LOUIS NARENS

ON QUALITATIVE AXIOMATIZATIONS FOR PROBABILITY THEORY*

Abstract. In the literature, there are many axiomatizations of qualitative probability. They all suffer certain defects: either they are too nonspecific and allow nonunique quantitative interpretations or are overspecific and rule out cases with unique quantitative interpretations. In this paper, it is shown that the class of qualitative probability structures with nonunique quantitative interpretations is not first order axiomatizable and that the class of qualitative probability structures with a unique quantitative interpretation is not a finite, first order extension of the theory of qualitative probability. The idea behind the method of proof is quite general and can be used in other measurement situations.

1. INTRODUCTION

Qualitative probability, whose formal roots go back to de Finetti [1937], is concerned with the axiomatizations of order relations, $\succsim$, on Boolean algebras that are compatible with probability functions on the same algebra. $\succsim$ is interpreted as "least as likely as", and "compatibility" here means the relevant probability functions P on the Boolean algebra homomorphically imbeds $\succsim$ into the numerical ordering $\geqslant$ on the reals, i.e., if $A \succsim B$ then $P(A) \geqslant P(B)$. The goal of the qualitative approach is to provide a sound axiomatic basis for the classical quantitative Kolmogorov [1933] theory of probability in terms of the more direct and basic ordering, $\succsim$. In addition, the qualitative approach provides a powerful method for the scrutinization and revelation of underlying assumptions of probability theory, is a link to empirical probabilistic concerns, and is a point of departure for the formulation of alternative probabilistic concepts.

DEFINITION 1. Let $\mathscr{A} = \langle \mathscr{E}, \cup, \cap, X, \phi \rangle$ be a Boolean algebra where $\cup$ is the join operation, $\cap$ the meet operation, X the maximal element, and ϕ the minimal element. (We will also use the symbols "$\cup$" and "$\cap$" to denote the union and intersection of sets, and also use the symbol "ϕ" to denote the empty set. The context will make clear which interpretations of these symbols are intended.) A *probability function* for $\mathscr{A}$ is a function P from $\mathscr{E}$ into $[0, 1]$ such that

Journal of Philosophical Logic 9 (1980) 143–151. 0022–3611/80/0092–0143$00.90

(*i*) $P(X) = 1, P(\phi) = 0,$

and

(*ii*) for each A, B in $\mathscr{E}$, if $A \cap B = \phi$,

then

$$P(A \cup B) = P(A) + P(B).$$ □

The concept of probability function presented in Definition 1 is often called a *finitely additive* probability function since it doesn't satisfy the Kolmogorov axiom of *σ-additivity*, i.e., for each sequence A_i of elements of $\mathscr{E}$ such that $A_i \cap A_j = \phi$ for $i \neq j$,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

However, from empirical and philosophical points of view, σ-additivity is the most difficult of the Kolmogorov axioms to justify, and although it provides a rich structure for the mathematical theory, it is not of critical importance for the empirical and philosophical theory. Thus we follow the practice of many others and discard this assumption from our treatment of probability theory.

DEFINITION 2. $\mathscr{B} = \langle \mathscr{E}, \cup, \cap, X, \phi, \succsim \rangle$ is said to be a *Boolean algebra with an ordering relation* if and only if $\langle \mathscr{E}, \cup, \cap, X, \phi \rangle$ is a Boolean algebra and $\succsim$ is a reflexive binary relation on $\mathscr{E}$. $\succsim$ is called the (*qualitative*) *ordering* of $\mathscr{B}$. $\mathscr{B}$ is said to be *weakly ordered* if and only if $\succsim$ is a *weak ordering*, i.e., if and only if $\succsim$ is a transitive and connected relation. For each A, B in $\mathscr{E}$, we define $A \sim B$ and $A \succ B$ as follows:

$$A \sim B \text{ iff } A \succsim B \text{ and } B \succsim A,$$

and

$$A \succ B \text{ iff } A \succsim B \text{ and not } B \succsim A.$$ □

DEFINITION 3. Let $\mathscr{B} = \langle \mathscr{E}, \cup, \cap, X, \phi, \succsim \rangle$ be a Boolean algebra with an ordering relation. A *probability representation* for $\mathscr{B}$ is a probability function P for $\langle \mathscr{E}, \cup, \cap, X, \phi \rangle$ such that the following two conditions hold for each A, B in $\mathscr{E}$:

(*i*) if $A \succ B$ then $P(A) > P(B)$;

(*ii*) if $A \sim B$ then $P(A) = P(B)$.

