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A Meaningful Justification for the Representational Theory of Measurement

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Measurement, the means by which numbers enter science, is of fundamental importance to modern science. The relationship between its qualitative and quantitative aspects has generated many theories and much controversy. In 19th century geometry similar developments led mathematician Felix Klein to devise a theory for unifying qualitative and quantitative approaches to geometry. Klein's theory, which today is called the Erlanger program, was based on transformation groups. In this article, the Erlanger program is given a new foundation based on mathematical logic and is extended to science. The current dominant theory of measurement in the literature, the representational theory, is then justified in terms of the new foundation for the Erlanger program. Certain inferential techniques used in dimensional analysis and the related technique of possible psychophysical laws are also given justifications in terms of the new foundation.

1. INTRODUCTION

This article presents a new foundation for a familiar theory of measurement. The foundation takes "meaningfulness" as an undefined term, states axioms about it, and then shows that the only "meaningful" forms of measurement are equivalent to the representational theory; i.e., each *meaningful* set of measuring functions on a qualitative domain A has a characterization as a set of structure preserving mappings from a qualitative structure with domain A into a purely mathematical structure.

The axioms are designed to reflect the following common practice in science: mathematics may be freely employed in scientific formulation, deduction, and inference. Definitions that allow the free use of mathematics are called *scientific definitions*. The axioms say that an entity is meaningful with respect to a qualitative structure if and only if it is scientifically defined in terms of the qualitative

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structure. The set of meaningful entities generated by the qualitative structure is called a *scientific topic*.

Let $\mathfrak{X}$ be a qualitative structure describing a geometry. In mathematics, the geometric content of $\mathfrak{X}$ has been identified with the set of entities left invariant by the transformations that leave the primitives of $\mathfrak{X}$ invariant. For example, in the Euclidean geometry of the plane, the transformations that leave the primitive Euclidean concepts (e.g., point, line, circle, intersection, congruence) invariant are those that are composed of translations, rotations, and reflections. According to this view of geometry, the Euclidean concepts (e.g., midpoint of a line segment) are those that are left invariant by these transformations. This article presents theorems showing the equivalence of geometric content and scientific topic. Thus the theory of meaningfulness presented here also provides a new foundation for the concept of *geometric content*, which is a key component of the famous program developed by mathematician Felix Klein for geometry, called the Erlanger program.

Inferential techniques based on invariance have appeared in geometry, physics, and the behavioral sciences. While it is agreed that these techniques often produce important insights, the literature lacks reasonable justifications for them. Results of this article suggest that their *epistemology* is better understood in terms of scientific definability than its shown equivalent, invariance, while their *mathematical power* is best understood in terms of invariance.

2. KLEIN'S ERLANGER PROGRAM

DEFINITION 1. Let X be a nonempty set. By definition, a *permutation* on X is a one-to-one function from X onto X. A subset G of permutations on X is said to be a *permutation group* if and only if the identity function, ι, is in G, and if f and g are in G, then the functional composition of f and g, $f * g$, is in G, and the inverse of f, f^{-1}, is in G. In the literature, permutation groups are often called *transformation groups or transformational groups*.

For most of the history of mathematics, geometry was the study of physical space, which was conceptualized as Euclidean 3-dimensional geometry. However, by the middle of the 19th century, Euclidean geometry no longer received universal acceptance from mathematicians as being necessarily descriptive of physical space. A little later in the century, analysis provided the basis for modeling Euclidean and non-Euclidean geometries. In such analytic geometries, it was difficult to distinguish geometric concepts from nongeometric ones. Before, when geometry was based on physical space, metaphysical principles about physical reality could be invoked to define what was geometrical. However, in analytical geometry no analogous program could be carried out, because the methods of analysis did not provide the necessary insight into the geometrical nature of things. This problem led mathematician Felix Klein to develop a program for defining and manipulating the geometrical content within an analytic geometry.

Klein's method consisted of identifying a geometry with the invariants under a permutation group. The following quotation from Veblen and Young (1946) illustrates how geometers employed this identification to transfer results and concepts from one geometric structure to another:

> At each step we have helped ourselves forward by transferring the results of one geometry to another, combining these with easily obtained theorems of the second geometry, and thus extending our knowledge of both. This is one of the characteristic methods of modern geometry. It was perhaps first used with a clear understanding by O. Hesse [*Gesammelte Werke*, p. 531], and was formulated as a definite geometrical principle (Übertragungsprinzip) by F. Klein [1872].
>
> This principle of transference or of carrying over the results of one geometry to another may be stated as follows: *Given a set of elements* [e] *and a group G of permutations of these elements, and a set of theorems* [T] *which state relations left invariant by G. Let* [e'] *be another set of elements, and G′ a group ofpermutations of* [e']. *If there is a one-to-one reciprocal correspondence between* [e] *and* [e'] *in which G is simply isomorphic with G', the set of theorems* [T] *determines by a mere change of terminology a set of theorems* [T'] *which state relations among elements e′ which are left invariant by G′.*
>
> This principle becomes effective when the method by which [e] and G are defined is such as to make it easy to derive theorems which are not so easily seen for [e'] and G'. This has been abundantly illustrated in the present chapter ...
>
> DEFINITION. Given a set of elements [e] and a group G of permutations of [e], the set of theorems [T] which states relations among the elements of [e] which are left invariant by G and are not left invariant by any group of transformations containing G is called a *generalized geometry or a branch of mathematics.*
>
> This is, of course, a generalization of the definition of a geometry [by Klein, 1872] .... pp. 284–S85)

Instead of the terms *generalized geometry* or *branch of mathematics*, the more neutral term *scientific topic* is employed throughout this article. This is appropriate because the article is primarily concerned with scientific content.

Historically, the identification of a scientific topic with the invariants of a group has been a very potent epistemological principle in mathematics and physics. However, Klein, geometers, and others who have employed this identification have provided only very vague justifications for it. This article presents a generalization based on mathematical logic of Erlanger's approach to a scientific topic. The idea behind the generalization is that a scientific topic is propagated by a special form of definability.

3. MEANINGFULNESS

3.1. *Scientific Topics*

Fundamental to the theory presented in this article is that certain relations and concepts belong to a fragment of science and others do not. Those that belong will be called *meaningful* (with respect to the fragment) and those that do not *meaningless* (with respect to the fragment). A *theory of meaningfulness* consists of giving necessary or sufficient conditions for meaningfulness. Because theories of meaningfulness are metascientific and philosophical in nature, different theories arise naturally out of various epistemological and metaphysical positions typically taken about science. Several formal theories of meaningfulness are presented and discussed in Narens (2002).

This article presents a theory of meaningfulness that extends and generalizes the Erlanger program to scientific domains. The theory, which provides a precise description of scientific topic, is then used to justify the representational theory of measurement, currently the dominant theory of measurement.

The meaningfulness concept is based on the following four intuitive principles about scientific topics.

Principle 1. Scientific topics have both qualitative and quantitative aspects. The qualitative aspects are relations and concepts formulated entirely in terms of relations and concepts on the domain A of the topic. These relations and concepts may be higher order, e.g., a relationship between five binary relations on A. The quantitative aspects include pure mathematics and relationships, including higher-order ones, between the qualitative aspects and pure mathematics, e.g., the set of functions from A into the real numbers.

