assume that $x \neq 0$ and $y \neq 0$. These cases will follow by an analogous argument. We will first show that for some $a \in A_1$, $xy \succ' \Phi_1(a)\, y \succ' uv$. Let x_1 be such that $x > x_1 > 0$ and $x - x_1 < \epsilon$. Then since $x_1 y \succ uv$, we need only find some $a \in A_1$ such that $x > \Phi_1(a) > x_1$. Let $\delta = x - x_1$. Let $r = \inf\{\Phi_1(c) \mid \Phi_1(c) \geqslant x \text{ and } c \in A_1\}$ and $s = \sup\{\Phi_1(c) \mid x_1 \geqslant \Phi_1(c)$ and $c \in A_1\}$. Then $r - s \geqslant \delta$. By the definitions of *inf* and *sup*, let $c, d \in A_1$ be such that $\Phi_1(c) \geqslant r$, $s \geqslant \Phi_1(d)$, $\Phi_1(c) - r < \delta/100$. Let $\delta_1 = \Phi_1(c) - \Phi_1(d)$. Then $\delta + \delta/50 > \delta_1 \geqslant \delta$. Let (c, e, f, d) be a 1-trisplit of A_1. By Lemma 6.3, $\Phi_1(c) - \Phi_1(e) \leqslant \frac{1}{2}\delta_1$ and $\Phi_1(f) - \Phi_1(d) \leqslant \frac{1}{2}\delta_1$. Therefore, by Definition 6.4 $\frac{1}{2}\delta_1 > \Phi_1(e) - \Phi_1(f) = [\Phi_1(c) - \Phi_1(d)] - [\Phi_1(c) - \Phi_1(e)] - [\Phi_1(f) - \Phi_1(d)]$. Therefore, since $\Phi_1(c) - \Phi_1(d) = \delta_1$, either $\Phi_1(c) - \Phi_1(e) > \frac{1}{4}\delta_1$ or $\Phi_1(f) - \Phi_1(d) > \frac{1}{4}\delta_1$. Without loss of generality suppose that $\Phi_1(c) - \Phi_1(e) > \frac{1}{4}\delta_1$. Then $\frac{1}{2}\delta_1 \geqslant \Phi_1(c) - \Phi_1(e) > \frac{1}{4}\delta_1$. It then easily follows that $x > \Phi_1(c) > x$, which we wanted to show. Thus, if $xy \succ' uv$ then for some $a \in A_1$, $xy \succ' \Phi_1(a)y \succ' uv$. Similarly, it can be shown that since $\Phi_1(a)y \succ' uv$, for some $b \in A_2$, $xy \succ' \Phi_1(a)y \succ' \Phi_1(a)\,\Phi_2(b) \succ' uv$.

For $i = 1, 2$, define α_i on B_i as follows: $\alpha_i(x) = \sup_{a \in D} \Psi_i(a)$ where $D = \{a \mid a \in A_i$ and $x \geqslant \Phi_i(a)\}$. Note that for each $ab \in A$, $\alpha_1[\Phi_1(a)] = \Psi_1(a)$ and $\alpha_2[\Phi_2(b)] = \Psi_2(b)$. Suppose that $xy \succ' uv$. Let $ab \in A$ be such that $xy \succ' \Phi_1(a)\,\Phi_1(b) \succ' uv$. Then by the definition of α_1 and α_2, $\alpha_1(x) + \alpha_2(y) \geqslant \alpha_1[\Phi_1(a)] + \alpha_2[\Phi_2(b)] = \Psi_1(a) + \Psi_2(b) > \alpha_1(u) + \alpha_2(v)$. In other words, α_1, α_2 is a set of strict representation functions for $\langle B, \succsim' \rangle$. But $\langle B, \succsim' \rangle$ satisfies Luce's axioms for additive conjoint structures [see Krantz *et al.*, 1971, Chapter 6]. It follows from the uniqueness theorem for such structures [see Krantz *et al.*, 1971, Chapter 6] that the only set of strict representation functions β_1, β_2 for $\langle B, \succsim' \rangle$ such that $\beta_1(0) = \beta_2(0) = 0$ and $\beta_1(1) = \beta_2(1) = 1$ is the set of identity functions. Thus for each $xy \in B$, $\alpha_1(x) = x$ and $\alpha_2(y) = y$. Therefore, for $i = 1, 2$ and each $a \in A_i$, $\Phi_i(a) = \alpha_i[\Phi_i(a)] = \Psi_i(a)$.