P is said to be a *weak probability representation* for $\mathscr{B}$ if and only if P is a probability function for $\langle \mathscr{E}, \cup, \cap, X, \phi \rangle$ and the following two conditions hold for all A, B in $\mathscr{E}$:

(*i'*) if $A \succ B$ then $P(A) \geqslant P(B)$;

(*ii'*) if $A \sim B$ then $P(A) = P(B)$. □

Weak probability representations naturally arise in probability theory. For example, for the closed interval $[0, 1]$ with the uniform distribution, the event $[0, \frac{1}{2}]$ is a little more likely to occur than the event $[0, \frac{1}{2}] - \{\frac{1}{2}\}$, although the probability of both events is $\frac{1}{2}$. Intuitively, the event $\{\frac{1}{2}\}$ has some chance – an infinitesimal chance – of occurring. Weak probability representations are useful in handling non-Archimedean situations like this.

The theory of Boolean algebras with an ordering relation is easily formulatable in first order predicate calculus. Scott [1964] gave a set of necessary and sufficient first order conditions for *finite* Boolean algebras with ordering relations to have probability representations. These conditions, called the *finite cancellation axioms*, are infinite in number and it has been shown by Scott and Suppes [1958] that no finite subset of them will imply the existence of a probability representation. For infinite Boolean algebras with ordering relations, the finite cancellation axioms are not sufficient for the existence of a probability representation; some Archimedean condition must be added to insure that non-null, infinitesimally likely events cannot occur. Since Archimedean conditions are not axiomatizable in first order languages (Narens [1974b]), there is no extension of Scott's representation theorem to infinite structures. However, Narens [1974a] showed that the finite cancellation axioms imply the existence of weak probability representations. This latter result shows that for the first order theory of Boolean algebras with ordering relations, *weak probability representation* rather than *probability representation* is the natural quantitative concept. These considerations give rise to the following definition:

DEFINITION 4. Let $\mathscr{B} = \langle \mathscr{E}, \cup, \cap, X, \phi, \succsim \rangle$ be a Boolean algebra with an ordering relation. $\mathscr{B}$ is said to be a *qualitative probability structure* if and only if it has a weak probability representation.

The following theorem immediately follows from Narens [1974a].

THEOREM A. *The class of qualitative probability structures is first order axiomatizable.*

However, if qualitative probability is to be the qualitative version of probability functions then the above approach is clearly inadequate since weak probability representations are in general not unique. Axiomatizations of more restricted forms of qualitative probability that have unique probability representations have been given by de Finetti [1937], Koopman [1940a, b], Savage [1954], and Luce [1967]. All of these axiomatizations use an Archimedean condition and are sufficient but not necessary for the existence of a unique probability representation. Narens [1974a] gives sufficient but not necessary first order axioms for the existence of a unique weak probability representation. The problems and methods of obtaining uniqueness results in qualitative probability are considered in Cohen [1978].

In this paper, we will show that the class of qualitative probability structures with nonunique weak probability representations is not first order axiomatizable, and that no finite set of axioms added to a first order axiomatization of the theory of qualitative probability structures will yield an axiomatization of the class of qualitative probability structures with unique representations.

2. THE THEOREMS

We will first construct a sequence of structures based upon subsets of [0, 1]. This sequence will be used in an important way in the proof of Theorem 1.

Recall that Borel subsets of [0, 1] form equivalence classes, where two Borel measurable subsets are considered equivalent if and only if they are identical except for a subset of Borel measure 0. We will follow the usual mathematical practice of ignoring subsets of Borel measure 0. This practice, which harmlessly confuses notation a little, allows for simpler and more readable notation. Thus, for probabilistic purposes, the Borel sets (0, 1) and [0, 1] are considered identical, and the Borel sets $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are considered disjoint.

For each positive integer n and each positive integer k such that $1 \leqslant k \leqslant n$, let

$$A_n^k = \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

and let $\langle \mathscr{E}_n, \cup, \cap, [0, 1], \phi \rangle$ be the Boolean algebra generated by the Borel sets $A_n^1, A_n^2, \ldots, A_n^n$. Define the binary relation $\succsim_n$ on $\mathscr{E}_n$ as follows: for each A, B in $\mathscr{E}_n$,

$$A \succsim B \text{ iff } \mu(A) \geqslant \mu(B),$$

where μ is the Borel measure on [0, 1]. Let P_n be the restriction of μ to $\mathscr{E}_n$. Then

$$\mathscr{B}_n = \langle \mathscr{E}_n, \cup, \cap, [0, 1], \phi, \succsim_n \rangle$$

is a weakly ordered qualitative probability structure with probability representation P_n. It is easy to show that $P_n(A_n^k) = 1/n$ for $k = 1, \ldots, n$, that P_n is the unique probability representation for $\mathscr{E}_n$, and that P_n takes on the values and only the values k/n for $k = 1, \ldots, n$. Also, it immediately follows from the definition of $\mathscr{B}_n$ that $\mathscr{B}_n$ is a substructure of $\mathscr{B}_{(n+1)!}$.