Principle 2. The domain A of the scientific topic is a qualitative set belonging to the topic.

Principle 3. The scientific topic is closed under scientific definition; i.e., if $a_1, ..., a_n$ belong to the topic and a is defined scientifically in terms of $a_1, ..., a_n$, then a belongs to the scientific topic.

Principle 4. Pure mathematics may be used freely in scientific definitions.

Principles 1 and 2 reflect a common practice in mathematical science of using pure mathematics in the formulations of scientific concepts and the derivations of scientific results. Because of the nonempirical, nonqualitative nature of pure mathematics, Principles 3 and 4 allow relations and concepts to belong to the scientific topic that are neither qualitative nor empirical. Thus the concept of meaningfulness presented in this article should not identified with either empiricalness or qualitativeness.

3.2. *Formal Development*

Convention 1. Throughout this article the following conventions are observed: $\mathbb{R}$ denotes the set of real numbers, $\mathbb{I}^+$ the set of positive integers, $\varnothing$ the empty set, and $\in$ the set-theoretic membership relation. When Z is a set, the notation $\langle Z, \in \rangle$ will, by convention, denote the structure $\langle Z, G' \rangle$, where G' is the restriction of $\in$ to Z.

For each set x, $\mathfrak{P}(x)$ denotes the power set of x, i.e., the set of all subsets of x,

A denotes a nonempty set of objects. A is called the scientific domain and elements of A are called atoms. It is assumed that each atom in A is not a set. A is to be interpreted as a scientific domain of objects for the fragment of science under consideration.

W denotes the set of *nonmathematical entities* based on A. W is defined inductively as follows: $W_1 = A$, and for each positive integer n,

$$W_{n+1} = W_n \cup \mathfrak{P}(W_n) - \{\varnothing\},$$

and

$$W = \bigcup_{n \in \mathbb{I}^+} W_n.$$

In the intended interpretation, the nonmathematical entities belonging to the scientific topic will form a subset of W.

It is well known that for n in $\mathbb{I}^+$ and sets s and t in W the concepts of the set of n-ary relations on s and the set of functions from s into t can be formulated in terms of the entities in W and the $\in$-relation of set theoretic membership and are in W. For example, an ordered pair (x, y) of elements x and y of W is defined to be the set

$$a = \{\{x\}, \{x, y\}\},$$

with the first element of the ordered pair, x, being the element of W belonging to both elements of a and the second element of the ordered pair, y, being the element of W belonging to exactly one element of a. An ordered triplet (x, y, z) is defined to be $((x, y), z)$ and ordered n-tuples for $n > 3$ have analogous definitions. Functions are certain kinds of sets of ordered pairs, and n-ary relations are sets of n-tuples. Similarly all the usual concepts associated with functions and relation, e.g., domain, range, composition of two functions, can be formulated for functions and relations in W in terms of entities of W and the $\in$-relation.

CONVENTION 2. For the purposes of this article, *pure mathematics* will correspond to certain sets that are built out of the empty set using the $\in$-relation and principles of set theory. Formally, let

$$P_1 = \varnothing,$$

and for each n in $\mathbb{I}^+$, let

$$P_{n+l} = P_n \cup \mathfrak{P}(P_n).$$

Let

$$P_1^* = \bigcup_{n \in \mathbb{I}^+} P_n.$$

For each n in $\mathbb{I}^+$, let

$$P_{n+1}^* = P_n^* \cup \mathfrak{P}(P_n^*),$$

and let

$$P = \bigcup_{n \in \mathbb{I}^+} P_n^*$$

For the purposes of this article, P is called the *set of pure sets* and the elements of P are called *pure sets*.

$\langle P, \in \rangle$ is a very powerful mathematical system. It is well known that most of ordinary mathematics is formulable in $\langle P, \in \rangle$ in the following sense: Structures and concepts of ordinary mathematics have isomorphic counterparts in $\langle P, \in \rangle$. For example, the finite ordinals, the set of finite ordinals, ω, and the finite ordinal successor function $a^+ = a \cup \{a\}$ are elements of P_2^* and therefore elements of P. These satisfy G. Peano's well-known axiom system for the natural numbers. Then, by using the well-known constructions of Peano and Dedekind, it is easy to show that the operations of addition and multiplication are definable on ω (within $\langle P, \in \rangle$) and that the resulting algebraic structure—which we will call the *set-theoretic natural numbers*—is isomorphic to the natural numbers with its usual operations of addition and multiplication. By taking ordered pairs of set-theoretic natural numbers and (in terms of the set-theoretic arithmetical operations of addition and multiplication) defining the appropriate operations of addition and multiplication on them (which is easy to do within $\langle P, \in \rangle$), another algebraic structure results—which we will call the *set-heoretic integers*—that is isomorphic to the usual system of integers with its usual operations of addition and multiplication. Similarly, by taking ordered pairs of the set-theoretic integers and defining for them the appropriate operations of addition and multiplication, the structure of *set-theoretic rational numbers* can be constructed, and it is easy to show that it is isomorphic to the rational numbers under addition and multiplication. By using similar, but little more involved constructions, the structure of *set-theoretic real numbers* can be constructed out of the set-theoretic rational numbers, and it can be shown that it is isomorphic to the structure of real numbers with its usual operations of addition and multiplication. And, of course, all these latter constructions can also be carried out in $\langle P, \in \rangle$. Then structures based on the real numbers, e.g., the space of continuous functions from the real numbers into itself, can be constructed in $\langle P, \in \rangle$, etc.

In set-theory the concept of pure set is much larger than P, e.g., it includes P, $\mathfrak{P}(P)$, $\mathfrak{P}(\mathfrak{P}(P))$, etc., and in fact it is too large to be realized as a set. However, for the purposes of this article, it is convenient to have the concept of pure set realizable by a set, and P is a good choice for this. If, for some reason, a larger set of pure sets were needed to represent a portion of mathematics that one might use in science, then that larger set could be substituted for P without changing in any essential way the theory presented in this article.

Convention 3. For the purposes of this article, ordinary mathematics will be identified with the structure of pure sets, $\langle P, \in \rangle$.

In order to effectively use pure mathematics in scientific applications involving a scientific domain A, the following axiom is employed to guarantee that A is imbeddable into a pure set.

Axiom 1. *There exists a one-to-one function from A onto an element of P.*

Convention 4. Both $\langle W, \in \rangle$ and $\langle P, \in \rangle$ are substructures of the *set-theoretic fragment* $\langle V, \in \rangle$, where V is defined inductively as

$$V_1 = A,$$

and for each n in $\mathbb{I}^+$, let

$$V_{n+1} = V_n \cup \mathfrak{P}(V_n),$$

and let

$$V_1^* = \bigcup_{n \in \mathbb{I}^+} V_n.$$

For each n in $\mathbb{I}^+$, let

$$V_{n+1}^* = V_n^* \cup \mathfrak{P}(V_n^*),$$

and let

$$V = \bigcup_{n \in \mathbb{I}^+} V_n^*.$$

For the purposes of this article, V is called the *set of scientific entities* and the elements of V are called *scientific entities*. Elements x of V such that there exists y in V such that $y \in x$ are called *(scientific) sets*. The only nonsets in V are the elements of A, i.e., are the atoms. The set of pure sets P and the set of nonmathematical entities W are proper subsets of V. (Note that, in particular, $\varnothing$ is not an element of W.)