THEOREM 6.4. *Let* $A = \times_{i=1}^{n} A_i$, $\succsim$ *be a weak ordering on* A, *and* $\langle A, \succsim \rangle$ *satisfy the finite cancellation axioms, be trisplittable and be Archimedean. Then* $\langle A, \succsim \rangle$ *is an additive conjoint structure with a set of strict representation functions* $\Phi_1, \ldots, \Phi_n$. *Furthermore,* $\Psi_1, \ldots, \Psi_n$ *is another set of representation functions for* $\langle A, \succsim \rangle$ *if and only if there is a positive real number* r *and real numbers* $t_1, \ldots, t_n$ *such that for* $i = 1, \ldots, n$, $\Phi_i = r\Psi_i + t_i$.

Proof. To simplify notation, we will assume that $A = A_1 \times A_2$. By Theorem 6.1, let $\langle {}^*\mathrm{Re}, +, \cdot, \succsim \rangle$ be an ordered field extension of the reals and α_1, α_2 functions from A_1, A_2 into *Re such that for each xy, uv in A,

$$xy \succ uv \text{ if and only if } \alpha_1(x) + \alpha_2(y) \succ \alpha_1(u) + \alpha_2(v).$$

Let b, a in A_1 be such that $b \succ_1 a$, and let c be in A_2. Define β_1, β_2 as follows: $\beta_1 = [\alpha_1 - \alpha_1(a)]/[\alpha_1(b) - \alpha_1(a)]$ and $\beta_2 = \alpha_2 - \alpha_2(c)$. Then it is easy to show that $\beta_1(a) = 0$, $\beta_1(b) = 1$, $\beta_2(c) = 0$, and for each xy, uv in A,

$$xy \succ uv \text{ if and only if } \beta_1(x) + \beta_2(y) \succ \beta_1(u) + \beta_2(v).$$

We will now show that for each xy in A, $\beta_1(x)$ and $\beta_2(y)$ are finite. First, suppose that $y \in A_2$ and y is not finite. A contradiction will be shown. Without loss of generality, suppose that $\beta_2(y) \succ 0$. Then $\beta_2(y) \succ t$ for each $t \in \mathrm{Re}$. Since $\langle A, \succsim \rangle$ is trisplittable, let (y, e, f, c) be a 1-trisplit of y, c. Then it can be shown by a proof similar to Lemma 6.3 that $\frac{1}{2}[\beta_2(y) - \beta_2(c)] = \frac{1}{2}\beta_2(y) \succ \beta_2(y) - \beta_2(e)$. Since $\beta_2(y) \succ 2t$ for each $t \in \mathrm{Re}$, $\frac{1}{2}\beta_2(y) \succ t$ for each $t \in \mathrm{Re}$. Since $\beta_2(e) \succ \beta_2(y) - \frac{1}{2}\beta_2(y) = \frac{1}{2}\beta_2(y)$, we can conclude that $\beta_2(e) \succ t$ for each $t \in \mathrm{Re}$. That is, $\beta_2(e)$ is not finite. We will now show that

$\beta_2(y) - \beta_2(e) \gtrsim 1$. Suppose not. Then $\beta_2(y) - \beta_2(e) < 1$. and since $\beta_2(y) - \beta_2(e) \succ \beta_2(e) - \beta_2(f)$, $2 \succ [\beta_2(y) - \beta_2(e)] + [\beta_2(e) - \beta_2(f)] = \beta_2(y) - \beta_2(f) \succ \beta_2(f) - \beta_2(c)$. In other words, $4 \succ [\beta_2(y) - \beta_2(f)] + [\beta_2(f) - \beta_2(c)] = \beta_2(y) - \beta_2(c) = \beta_2(y) \succ 0$. But this is impossible since $\beta_2(y)$ is not finite. Thus, $\beta_2(y) - \beta_2(e) \gtrsim 1 = \beta_1(b) - \beta_1(a)$. In summary, if $y \in A_2$ and $\beta_2(y)$ is not finite and $\beta_2(y) \succ 0$ then there is a $y_1 \in A_2$ such that $y \succ_2 y_1 \succ_2 c$, $\beta_2(y_1)$ is not finite and $ay \gtrsim by_1$. Therefore, by repeating this argument, we can find $y_1, y_2, y_3, y_4, \ldots$ such that $y \succ_2 y_1 \succ_2 y_2 \succ_2 \ldots$, and for each $i \in \mathrm{I}^+$, $y_i \succ_2 c$ and $ay_i \gtrsim by_{i+1}$. This contradicts the Archimedean axiom for $\langle A, \gtrsim\rangle$. Therefore, we have shown that if $xy \in A$ then $\beta_2(y)$ is finite. To show that $\beta_1(x)$ is finite for each xy in A, we only have to apply a similar argument. (Note that since for each element x of A_2 $\beta_2(x)$ is finite and that there are elements g, h in A_2 such that $g > h$, the violation of the Archimedean axiom takes the form: there are $x_1, x_2, \ldots$ such that $x_1 \succ_1 x_2 \succ_1 \cdots$ and for each $i \in \mathrm{I}^+$, $x_i h \gtrsim x_{i+1} g$.) In summary, for each $xy \in A$, $\beta_1(x)$ and $\beta_2(y)$ are finite. Therefore, for each $xy \in A$, let $\Phi_1(x) = {}^\circ\beta_1(x)$ and $\Phi_2(y) = {}^\circ\beta_2(y)$. Then by using the methods of Theorem 6.2, one can easily verify that Φ_1, Φ_2 is a set of representation functions for $\langle A, \gtrsim\rangle$. Note that $1 = \Phi_1(b) > \Phi_1(a) = 0$.