Let $\mathscr{E} = \bigcup_{i=1}^{\infty} \mathscr{E}_i$ and $\succsim = \bigcup_{i=1}^{\infty} \succsim_i$. Let P be the restriction of the Borel measure μ to $\mathscr{E}$. Then for each A in $\mathscr{E}$, if $A \in \mathscr{E}_n$ for some n, then $P(A) = P_n(A)$. Using this latter result, it is to verify that

$$\mathscr{B} = \langle \mathscr{E}, \cup, \cap, [0, 1], \phi, \succsim \rangle$$

is a weakly ordered, qualitative probability structure, and that P is a probability representation for $\mathscr{B}$. Also, it is easy to show that P takes on and only takes on values of the form k/n where k and n are positive integers such that $k \leqslant n$.

Let α be an irrational number and $0 < \alpha < 1$. Let $Z = [0, \alpha]$, $\langle \mathscr{F}_n, \cup, \cap, [0, 1], \phi \rangle$ be the Boolean algebra generated by $\mathscr{E}_n$ and $\{Z\}$, and let $\langle \mathscr{F}, \cup, \cap, [0, 1], \phi \rangle$ be the Boolean algebra generated $\mathscr{E}$ and $\{Z\}$. Then it easily follows that $\mathscr{F} = \bigcup_{i=1}^{\infty} \mathscr{F}_i$. Let Q be the restriction of the Borel measure μ on [0, 1] to $\mathscr{F}$. Define the binary relation $\succsim'$ on $\mathscr{F}$ as follows: for each A, B in $\mathscr{F}$,

$$A \succsim' B \text{ iff } Q(A) \geqslant Q(B).$$

Let

$$\mathscr{B}' = \langle \mathscr{F}, \cup, \cap, [0, 1], \phi, \succsim' \rangle.$$

Then it easily follows that $\mathscr{B}'$ is a weakly ordered, qualitative probability structure and Q is a weak probability representation for $\mathscr{B}'$. Let $\succsim'_n$ be the restriction of $\succsim'$ to $\mathscr{F}_n$. Let

$$\mathscr{B}'_n = \langle \mathscr{F}_n, \cup, \cap, [0, 1], \phi, \succsim'_n \rangle,$$

and let Q_n be the restriction of Q to $\mathscr{F}_n$. Then $\mathscr{B}'_n$ is a weakly ordered, qualitative probability structure, and Q_n is a weak representation for $\mathscr{B}'_n$, and $Q_n(Z)$ is the irrational number α. Furthermore, $\mathscr{B}'_n \subseteq \mathscr{B}'_{(n+1)!}$ and $\mathscr{B}' = \bigcup_{i=1}^{\infty} \mathscr{B}'_{i!}$ (i.e., $\mathscr{B}'_n$ is a substructure of $\mathscr{B}_{(n+1)!}$ and $\mathscr{B}'$ is the union of the chain of structures $\mathscr{B}'_{i!}$).

Observe that $A_n^k, k = 1, \ldots, n$, are atomic elements of the Boolean algebra $\mathscr{E}_n$, and that in the structure $\mathscr{B}_n, A_n^k \sim A_n^j$ for all j, k such that $1 \leqslant j, k \leqslant n$. It easily follows from this that all weak probability representations for $\mathscr{B}_n$ must be identical, i.e., that P_n is the unique weak probability representation for $\mathscr{B}_n$. Since $\mathscr{E} = \bigcup_{i=1}^{\infty} \mathscr{E}_i$ and for each positive integer n, $\mathscr{B}_n$ is a substructure of $\mathscr{B}$, it follows that P is the unique weak probability representation for $\mathscr{B}$.

For each positive integer n, $\mathscr{B}'_n$ does not have a unique probability representation. There are many ways to show this, but perhaps the easiest for our purposes is to appeal to the method given in Chapter 9 of Krantz *et al.* [1971] for constructing probability representations for finite qualitative probability structures. This method (which uses solutions to finite sets of homogeneous linear inequalities) allows one to always construct for finite qualitative probability structures probability representations that take on only rational values, and since Q_n takes on the irrational value α, there must be at least two distinct probability representations for $\mathscr{B}'_n$.