The set-theoretic fragment $\langle V, \in \rangle$ contains additional concepts that are neither nonmathematical nor purely mathematical. E.g., if f is a function from a nonempty subset of A into the set-theoretic reals, then f is neither nonmathematical nor purely mathematical.

The set-theoretic fragment $\langle V, \in \rangle$ with the distinctions of nonmathematical and purely mathematical entities is a sufficient instantiation of Principle 1. To obtain Principles 2 to 4, an undefined term "meaningfulness" is introduced, and in terms of it the essential properties of a scientific topic encompassed by Principles 2 to 4 are formally introduced.

3.3. *Scientific Definability*

DEFINITION 2. Let $M(x)$ be a 1-place predicate. "$M(x)$" is to be read as "x is meaningful." Throughout the rest of this article, for purposes of readability "$M(x)$" and "x is meaningful" will often be used interchangeably.

It is assumed that the reader is familiar with the concept of first order language.

Let $\mathbf{L}(\mathbf{A}, \in)$ be a first order language (with identity), with an individual constant symbol, $\mathbf{A}$, and a binary predicate symbol, $\in$. (Except for the above, there are no other individual constant symbols or predicate symbols in $\mathbf{L}(\mathbf{A}, \in)$.) $\mathbf{L}(\mathbf{A}, \in)$ is used to describe the structure $\langle V, A, \in \rangle$. (In particular, $\mathbf{A}$ will be interpreted as A and $\in$ as the restriction of the $\in$-relation to V.) Let $a_1, \ldots, a_n$ and a be elements of V. Then a is said to be *scientifically defined in terms of* $a_1, \ldots, a_n$ if and only if there is a formula

$$\Theta(x, x_1, \ldots, x_n, y_1, \ldots, y_m)$$

of $\mathbf{L}(\mathbf{A}, \in)$ and pure sets $b_1, \ldots, b_m$ such that the following is a true statement about the structure $\langle V, A, \in \rangle$:

$\Theta(a, a_1, \ldots, a_n, b_1, \ldots, b_m)$ and for all x, if $\Theta(x, a_1, \ldots, a_n, b_1, \ldots, b_m)$, then $x = a$.

Note that M is not a predicate symbol of $\mathbf{L}(\mathbf{A}, \in)$.

The following axiom, which relates meaningfulness and scientific definability, is an instantiation of Principles 3 and 4:

AXIOM 2. *For all entities $a, a_1, \ldots, a_n$ in V, if $M(a_1), \ldots, M(a_n)$ and a is scientifically defined in terms of $a_1, \ldots, a_n$, then $M(a)$.*

Meaningful entities will be identified with the entities that belong to the scientific topic. Because A is the interpretation of $\mathbf{A}$, A is meaningful and thus Principle 2 holds:

LEMMA 1. *Assume Axiom 2. Then the following two statements are true:*

1. $M(A)$.
2. *For each pure set p, $M(p)$.*

Proof. Let p be an arbitrary pure set. Then the lemma immediately follows from Axiom 2 and the formulas $x = \mathbf{A}$ and $x = p$.

By themselves Axioms 1 and 2 do not provide satisfactory means for describing meaningful entities. This deficiency is eliminated by adding another axiom (Axiom 3 below). The addition of this axiom allows meaningfulness to be determined through a structure of primitive relations based on A.

3.4. *First-Order Structures*

DEFINITION 3. Let A be a nonempty set. Elements of A are called 0-*ary relations* on A. n-ary relations on A, $n \in \mathbb{I}^+$, are subsets of A^n (the Cartesian product of A with itself n times). In particular, 1-ary relations on A are subsets of A.

R is said to be a *first-order relation* on A if and only if R is a 0-ary relation or n-ary relation on A for some n in $\mathbb{I}^+$.

Note that the first-order relations on A include elements of A as well as A itself.

DEFINITION 4. $\mathfrak{X}$ is said to be a *first-order structure* if and only if $\mathfrak{X}$ is of the form

$$\mathfrak{X} = \langle A, R_j \rangle_{j \in J},$$

where A is a nonempty set, J is a pure set, and for each j in J, R_j is a first-order relation on A.

Let $\mathfrak{X} = \langle A, R_j \rangle_{j \in J}$ be a first-order structure. Then, by definition, the *primitives* of $\mathfrak{X}$ consist of A and R_j, $j \in J$. Also by definition, A is called the *domain* of $\mathfrak{X}$.

Let $\mathfrak{X} = \langle X, R_j \rangle_{j \in J}$ and $\mathfrak{Y} = \langle Y, S_k \rangle_{k \in K}$ be first-order structures such that for each j in J, R_j is a $m(j)$-ary relation. Then f is said to be an isomorphism of $\mathfrak{X}$ *onto*

$\mathfrak{Y}$ if and only if f is a one-to-one function from X onto Y, $J = K$, and for all $j \in J$ and all $x_1, \ldots, x_{m(j)}$ in X,

$$R_j[x_1, \ldots, x_{m(j)}] \quad \text{iff} \quad S_j[f(x_1), \ldots, f(x_{m(j)})].$$

g is said to be a *homomorphism* of $\mathfrak{X}$ *into* $\mathfrak{Y}$ if and only if g is a function from X into Y, $J = K$, and for all $j \in J$ and all entities $x_1, \ldots, x_{m(j)}$,

$$R_j[x_1, \ldots, x_{m(j)}] \quad \text{iff} \quad S_j[g(x_1), \ldots, g(x_{m(j)})].$$

(Note the use of "iff" in the above definition of "homomorphism.")

DEFINITION 5. Let $\mathfrak{X} = \langle A, R_j \rangle_{j \in J}$ be a first-order structure such that for each j in J, R_j is a $m(j)$-ary relation. Then f is said to be an *automorphism* of $\mathfrak{X}$ if and only if it is an isomorphism of $\mathfrak{X}$ onto $\mathfrak{X}$, i.e., if and only if f is a one-to-one function from A onto A such that for each j in J and each $a_1, \ldots, a_{m(j)}$ in A,

$$R_j[a_1, \ldots, a_{m(j)}] \quad \text{iff} \quad R_j[f(a_1), \ldots, f(a_{m(j)})].$$

DEFINITION 6. Let f be a one-to-one function from A onto a set N and let R be an n-ary relation on A. Then, by convention, $f(R)$ is the n-ary relation S on N such that for all $y_1, \ldots, y_n$ in N,

$S(y_1, \ldots, y_n)$ if and only if there exist $x_1, \ldots, x_n$ in A such that $f(x_1) = y_1, \ldots, f(x_n) = y_n$ and $R(x_1, \ldots, x_n)$.