We will now show that Φ_1, Φ_2 are a set of strict representation functions $\langle A, \gtrsim\rangle$. Assume not, i.e., assume that a_1a_2, b_1b_2 are in A, $a_1a_2 \succ b_1b_2$, and $\Phi_1(a_1) + \Phi_2(a_2) = \Phi_1(b_1) + \Phi_2(b_2)$. A contradiction will be shown. Since $\langle A, \gtrsim\rangle$ is trisplittable, at least one of the following two cases hold:

Case 1. Let $e_1 \in A_1$ be such that $a_1 \succ_1 e_1$ and $e_1a_2 \succ b_1b_2$. Since $a_1a_2 \succ e_1a_2 \succ b_1b_2$, $\Phi_1(a_1) + \Phi_2(a_2) = \Phi_1(e_1) + \Phi_2(a_2) = \Phi_1(b_1) + \Phi_2(b_2)$. Thus $\Phi_1(a_1) = \Phi_1(e_1)$. Since $\langle A, \gtrsim\rangle$ is trisplittable, let (b, u, v, a) be a 1-trisplit of b, a. (Recall that $\Phi_1(b) > \Phi_1(a)$.) Since $u - a \succ_1 b - u$, by Definition 6.3 let d, f be in A_2 and such that $d \succ_2 f$ and $ud \succ bf$. Since $\Phi_1(u) + \Phi_2(d) \geqslant \Phi_1(b) + \Phi_2(f)$ and by Lemma 6.2 $\Phi_1(b) > \Phi_1(u)$, we can conclude that $\Phi_2(d) > \Phi_2(f)$. Since $\langle A, \gtrsim\rangle$ is trisplittable, let $d_1, d_2, \ldots$, and $c_1, c_2, \ldots$, be such that for each $i \in \mathrm{I}^+$, (d, d_{i+1}, c_i, d_i) is a 2-trisplit of d, f. Since by Lemma 6.2 $\Phi_2(d_{i+1}) - \Phi_2(d_i) > 0 = \Phi_1(a_1) - \Phi_1(e_1)$ for each $i \in \mathrm{I}^+$, we can conclude that $\Phi_1(e_1) + \Phi_2(d_{i+1}) > \Phi_1(a_1) + \Phi_2(d_i)$ for each $i \in \mathrm{I}^+$. Since $\gtrsim$ is a weak order, $e_1d_{i+1} \succ a_1d_i$ for each $i \in \mathrm{I}^+$. Since $d \succ_2 d_i$ for each $i \in \mathrm{I}^+$, this contradicts the Archimedean axiom (Definition 2.4).

Case 2. Let $e_2 \in A_2$ be such that $a_1e_2 \succ b_1b_2$. By an argument similar to Case 1 the Archimedean axiom can be contradicted.

7. Approximation by Finite Structures

Definition 7.1. For each $i \in \mathrm{I}^+$ let Y_i be a nonempty set and F_i a function from Y_i into Re. Suppose that for each $i, j \in \mathrm{I}^+$ such that $i < j$, $Y_i \subseteq Y_j$ and $Y = \bigcup_{i\in\mathrm{I}^+} Y_i$. Then, by definition, $F = \lim_{i\to\infty} F_i$ if and only if F is a function from Y into Re such that for each $x \in Y$ and each positive real number ϵ, there is a $q \in \mathrm{I}^+$ such that $x \in Y_q$ and for each $i \geqslant q$, $|F_i(x) - F(x)| < \epsilon$.