However, the infinite structure $\mathscr{B}'$ does have a unique probability representation. This can easily be seen by observing that P is the unique weak probability representation for $\mathscr{B}$ and P takes on every rational value in $[0, 1]$. Thus for each event B in $\mathscr{F}$, there exist sequences of events B_i, C_i in $\mathscr{E}$ such that $\lim[P(B_i) - P(C_i)] = 0$ and $B_i \succsim' B \succsim' C_i$, and it therefore follows that for each weak probability representation R of $\mathscr{B}'$, $R(B) = \lim P(B_i)$, and thus the uniqueness of R follows from the uniqueness of P.

Letting $\mathscr{B}' = \mathscr{C}$ and $\mathscr{B}'_{k!} = \mathscr{C}_k$, we summarize these results in the following lemma:

LEMMA 1. *There exists a weakly ordered qualitative probability structure* $\mathscr{C}$ *and a sequence of weakly ordered qualitative probability structures* $\mathscr{C}_k$ *such that the following four conditions hold for all positive integers i and j:*

(i) *if* $i > j$ *then* $\mathscr{C}_i \supseteq \mathscr{C}_j$;

(ii) $\mathscr{C} = \cup_{k=1}^{\infty} \mathscr{C}_k$;

(iii) *each* $\mathscr{C}_k$ *has a nonunique weak probability representation;*

(iv) $\mathscr{C}$ *has a unique weak probability representation that takes on all rational values in* [0, 1].

We are now in the position to prove the main result:

THEOREM 1. *The class of qualitative probability structures with nonunique weak probability representations is not first order axiomatizable.*

Proof. Suppose Δ is a set of first order axioms for the class of qualitative probability structures with nonunique weak probability representations. A contradiction will be shown. Let $\mathscr{C}$ and $\mathscr{C}_i$ be as in Lemma 1. Let $\mathscr{U}$ be a nonprincipal ultrafilter of the Boolean algebra of subsets of the positive integers, I^+, and let $^*\mathscr{C}$ be the $\mathscr{U}$-ultraproduct of $\{\mathscr{C}_i\}_{i \in I^+}$. Then $\mathscr{C}_i$ satisfies all the axioms of Δ for each i in I^+, by Łoś's Theorem, $^*\mathscr{C}$ has two distinct weak probability representations. Similarly, $^*\mathscr{C}$ is weakly ordered since by hypothesis $\mathscr{C}_i$ weakly ordered for each i in I^+.

Let

$$\mathscr{C} = \langle \mathscr{G}, \cup, \cap, X, \phi, \succsim \rangle,$$

$$^*\mathscr{C} = \langle ^*\mathscr{G}, {}^*\cup, {}^*\cap, X, \phi, {}^*\succsim \rangle,$$

$$\mathscr{C}_i = \langle \mathscr{G}_i, \cup_i, \cap_i, X, \phi, \succsim_i \rangle,$$

and let d, e, f be arbitrary elements of $\mathscr{G}$. Let ι_d be the function from I^+ into $\mathscr{G}$ such that for all j in I^+, $\iota_d(j) = d$. Since $\mathscr{C}_i \subseteq \mathscr{C}_{i+1}$ for all i in I^+ and $\mathscr{C} = \cup_{i=1}^{\infty} \mathscr{C}_i$, it follows that d, e, and f are in $\mathscr{C}_j$ for all but finitely many j in I^+. It also follows that $d \cup e = f$ if and only if $d \cup_j e = f$ for all but finitely many j in I^+. Thus since $\mathscr{U}$ is a non-principal ultrafilter, it is easy to establish that the function F from $\mathscr{G}$ into $^*\mathscr{G}$ defined by

$$F(c) = \text{the } \mathscr{U}\text{-equivalence class that contains } \iota_c$$

is an isomorphic imbedding of $\mathscr{C}$ into $^*\mathscr{C}$.

Since the previous paragraph establishes that $\mathscr{C}$ is isomorphically imbeddable in $^*\mathscr{C}$, we may assume, without loss of generality, that $\mathscr{C}$ is a substructure of $^*\mathscr{C}$. By hypothesis, let R be the unique weak probability representation of $\mathscr{C}$, and let R_1 and R_2 be two distinct weak probability representations for $^*\mathscr{C}$. Since R is unique, $R = R_1 \upharpoonright \mathscr{G} = R_2 \upharpoonright \mathscr{G}$. Since $R_1 \neq R_2$, let a in $^*\mathscr{G}$ be such that $R_1(a) \neq R_2(a)$. Without loss of generality, suppose $R_1(a) > R_2(a)$. Since by hypothesis R takes on all rational values in $[0, 1]$, let b in $\mathscr{G}$ be such that $R_1(a) > R(b) > R_2(a)$. Since $R_1(b) = R(b) = R_2(b)$, it then follows that

(1) $\qquad R_1(a) > R_1(b)$

and

(2) $\qquad R_2(b) > R_2(a)$.