LEMMA 2. *Let $\mathfrak{X} = \langle A, R_j \rangle_{j \in J}$ be a first-order structure and f be a one-to-one function from A onto N. Then f is an isomorphism of $\mathfrak{X}$ onto $\langle N, f(R_j) \rangle_{j \in J}$.*

Proof. Immediate from Definition 6.

3.5. *Invariance under Permutations*

A permutation f on A (Definition 1) can be extended naturally to subsets of A as follows: for each subset x of A, let

$$f(x) = \{f(a) \mid a \in x\}.$$

Using a similar idea, permutations on A can be extended naturally to V as follows:

DEFINITION 7. Let f be a permutation on A. Using the notation of Convention 4, f will be extended to a function $\bar{f}$ on V as follows: For each x in $V_1 = A$, let

$$f_1(x) = f(x),$$

and for each n in $\mathbb{I}^+$ and each y in $V_{n+1} - V_n$, let

$$f_{n+i}(y) = \{f_n(z) \mid z \in y\},$$

and let

$$f^* = \bigcup_{n \in \mathbb{I}^+} f_n.$$

For each x in V_1^*, let

$$f_1^*(x) = f^*(x),$$

and for each n in $\mathbb{I}^+$ and each y in $V_{n+1}^* - V_n^*$, let

$$f_{n+1}^*(y) = \{f_n^*(z) \mid z \in y\},$$

and let

$$\bar{f} = \bigcup_{n \in \mathbb{I}^+} f_n^*.$$

$\bar{f}$ is called the *extension* of f to V.

Let f be a permutation on A. Note that by Definition 7,

$$\bar{f}(\varnothing) = \{f(x) \mid x \in \varnothing\} = \varnothing.$$

THEOREM 1. *Let f be a permutation on A, and let f be the extension of f to V. Then the following six statements are true:*

1. $\bar{f}(A) = A$.
2. *For all x and y in V,*

$$x \in y \quad \textit{iff} \quad \bar{f}(x) \in \bar{f}(y).$$

3. *For all pure sets p, $\bar{f}(p) = p$.*
4. *For all ordered n-tuples $(a_1, \ldots, a_n)$ of elements of V,*

$$\bar{f}((a_1, \ldots, a_n)) = (\bar{f}(a_1), \ldots, \bar{f}(a_n)).$$

5. *$\bar{f}$ is a one-to-one function from V onto V.*
6. $\bar{f}^{-1} = \overline{f^{-1}}$.

The proof of Theorem 1 is not difficult. It follows by Definition 7 and an inductive argument. For details, see Jech (1973) or Chapter 3 of Narens (n.d.).

LEMMA 3. *Let f be a permutation on A, $\bar{f}$ be the extension of f to V, and $a_1, \ldots, a_n$ be elements of V such that for $i = 1, \ldots, n$,*

$$\bar{f}(a_i) = a_i.$$

Let a be in V,

$$\Theta(x, x_1, \ldots, x_n, y_1, \ldots, y_m)$$

be a formula of $\mathbf{L}(\mathbf{A}, \in)$, and $b_1, \ldots, b_m$ be pure sets in V. Suppose the following statement is true about the structure $\langle V, A, \in \rangle$:

> *$\Theta(a, a_1, \ldots, a_n, b_1, \ldots, b_m)$ and for all x, if $\Theta(x, a_1, \ldots, a_n, b_1, \ldots, b_m)$, then $x = a$.*

Then $\bar{f}(a) = a$.

Proof. By hypothesis, $\bar{f}(a_i) = a_i$, for $i = 1, \ldots, n$ and by Statement 3 of Theorem 1, $\bar{f}(b_k) = b_k$ for $k = 1, \ldots, m$. Thus by Statements 1 to 3 of Theorem 1, $\bar{f}$ is an automorphism of

$$\mathfrak{B} = \langle V, A, \in, a_1, \ldots, a_n, b_1, \ldots, b_m \rangle.$$

It is well known that automorphisms preserve definable relations (e.g., see Corollary 22D on page 93 of Enderton, 1972). Application of this to $\mathfrak{B}$ yields the following principle: For each automorphism φ of $\mathfrak{B}$ and each element e in V, if e is defined in terms of the primitives of $\mathfrak{B}$ through first-order logic, then $\varphi(e)$ has the same definition in terms of the primitives of $\mathfrak{B}$. The conclusion of the lemma follows by applying the above principle to $\mathfrak{B}$ and $\bar{f}$.

The following theorem links Klein's Erlanger program (which is based on invariance) with Axiom 2 (which is based on scientific definability).

THEOREM 2. *Suppose G is a permutation group on A. Define the predicate $M(x)$ as follows: For each a in V,*

$$M(a) \quad \textit{iff} \quad \bar{f}(a) = a \quad \textit{for each} \quad f \quad \textit{in} \quad G.$$

Then Axiom 2 is true.

Proof. First note that for the structure $\langle V, A, \in \rangle$ an element a of V is definable in terms $a_1, \ldots, a_n$ through the first-order language $\mathbf{L}(\mathbf{A}, \in)$ if and only if a is definable in terms of $A, a_1, \ldots, a_n$ through the first-order language $\mathbf{L}(\mathbf{A}, \in)$. By hypothesis and Statement 1 of Theorem 1, $M(A)$. Axiom 2 is then an immediate consequence of the hypothesis of the theorem and Lemma 3.

CONVENTION 5. Let f be an arbitrary permutation on A, $\bar{f}$ the extension of f to V, and n a positive integer. Throughout the rest of this article, n-ary relations R on A will be identified with the set t of n-tuples $(a_1, \ldots, a_n)$ such that $R(a_1, \ldots, a_n)$. By Statement 4 of Theorem 1, for each ordered n-tuple $(x_1, \ldots, x_n)$ of elements of A,

$$\bar{f}[(x_1, \ldots, x_n)] = (\bar{f}(x_1), \ldots, \bar{f}(x_n)) = (f(x_1), \ldots, f(x_n).$$

Therefore,

$$\bar{f}(t) = \{(f(a_1), \ldots, f(a_n)) \mid (a_1, \ldots, a_n) \in t\}.$$

Thus f is an automorphism of $\langle A, R \rangle$ if and only if $\bar{f}(t) = t$. Elements b of V such that $\bar{f}(b) = b$ are often said to be *left invariant* by $\bar{f}$.

The following lemma is useful in later proofs.