DEFINITION 7.2. Let $\mathscr{E}$ be an algebra of subsets on X and $\gtrsim$ be a reflexive relation on $\mathscr{E}$. Then $\langle X, \mathscr{E}, \gtrsim\rangle$ is said to be a *finite qualitative probability structure* if and only if $\langle X, \mathscr{E}, \gtrsim\rangle$ is a qualitative probability structure and $\mathscr{E}$ is a finite set.

Let $A = \times_{i=1}^{n} A_i$ and $\gtrsim$ be a reflexive relation on A. Then $\langle A, \gtrsim\rangle$ is said to be a *finite additive conjoint structure* if and only if $\langle A, \gtrsim\rangle$ is an additive conjoint structure and A is a finite set.

THEOREM 7.1. *For each $i \in \mathrm{I}^+$, suppose that $\langle X, \mathscr{E}_i, \gtrsim_i\rangle$ is a finite probability structure and that P_i is a probability representation for $\langle X, \mathscr{E}_i, \gtrsim_i\rangle$. Suppose that for each $i, j \in \mathrm{I}^+$ such that $i < j$, $\mathscr{E}_i \subseteq \mathscr{E}_j$ and $\gtrsim_i \subseteq \gtrsim_j$. Let $\mathscr{E} = \bigcup_{i \in \mathrm{I}^+} \mathscr{E}_i$ and $\gtrsim = \bigcup_{i \in \mathrm{I}^+} \gtrsim_i$. Then the following three propositions are true*:

(1) *$\langle X, \mathscr{E}, \gtrsim\rangle$ has a weak probability representation*;

(2) *if $\gtrsim_i$ is a weak order on $\mathscr{E}_i$ for each $i \in \mathrm{I}^+$ and $\langle X, \mathscr{E}, \gtrsim\rangle$ is trisplittable, then $\langle X, \mathscr{E}, \gtrsim\rangle$ has an unique weak probability representation P and* $\lim_{i\to\infty} P_i = P$; *and*

(3) *if $\gtrsim_i$ is a weak order on $\mathscr{E}_i$ for each $i \in \mathrm{I}^+$ and $\langle X, \mathscr{E}, \gtrsim\rangle$ is trisplittable and Archimedean, then $\langle X, \mathscr{E}, \gtrsim\rangle$ has an unique probability representation P and $P = \lim_{i\to\infty} P_i$.*

Proof. (1) Since for each $i \in \mathrm{I}^+ \langle X, \mathscr{E}_i, \gtrsim_i\rangle$ is a finite probability structure, for each $i \in \mathrm{I}^+ \langle X, \mathscr{E}_i, \gtrsim_i\rangle$ satisfies the finite cancellation axioms. Thus, $\langle X, \mathscr{E}, \gtrsim\rangle$ satisfies the finite cancellation axioms. Therefore, by Theorem 5.2 $\langle X, \mathscr{E}, \gtrsim\rangle$ has a weak probability representation.