Since $^* \succsim$ is a weak ordering on $^*\mathscr{G}$ and R_1 and R_2 are weak probability representations, it follows from Equation (1) that $a \,^*\!\succ b$ and from Equation (2) that $b \,^*\!\succ a$, which is impossible. □

THEOREM 2. *Let* Γ *be a set of first order axioms for the theory of qualitative probability structures (e.g.,* Γ *be the set of finite cancellation axioms of Scott* [1964]). *Then there does not exist a finite set of first order axioms* Σ *such that* $\Gamma \cup \Sigma$ *is an axiomatization for the class of qualitative probability structures with unique weak probability representations.*

Proof. Suppose Σ is finite and $\Gamma \cup \Sigma$ is a first order axiomatization for the theory of qualitative probability structures with unique weak probability representations. A contradiction will be shown. Since Σ is finite, let $\Sigma = \{\theta_1, \ldots, \theta_n\}$. Let θ be the conjunction of $\theta_1, \ldots, \theta_n$, i.e., $\theta = \theta_1 \wedge \ldots \wedge \theta_n$. Then $\Gamma \cup \{\neg\theta\}$ axiomatizes the theory of qualitative probability structures with nonunique weak probability representations, and this is impossible by Theorem 1. □

3. DISCUSSION

Qualitative axiomatizations of quantitative models is an important part of measurement theory, and in fact Krantz, Luce, Suppes, and Tversky's

Foundations of Measurement consists almost entirely of such axiomatizations. Since uniqueness of quantitative interpretations also plays a central role in measurement theory, axiomatic problems involving uniqueness have intrinsic theoretical interest. The method of proof for Theorems 1 and 2 does not rely in an essential way upon probabilistic concerns; rather it is the convergence and uniqueness/nonuniqueness properties of the sequence of structures mentioned in Lemma 1 that allow the proofs to work. Since there are other important measurement situations where sequences with such properties can be found, the method of proof presented in this paper can be extended to these other situations and analogous theorems obtained.

University of California, Irvine

NOTE

*This research was partially supported by the National Science Foundation grant NSF BNS7702911 and by the joint NSF-NIE grant NSF SED 78-22271 to the University of California, Irvine.

REFERENCES

Cohen, M.: 'On the characterization of qualitative probability spaces', manuscript, 1978.

de Finetti, B.: 1937, 'La prevision: ses lois logiques, ses sources subjectives,' *Ann. Inst. H. Poincaré*, 7, 1–68. Translated into English in H. E. Kyburg, Jr., & H. E. Smokler (eds.), *Studies in Subjective Probability*, Wiley, New York, 1964, pp. 93–158.

Kolmogorov, A. N.: 1933, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer, Berlin. English translation by N. Morrison, *Foundations of the Theory of Probability*, Chelsea, New York, 1956.

Koopman, B. O.: 1940a, 'The bases of probability,' *Bull. Amer. Math. Soc.* 46, 763–774. Reprinted in H. E. Kyburg, Jr., & H. E. Smokler (eds.), *Studies in Subjective Probability*, Wiley, New York, 1964, pp. 159–172.

Koopman, B. O.: 1940b, 'The axioms and algebra of intuitive probability,' *Ann. of Math.* 41, 269–292.

Krantz, D. H., Luce, R. D., Suppes, P., & Tversky, A.: 1971, *Foundations of Measurement, Vol. I*, Academic Press, New York.

Luce, R. D.: 1967, 'Sufficient conditions for the existence of a finitely additive probability measure,' *Ann. Math. Statist.* 38, 780–786.

Narens, L.: 1974a, 'Minimal conditions for additive conjoint measurement and qualitative probability,' *J. Math. Psychol.* 11, 404–430.

Narens, L.: 1974b, 'Measurement without Archimedean axioms,' *Phil. Sci.* 41, 374–393.

Savage, L. J.: 1954, *The Foundations of Statistics*, Wiley, New York.

Scott, D.: 1964, 'Measurement models and linear inequalities,' *J. Math. Psychol.* 1, 233–247.

Scott, D. & Suppes, P.: 1958, 'Foundational aspects of theories of measurement,' *J. Symbolic Logic* 23, 113–128.