LEMMA 4. *Let $\mathfrak{X} = \langle A, R_j \rangle_{j \in J}$ be a first-order structure, α be an automorphism of $\mathfrak{X}$, and $\{Q_1, \ldots, Q_n\}$ be a finite subset of $\{R_j \mid j \in J\}$. Then the following two statements are true:*

1. *Noting that $\langle A, R_j \rangle_{j \in J}$ is the ordered pair (A, F), where F is the function on J such that $F(j) = R_j$ for each j in J, it follows that $\bar{\alpha}(\mathfrak{X}) = \mathfrak{X}$.*
2. *α is an automorphism of $\langle A, Q_1, \ldots, Q_n \rangle$.*

Proof. 1. Let j be an arbitrary element of J and let R be $F(j)$. Then $\bar{\alpha}(R) = R$, because R is a primitive of $\mathfrak{X}$ and α is an automorphism of $\mathfrak{X}$. By Definition 4, J is a pure set. Thus j is a pure set, and by Statement 2 of Theorem 1, $\bar{\alpha}(j) = j$. Thus for the ordered pair $j, F(j)),$

$$\bar{\alpha}(j, F(j))) = (\bar{\alpha}(j)), \bar{\alpha}(F(j)) = (j, R) = j, F(j)).$$

Since F is the set of such ordered pairs (as j varies within J), it follows from Definition 7 that $\bar{\alpha}(F) = F$. Because $\bar{\alpha}(F) = F$, it then follows that $\bar{\alpha}((A, F)) = (\bar{\alpha}(A), \bar{\alpha}(F))$; i.e., $\bar{\alpha}(\mathfrak{X}) = \mathfrak{X}$.

2. Immediate from Definition 5.

4. REPRESENTATIONAL THEORY OF MEASUREMENT

4.1. Representational Theory

In the measurement literature, the term *scale* is used to denote a specific way of measuring, as in *the centimeter scale*, as well as a set of specific ways of measuring, as in *a ratio scale for length*. Throughout this article, the latter usage of scale will be retained and the former will be replaced by other terms. The following is a very broad definition of scale.

DEFINITION 8. A *scale* on A is a nonempty set of functions from A into a pure set.

A *theory of measurement* consists of a precise specification of how a scale is formed. The currently dominant approach to measurement in the literature is the representational theory. The representational theory really comprises many related theories of measurement. What they have in common is that they require a scale to be a set of structure preserving mappings (e.g., a set of isomorphisms or homomorphisms) from some qualitative or empirically based structure into a structure from pure mathematics. For this article, the following version of the representational theory is assumed:

DEFINITION 9. $\mathfrak{X}$ is said to be a *qualitative structure* if and only if $\mathfrak{X}$ is a first-order structure on A.

Suppose $\mathfrak{X} = \langle A, R_j \rangle_{j \in J}$ is a qualitative structure. Then

$$\mathfrak{N} = \langle N, S_j \rangle_{j \in J}$$

is said to be a *mathematical representing structure* for $\mathfrak{X}$ if and only if N is a pure set and there exists a one-to-one homomorphism of $\mathfrak{X}$ into $\mathfrak{N}$.

DEFINITION 10. $\mathscr{S}$ is said to be a *representational scale* if and only if there exists a qualitative structure $\mathfrak{X}$ and a mathematical representing structure $\mathfrak{N}$ for $\mathfrak{X}$ such that $\mathscr{S}$ is a subset of one-to-one homomorphisms from $\mathfrak{X}$ into $\mathfrak{N}$.

The representational theory was first formulated by Scott and Suppes (1958). They justified this approach to measurement as follows:

> A primary aim of measurement is to provide a means of convenient computation. Practical control or prediction of empirical phenomena requires that unified, widely applicable methods of analyzing the important relationships between the phenomena be developed. Imbedding the discovered relations in various numerical relational systems is the most important such unifying method that has yet been found. (Scott & Suppes, 1958, pp. 116–117)

4.2. Representational Concepts of Meaningfulness

The imbedding by a one-to-one homomorphism φ of a fragment of science $\mathscr{F}$ captured by a structure $\mathfrak{X}$ of observable primitives with domain A into a portion $\mathfrak{N}$ of pure mathematics allows the full power and results to pure mathematics be brought to bear in the analysis of $\mathscr{F}$. In science it is important to have a criterion for the interpretation of a relation S on the domain of $\mathfrak{N}$ to be scientific significant for the fragment of science, i.e., to have a criterion for $\varphi^{-1}(S)$ to be scientifically significant about $\mathscr{F}$. The seeking of such criteria has become known as the meaningfulness problem, and over time representational theorists have suggested various ways to deal with it. One of these is the Erlanger program's concept of meaningfulness based on permutation invariance specialized to qualitative measurement structures; another is a meaningfulness concept due to Pfanzagl (1968), which the following definition calls $\mathscr{S}$-*invariance*.[1]

Definition 11. Let $\mathscr{S}$ be a scale on A and R be a n-ary relation on A. Then R is said to be $\mathscr{S}$-*invariant* if and only if there exists a purely mathematical n-ary relation S such that for all φ in $\mathscr{S}$ and all $a_1, ..., a_n$ in A,

$$R(a_1, ..., a_n) \quad \text{iff} \quad S(\varphi(a_1), ..., \varphi(a_n)).$$

Because Definition 11 does not assume that the scale $\mathscr{S}$ results from the representational theory, it may be viewed as a way of extending Pfanzagl's meaningfulness concept to nonrepresentational theories of measurement such as Stevens' theory (Stevens, 1951) or Niederée's theory (Niederée, 1992).

The representational theory does not provide principled ways of selecting mathematical representing structures for a qualitative structure. This clearly presents a problem for the representational use of $\mathscr{S}$-invariance as a meaningfulness concept, since, as the following example shows, there exist a qualitative structure $\mathfrak{X}$, a first-order relation R on A, and scales $\mathscr{S}$ and $\mathscr{T}$ that are the sets of one-to-one homomorphisms $\mathfrak{X}$ into respectively mathematical representing structures $\mathfrak{N}$ and $\mathfrak{M}$ such that R is $\mathscr{S}$-invariant but not $\mathscr{T}$-invariant.

Example. Let $A = \{a, b, c\}$, $\mathfrak{X} = \langle A, \succ \rangle$, where

$$x \succ y \text{ iff } (x = a \text{ and } y = b) \text{ or } (x = b \text{ and } y = c),$$

[1] For descriptions of additional meaningfulness concepts considered by the representational theory, see Narens (1981); Narens (1985, Chap. 2, Sect. 14); Luce Krantz, Suppes, and Tversky (1990, Chap. 22); and Niederee (1994). See also Chiang (1995, 1997, 1998).

and let $\mathfrak{N} = \langle\{3, 2, 1\}\rangle$ and $\mathfrak{M} = \langle\mathbb{R}^+, \rangle$. Let $R(z)$ be the following relation on A: $R(z)$ if and only if $z = b$. Let $S(z)$ be the following relation on $\{3, 2, 1\}$: $S(z)$ if and only if $z = 2$. Then the set $\mathscr{S}$ of one-to-one homomorphisms of $\mathfrak{X}$ into $\mathfrak{N}$ consists of the single function φ, where $\varphi(a) = 3, \varphi(b) = 2$, and $\varphi(c) = 1$. Thus R is $\mathscr{S}$-invariant, because for each x in A,

$$R(x) \quad \text{iff} \quad S(\varphi(x)).$$

Let $\mathscr{T}$ be the set of one-to-one homomorphisms of $\mathfrak{X}$ into $\mathfrak{M}$. It will be shown by contradiction that R is not $\mathscr{T}$-invariant. For suppose R were $\mathscr{T}$-invariant. Then a relation T on $\mathbb{R}^+$ can be found such that for all x in A and all ψ in $\mathscr{T}$,

$$R(x) \quad \text{iff} \quad T(\psi(x)). \tag{1}$$

Let φ be as before, i.e., $\varphi(a) = 3, \varphi(b) = 2$, and $\varphi(c) = 1$. Then φ is in $\mathscr{T}$. Thus by Eq. (1) applied to φ,

$$T(z) \quad \text{iff} \quad z = 2. \tag{2}$$

Let θ be the following function on A: $\theta(a) = 4, \theta(b) = 3$, and $\theta(c) = 1$. Then θ is also in $\mathscr{T}$, and applying Eq. (1) to θ, we obtain

$$T(z) \quad \text{iff} \quad z = 3,$$

contradicting Eq. (2).