(2) Since for each $i \in \mathrm{I}^+$, $\gtrsim_i$ is a weak order on $\mathscr{E}_i$ and $\gtrsim = \bigcup_{i \in \mathrm{I}^+} \gtrsim_i$, it is easy to show that $\gtrsim$ is a weak order on $\mathscr{E}$. Since $\langle X, \mathscr{E}, \gtrsim\rangle$ is trisplittable, by Theorem 5.5 let P be the unique weak probability representation for $\langle X, \mathscr{E}, \gtrsim\rangle$. Suppose that $\lim_{i\to\infty} P_i \neq P$. We will show a contradiction. Since $\lim_{i\to\infty} P_i \neq P$, let $a \in \mathscr{E}$, J be an infinite subset of I^+, and ϵ a positive real number such that for each $i \in J$, $|P(a) - P_i(a)| \geqslant \epsilon$. Let $\mathscr{F} = \{\alpha \mid J - \alpha \text{ is a finite subset of } J\}$. Then it is easy to show that $\mathscr{F}$ has the finite intersection property. By Theorem 4.1, let $\mathscr{U}$ be an ultrafilter on J such that $\mathscr{U} \supseteq \mathscr{F}$. Let $\langle {}^*\mathrm{Re}, +, \cdot, \gtrsim\rangle$ be the $\mathscr{U}$-ultrapower of $\langle \mathrm{Re}, +, \cdot, \geqslant\rangle$. For each $x \in \mathscr{E}$, let Q_x be the function from J into Re such that for each $i \in J$, $Q_x(i) = P_i(x)$ if $x \in \mathscr{E}_i$ and $Q_x(i) = 0$ if $x \notin \mathscr{E}_i$. Then for each $x \in \mathscr{E}$, $Q_x \in {}^*[0, 1]$ where ${}^*[0, 1] = \{F \in {}^*\mathrm{Re} \mid 0 \lesssim F \lesssim 1\}$. Since for each $x \in \mathscr{E}$ Q_x is finite (Definition 4.5), let Q be the function from $\mathscr{E}$ into Re such that for each $x \in \mathscr{E}$, $Q(x) = {}^\circ(Q_x)$. Then by Theorem 4.5, for each $x \in \mathscr{E}$, $Q(x) \in [0, 1]$. We will now show that Q is a weak probability representation for $\langle X, \mathscr{E}, \gtrsim\rangle$. (i) Since $\{i \in J \mid P_i(X) = 1\} = \{i \in J \mid Q_X(i) = 1\} = J \in \mathscr{U}$, $Q_X \sim 1$. Therefore, by Theorem 4.5 $Q(X) = {}^\circ(Q_X) = 1$. Similarly, $Q(\varnothing) = 0$. (ii) Suppose that $x, y \in \mathscr{E}$ and $x > y$. Then let $q \in J$ be such that for each $i \in J$ such that $i \geqslant q$, $x, y \in \mathscr{E}_i$.

Then $\{i \mid i \in J \text{ and } Q_x(i) > Q_y(i)\} \supseteq \{i \geqslant q \mid i \in J \text{ and } P_i(x) > P_i(y)\} = \{i \mid i \in J$ and $i \geqslant q\} \in \mathscr{U}$. Therefore, as members of *Re, $Q_x \succ Q_y$. Therefore, by Theorem 4.5 $Q(x) = {}^\circ(Q_x) \geqslant {}^\circ(Q_y) = Q(y)$. That is, $Q(x) \geqslant Q(y)$. Similarly it can be shown that if $u, v \in \mathscr{E}$ and $u \sim v$ then $Q(u) = Q(v)$. (iii) Suppose that $w, z \in \mathscr{E}$ and $w \cap z = \varnothing$. Let $p \in J$ be such that for each $i \in J$ and such that $i \geqslant p$, $w, z \in \mathscr{E}_i$. Then $\{i \mid i \in J$ and $Q_{w \cup z}(i) = Q_w(i) + Q_z(i)\} \supseteq \{i \geqslant p \mid i \in J \text{ and } P_i(w \cup z) = P_i(w) + P_i(z)\} = \{i \mid i \in J$ and $i \geqslant p\} \in \mathscr{U}$. Thus, $Q_{w \cup z} \mathrel{\dot\sim} Q_w + Q_z$. Therefore, by Theorems 4.5 and 4.6 $Q(w \cup z) = {}^\circ(Q_{w \cup z}) = {}^\circ(Q_w + Q_z) = {}^\circ(Q_w) + {}^\circ(Q_z) = Q(w) + Q(z)$. That is, $Q(w \cup z) = Q(w) + Q(z)$. By (i), (ii), and (iii) we have shown that Q is a weak probability representation for $\langle X, \mathscr{E}, \succsim \rangle$. Since for each $i \in J \mid P_i(a) - P(a)| \geqslant \epsilon$, $\{i \in J \mid |Q_a(i) - P(a)| \geqslant \epsilon\} = \{i \in J \mid |P_i(a) - P(a)| \geqslant \epsilon\} = J \in \mathscr{U}$. Thus, in $\langle *\mathrm{Re}, +, \cdot, \succsim \rangle$, $|Q_a - P(a)| \succsim \epsilon$. Since ${}^\circ(Q_a) = Q(a)$, by Theorem 4.4 we can conclude that $|{}^\circ(Q_a) - Q(a)| < \epsilon/2$. Therefore, $|Q(a) - P(a)| > \epsilon/2$. That is, $Q \neq P$. This is impossible since P is the unique weak probability representation for $\langle X, \mathscr{E}, \succsim \rangle$. Therefore, we can conclude that $\lim_{i \to \infty} P_i = P$.