Because of various kinds of difficulties inherent in the meaningfulness part of the homomorphism approach to representational theory of measurement, Narens (1981, 1985, 2002) decided to base the representational theory on scales of isomorphisms. When this is done, the kind of difficulty illustrated in the above example disappears. (See the equivalence of Statements 1 and 2 in Theorem 5 below.)

4.3. Dimensional Analysis and the Possible Psychophysical Laws

Luce (1978) provided a measurement-theoretic foundation for an important part of dimensional analysis of physics in terms of a qualitative structure of primitives $\mathfrak{X}$. Luce showed that dimensionally invariant relations on the domain of $\mathfrak{X}$ were those that were invariant under the automorphisms of $\mathfrak{X}$. Such invariant relations he termed "meaningful." Luce viewed this form of meaningfulness as a qualitative version of the quantitative meaningfulness concept employed by Stevens (1946, 1951) for selecting statistics appropriate to a measurement situation. He did not, however, connect it with the Erlanger program or methods of geometric inference. Narens (1988) investigated the interconnections between an important inferential technique of dimensional analysis, the Erlanger program, and a definability concept of meaningfulness:

One of the main applications of the Erlanger program's meaningfulness concept has been to rule out nonmeaningful entities from consideration. This practice can be intuitively justified by [Lemma 3] as follows:

Suppose in a particular setting we are interested in finding the functional relationship of the qualitative variables x, y, and z. We believe that the primitive relations (which are known) completely characterize the current situation. Furthermore, our understanding (or insight) about the situation tells us that x must be a function of y and z. (This is the typical case for an application of dimensional analysis in physics.) This unknown function—which we will call "the desired function"—must be determined by the primitive relations and the qualitative variables x, y, and z. Therefore, it should somehow be "definable" from the relations and variables. Even though the exact nature of the definability condition is not known, (it can be argued that) it must be [at least as weak as scientific definability]. Thus by [Lemma 3] we know that any function relating the variable x to the variables y and z that is not invariant under the automorphisms of the primitives cannot be the desired function. In many situations, this knowledge of knowing that functions not invariant under the automorphisms of the primitives cannot be the desired function can be used to effectively find or narrow down the possibilities for the desired function. (Narens, 1988, p. 70.)

Drawing upon principles inherent in dimensional analysis in physics, Luce (1959) formulated related principles for the psychophysical case of a function of an observed physical quantity onto an unobserved psychological quantity and showed that his principles greatly delimited the possible mathematical forms of the function. He termed functions that satisfied his principles, "possible psychophysical laws," and an important literature developed in the behavioral sciences that extended and applied his methods. Falmagne and Narens (1983), Falmagne (1985), Roberts and Rosenbaum (1986), and Aczél, Roberts, and Rosenbaum (1986) connected various extensions of Luce's possible psychophysical laws with the concepts of meaningfulness related to the representational theory. (An alternative approach to meaningfulness and dimensional analysis is given in Dzhafarov, 1995.)

5. MEASUREMENT THROUGH MEANINGFULNESS

5.1. Meaningful Scales

The traditional form of the representational theory of measurement proceeds as follows: (1) A qualitative structure $\mathfrak{X}$ is selected to capture the domain A of interest; (2) a mathematical representing structure $\mathfrak{R}$ is selected to measure $\mathfrak{X}$ in terms of the scale $\mathscr{S}$ of homomorphisms of $\mathfrak{X}$ into $\mathfrak{R}$; and meaningfulness is identified with a form of invariance associated with $\mathscr{S}$, e.g., with $\mathscr{S}$-invariance. This section inverts most of the process: (i) As before, a qualitative structure $\mathfrak{X}$ is selected to capture the domain A of interest; (ii) a theory of meaningfulness which assumes that meaningful entities are scientifically defined in terms of $\mathfrak{X}$ and its primitives is assumed; (iii) in terms of this theory of meaningfulness, the concept of a *meaningful scale* is formulated; and (iv) it is shown that for each meaningful scale $\mathscr{S}$ there exists a mathematical representing structure $\mathfrak{R}$ such that $\mathscr{S}$ is a scale consisting of homomorphisms of $\mathfrak{X}$ into $\mathfrak{R}$.

DEFINITION 12. Assume Axiom 2. Then $\mathcal{S}$ is said to be a *meaningful scale* if and only if $\mathcal{S}$ is a scale (Definition 8) and $M(\mathcal{S})$. (Note that in this definition, like in Axiom 2, the predicate M is treated as an undefined term.)

5.2. *The Representational Theory*

DEFINITION 13. $\mathfrak{X}$ is said to be a *structure with meaningful primitives* if and only if $\mathfrak{X}$ is a qualitative structure and each primitive of $\mathfrak{X}$ is meaningful.

LEMMA 5. *Let* $\mathfrak{X}\langle A, R_j\rangle_{j\in J}$ *be a qualitative structure. Suppose* $\mathfrak{X}$ *is meaningful. Then* $\mathfrak{X}$ *is a structure with meaningful primitives.*

Proof. Formally, $\mathfrak{X}$ is the ordered pair (A, F), where F is the function on J such that for each j in J, $F(j) = R_j$. Since $\mathfrak{X}$ is meaningful by hypothesis, (A, F) is meaningful, and therefore F is meaningful. Then for each j in J, R_j is meaningful, because it is defined in terms of a pure set, j, and a meaningful entity, F. Thus $\mathfrak{X}$ is a structure of meaningful primitives.

DEFINITION 14. $\mathfrak{X}$ is said to *generate M* if and only if the following three conditions hold:

(i) $\mathfrak{X}$ is a first-order structure with meaningful primitives;

(ii) $\mathfrak{X}$ is meaningful; and

(iii) for all entities a, $M(a)$ if and only if a has a scientific definition in terms of $\mathfrak{X}$ and finitely many primitives of $\mathfrak{X}$.

M is said to be *qualitatively generated* if and only if M is generated by some $\mathfrak{X}$.

Observe that by Lemma 5, Condition (i) in Definition 14 is a consequence of Condition (ii) and is therefore from a logical point of view redundant.

AXIOM 3. *M is qualitatively generated.*

THEOREM 3. *Assume Axiom 2. Suppose* $\mathfrak{X} = \langle A, R_j\rangle_{j\in J}$ *is a qualitative structure,* $\mathfrak{X}$ *is meaningful,* $\mathfrak{X}$ *is a first-order structure,* $\mathcal{S}$ *is the set of homomorphisms of* $\mathfrak{X}$ *into (respectively, isomorphisms of* $\mathfrak{X}$ *onto)* $\mathfrak{N} = \langle N, S_j\rangle_{j\in J}$, N *is a pure set, and* $\mathcal{S} \neq \varnothing$. *Then* $\mathcal{S}$ *is a meaningful scale.*

Proof. $\mathfrak{X}$ is meaningful by hypothesis, and J, $\mathfrak{N}$, and N are pure sets. Thus $\mathcal{S}$ is meaningful, because it has a scientific definition in terms of $\mathfrak{X}$, A, J, $\mathfrak{N}$, and N as the set of homomorphisms of $\mathfrak{X}$ into (respectively, isomorphisms of $\mathfrak{X}$ onto) $\mathfrak{N}$.

The next theorem gives, under the assumption of Axioms 2 and 3, a method for constructing for each meaningful scale $\mathcal{S}$ a structure with meaningful primitives $\mathfrak{X}$ such that $\mathcal{S}$ is a set of homomorphisms of $\mathfrak{X}$ into a mathematical representing structure.

THEOREM 4. *Assume Axioms 2 and 3. Suppose* $\mathcal{S}$ *is a meaningful scale. By Axiom 3, let* $\mathfrak{Y} = \langle A, T_j\rangle_{j\in J}$ *generate M. Let*

$$N = \bigcup_{\varphi\in\mathcal{S}} \varphi(A),$$

and for each $j \in J$, let

$$S_j = \bigcup_{\varphi \in \mathcal{S}} \varphi(T_j),$$

and let

$$\mathfrak{N} = \langle N, S_j \rangle_{j \in J}$$

For each $j \in J$ define R_j on A as follows:

> *For each $a_1, \ldots, a_{m(j)}$ of A, $R_j(a_1, \ldots, a_{m(j)})$ if and only if there exists φ in $\mathcal{S}$ such that $S_j(\varphi(a_1), \ldots, \varphi(a_{m(j)}))$.*

Let

$$\mathfrak{X} = \langle A, R_j \rangle_{j \in J}$$

Then $\mathcal{S}$ is a set of homomorphisms of $\mathfrak{X}$ into $\mathfrak{N}$.

Proof. Because by hypothesis $\mathcal{S}$ is meaningful, R_j is meaningful by Axiom 2 for each j in J. Thus $\mathfrak{X}$ is a first-order structure with meaningful primitives. Since by hypothesis, the range of each element of $\mathcal{S}$ is a pure set, the union of these ranges, N, is a pure set. Let j be an arbitrary element of J. Without loss of generality, suppose R_j is a $m(j)$-relation. Let $a_1, \ldots, a_{m(j)}$ be $m(j)$ arbitrary elements of A, and ψ be an arbitrary element of $\mathcal{S}$. Then by the definition of R_j given above,

$$R_j(a_1, \ldots, a_{m(j)}) \qquad \text{iff} \quad S_j(\psi(a_1), \ldots, \psi(a_{m(j)}),$$

establishing that ψ is a, homomorphism of $\mathfrak{X}$ into $\mathfrak{N}$.

5.3. *Meaningfulness*

THEOREM 5. *Assume Axioms 1, 2, and 3. By Axiom 3, let $\mathfrak{X} = \langle A, R_j \rangle_{j \in J}$ generate M. Suppose R is a first-order relation on A. Then the following four statements are equivalent:*

(1) *For each scale $\mathcal{S}$, if $\mathcal{S}$ is the set of isomorphisms of $\mathfrak{X}$ onto a mathematical representing structure, then R is $\mathcal{S}$-invariant.*

(2) *There exists a scale $\mathcal{S}$ such that $\mathcal{S}$ is the set of isomorphisms of $\mathfrak{X}$ onto a mathematical representing structure and R is $\mathcal{S}$-invariant.*

(3) *R is meaningful.*

(4) *R is invariant under the automorphisms of $\mathfrak{X}$.*

Proof. (1) → (2). Assume (1). Then it needs to only be shown that there exists a set of isomorphisms of $\mathfrak{X}$ onto a mathematical representing structure. By Axiom 1, let φ be a one-to-one function from A onto a pure set N. Let $\mathfrak{N} = \langle N, \varphi(R_j) \rangle_{j \in J}$. Let $\mathcal{S}$ be the set of isomorphisms of $\mathfrak{X}$ onto $\mathfrak{N}$. By Lemma 2, $\varphi \in \mathcal{S}$. Thus $\mathcal{S} \neq \varnothing$ and therefore is a scale.

(2) → (3). Assume (2). Without loss of generality, suppose R is a n-ary relation. By hypothesis, let $\mathfrak{N}$ be a mathematical representing structure and $\mathscr{S}$ be the set of isomorphisms from $\mathfrak{X}$ onto $\mathfrak{N}$. Then $\mathscr{S}$ is meaningful by Theorem 3.

Because R is by hypothesis $\mathscr{S}$-invariant, let S be a n-ary relation on the domain of $\mathfrak{N}$ such that for all $x_1, ..., x_n$ in A,

$$R(x_1, ..., x_n) \quad \text{iff for all} \quad \varphi \in \mathscr{S}, S(\varphi(x_1), ..., \varphi(x_n)).$$

Because S is a pure set (of n-tuples of elements of the domain of $\mathfrak{N}$) and R is the unique relation such that all $x_1, ..., x_n$ in A,

$$R(x_1, ..., x_n) \quad \text{iff for all} \quad \varphi \in \mathscr{S}, S(\varphi(x_1), ..., \varphi(x_n)),$$

it follows from Lemma 1, the meaningfulness of $\mathscr{S}$, and Axiom 2 that R is meaningful.

(3) → (4). Assume (3). Since $\mathfrak{X}$ generates M, by Axiom 2 let $Q_1, ..., Q_k$ be primitives of $\mathfrak{X}$ such that R is scientifically definable in terms of $\mathfrak{X}$ and $Q_1, ..., Q_k$. Let α be an arbitrary automorphism of $\mathfrak{X}$. To show Statement (4) it is sufficient to show that $\bar{\alpha}(R) = R$, where $\bar{\alpha}$ is the extension α to V. By Lemma 4, $\bar{\alpha}(\mathfrak{X}) = \mathfrak{X}$ and $\bar{\alpha}(Q_i) = Q_i$, for $i = 1, ..., k$. Thus by Lemma 3, $\bar{\alpha}(R) = R$.