(3) Since for each $i \in \mathrm{I}^+$ $\succsim_i$ is a weak order on $\mathscr{E}_i$ and $\langle X, \mathscr{E}, \succsim \rangle$ is trisplittable, by (2) there is an unique weak probability representation P for $\langle X, \mathscr{E}, \succsim \rangle$ and $P = \lim_{i \to \infty} P_i$. Since $\langle X, \mathscr{E}, \succsim \rangle$ satisfies the finite cancellation axioms and is Archimedean, by Theorem 5.4, P is a probability representation for $\langle X, \mathscr{E}, \succsim \rangle$.

For notational simplicity, the following theorem for additive conjoint measurement will be stated for the case of two components. Similar theorems are true for the general case of n-components, $n \geqslant 2$. The proof of Theorem 7.2 is similar to the proof of Theorem 7.1 and will be omitted.

THEOREM 7.2. *For each* $i \in \mathrm{I}^+$, *let* $A_i = B_i \times C_i$, $\succsim_i$ *be a reflexive relation on* A_i, *and* $\langle A_i, \succsim_i \rangle$ *be a finite additive conjoint structure with a set of strict representation functions* Φ_i, Ψ_i. *Suppose that* $a_1 b_1$, $a_2 b_2$ *are in* A_1 *and for each* $i, j \in \mathrm{I}^+$ *such that* $i < j$, $A_i \subseteq A_j$, $\succsim_i \subseteq \succsim_j$, $A = \bigcup_{i \in \mathrm{I}^+} A_i$, $\succsim = \bigcup_{i \in \mathrm{I}^+} \succsim_i$, $\Phi_i(b_1) = 1$, *and* $\Phi_i(a_1) = \Psi_i(a_2) = 0$. *Then the following three propositions are true*:

(1) *If* $\langle A, \succsim \rangle$ *is bounded by* b_1, a_1, a_2 *then there is a set of representation functions for* $\langle A, \succsim \rangle$.

(2) *If* $\langle A, \succsim \rangle$ *is trisplittable and bounded by* b_1, a_1, a_2 *and for each* $i \in \mathrm{I}^+$, $\succsim_i$ *is a weak order on* A_i, *then there is an unique set of representation functions* Φ, Ψ *for* $\langle A, \succsim \rangle$ *such that* $\Phi(b_1) = 1$ *and* $\Phi(a_1) = \Psi(a_2) = 0$. *Furthermore,* $\Phi = \lim_{i \to \infty} \Phi_i$ *and* $\Psi = \lim_{i \to \infty} \Psi_i$.

(3) *If* $\langle A, \succsim \rangle$ *is trisplittable and Archimedean and for each* $i \in \mathrm{I}^+$, $\succsim_i$ *is a weak order on* A_i, *then there is an unique set of strict representation functions,* Φ, Ψ, *for* $\langle A, \succsim \rangle$ *such that* $\Phi(b_1) = 1$ *and* $\Phi(a_1) = \Psi(a_2) = 0$. *Furthermore,* $\Phi = \lim \Phi_i$ *and* $\Psi = \lim \Psi_i$.

It has been shown by Scott and Suppes [1958] that the finite cancellation axioms (Definition 2.6) are not derivable from the kth cancellation axiom for any $k \in \mathrm{I}^+$.

8. Historical Note

The formulation of the finite cancellation axioms (Definition 2.6) and the proof of the representation theorem (Theorem 2.5) appear in various forms in Scott [1964], Tversky [1964], and Adams [1965].

The discovery of necessary and sufficient conditions for finite qualitative probability structures is due to Kraft *et al.* [1959]. Scott later reformulated and proved these results in Scott [1964].

The ultraporudct construction was introduced in Łoš [1955]. Many uses of ordered field extensions of the reals for the elicitation of properties of the reals can be found in Robinson [1966].

Axioms, representation theorems, and uniqueness results for qualitative probability structures have been considered by Savage [1954], deFinetti [1937], Koopman [1940a, b], and Luce [1967]. All of these axiom systems use logically stronger assumptions than those that are presented in Section 5 to prove essentially the same theorems. The above papers only consider Archimedean structures.

Additive conjoint structures have been considered in various forms by Adams and Fagot [1959], Debreu [1960], Aczél, Belousov, and Hosszú [1960] and Aczél, Pickert, and Raó [1960], Luce and Tukey [1964], and Luce [1966]. The above papers use stronger assumptions than those presented in Section 6. Luce and Tukey [1964] consider a representation theorem for the non-Archimedean case.

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