(4) → (1). Assume (4). By Axiom 1, let φ be a one-to-one function from A onto a pure set N. Let $\mathfrak{N} = \langle N, \varphi(R_j) \rangle_{j \in J}$. Let $\mathscr{S}$ be the set of isomorphisms of $\mathfrak{X}$ onto $\mathfrak{N}$. By Lemma 2, $\varphi \in \mathscr{S}$. Thus $\mathscr{S} \neq \varnothing$ and therefore is a scale. Let ψ be an arbitrary element of $\mathscr{S}$. Then, because φ and ψ are isomorphisms of $\mathfrak{X}$ onto $\mathfrak{N}$, it easily follows that $\varphi^{-1} * \psi$ is an automorphism of $\mathfrak{X}$. Let $S = \varphi(R)$. Then it follows from Definition 6 and the hypothesis (4) that for all $x_1, ..., x_n$ in A,

$$\begin{aligned} R(x_1, ..., x_n) \quad &\text{iff} \quad R[\varphi^{-1} * \psi(x_1), ..., \varphi^{-1} * \psi(x_n)] \\ &\text{iff} \quad S[\varphi(\varphi^{-1} * \psi(x_1)), ..., \varphi(\varphi^{-1} * \psi(x_n))] \\ &\text{iff} \quad S[\psi(x_1), ..., \psi(x_n)], \end{aligned}$$

and thus, because ψ is an arbitrary element of $\mathscr{S}$, R is $\mathscr{S}$-invariant.

6. CONCLUDING REMARKS

Invariance has a number of important roles in mathematics and science. Klein (1872) explored one of these in his Erlanger program for geometry, where it was employed as a method for classification, discovery, and inference. In modern physics, related developments took place that were greatly influenced by the Erlanger program. However, before its systematic application in geometry, inferential techniques using invariance were employed by physicists, and one of these methods developed into what is today called *dimensional analysis*. Eventually dimensional analytic techniques found their way into the behavioral sciences.

Luce (1959) attempted to provide a mathematical and epistemological foundation for a behavioral variant of dimensional analysis with his theory of possible psychophysical laws. As the representational theory of measurement developed in the 1960s and the structure of physical units was given a rigorous measurement-theoretic foundation, it became possible to understand qualitatively many of the key components of physical dimensional analysis. Luce (1978) provided a qualitative treatment for the component called dimensional invariance. He showed dimensional invariance to be formally an instance of the Erlanger program in the following sense: The structure of physical units $\mathfrak{N}$ may be viewed as a numerical representing structure that results from the measurement of a qualitative structure $\mathfrak{X}$ through the scale $\mathscr{S}$ of isomorphisms of $\mathfrak{X}$ onto $\mathfrak{N}$, and each dimensionally invariant relation on the domain of $\mathfrak{N}$ corresponds through the isomorphisms in $\mathscr{S}$ to a first-order relation on the domain of $\mathfrak{X}$ that is left invariant by the automorphisms of $\mathfrak{X}$.

The Erlanger program provided no justification of the use of invariance under a group of permutations as a criterion for belonging to a scientific topic. Because of this, its methods often appear mysterious. For me, this mystery is dispelled by the identification of a scientific topic with the set of meaningful entities generated by a qualitative structure (Axioms 2 and 3) and the equivalence of meaningfulness with invariance under automorphisms (Statements 3 and 4 of Theorem 5).

Theorem 5 also shows the equivalence of meaningfulness with various concepts of invariance based on measurement scales. Thus Axioms 1 to 3 comprise a foundation for representational concepts of meaningfulness based on the concept of a scientific topic. They also provide through Theorems 3 and 4 a justification of the representational theory of measurement in terms of meaningfulness considerations.

Scientific definability (Definition 2) contains strong Pythagorean elements. One of these is its free use of pure mathematics for defining meaningful entities. The use of strong forms of pure mathematics in science has a long history. It also raises a number of epistemological issues, including the determination of the empirical contents of scientific mathematical expressions. The latter issue has sometimes been identified in the measurement literature with concepts of meaningfulness based on invariance. I believe such identifications to be in error. Similarly, the theory of meaningfulness comprising Axioms 1 to 3 should not be identified with empiricalness.

Scientific inquiry is a complicated issue with many overlapping parts. I believe meaningfulness belongs primarily to the theoretical part of scientific inquiry. Because of the overlap of the theoretical part of a science with its experimental and applied parts, meaningfulness often has important ramifications in the experimental and applied parts.

Meaningfulness is essentially a theoretical position about scientific content and its role in (theoretical) inference. For example, consider the case where by extra-scientific means (e.g., intuition, experience) a scientist is led to believe that a function $z = F(x, y)$ that he or she needs to describe from a subset of $A \times A$ into A is completely determined by the observable, first-order relations $R_1, \ldots, R_n$ on A. Then it is reasonable for the scientist to proceed under the hypothesis that F belongs to the scientific content of $\mathfrak{X} = \langle A, R_1, \ldots, R_n \rangle$, which for this discussion may be taken as the set of meaningful entities determined by Axioms 1 to 3. Thus the scientist assumes F has a scientific definition in terms of $\mathfrak{X}$ and its primitives. By Lemma 3, F is invariant under the automorphisms of $\mathfrak{X}$. Suppose the scientist knows enough properties about $\mathfrak{X}$ and has the mathematical skill to determine the automorphism group G of $\mathfrak{X}$. Then methods of analyses involving automorphisms may be employed to provide information helpful in characterizing F. There are several methods in the literature for accomplishing this.

Note that in the above process, scientific definability is used *to justify* F belonging to the appropriate topic, invariance is used *as a mathematical technique* to find helpful information for characterizing F, and that these two uses are connected *by a theorem* of mathematical logic. Also note that the scientist's belief that F belonged to the topic generated by $\mathfrak{X}$ is extra-scientific. Therefore, the deductions based on information obtained through the above process should be either checked by experiment or derived from accepted scientific theory and facts; i.e., they should be treated as scientific hypotheses that need corroboration. Thus, for the purposes of science, the above process is a method of generating hypotheses and not facts: If the scientist's extra-scientific beliefs are correct, then the generated hypotheses will be facts; however, the scientist has no *scientific guarantee* that his or her beliefs are correct.

The concept of meaningfulness and methods of inference based on it cease to have value when all entities in V are meaningful. For the theory of meaningfulness described by Axioms 1 to 3, this occurs when the qualitative structure has the identity function on A as its only automorphism. (This follows from the equivalence of Statements 3 and 4 of Theorem 5.) Such situations are plenteous in science, for example, the geometry of physical space-time as described by Einstein's general theory of relativity. They should not be taken as a refutation of Axioms 1 to 3 as a valid description of "scientific topic:" They only establish that the above concept of scientific topic ceases to be useful as a theoretical tool in some situations.

Because an entity may belong to one scientific topic and not to another, the concept of meaningfulness developed in this article is relative. In particular, it should not be identified with "having meaning." Although Axiom 3 is formulated in terms of a qualitative structure, meaningfulness should not be identified with qualitativeness, because it allows the free use of pure mathematics in the specification of meaningful entities.

I prefer to reserve the term "lawfulness" for particular kinds of meaningful relationships that display an extra form of invariance. I believe that "laws" obtained through the methods of possible psychophysical laws and dimensional analysis possess this extra kind of invariance, but a proper discussion of the subject is outside the scope of this article. (Chapter 6 of Narens, 2002, is devoted to this issue.)

In science, relationships are often formulated across subtopics. For purposes of interpretation and theory, it is often important to know if such relationships belong to one or another subtopic. For example, in behavioral psychophysics, the situation under consideration often has a characterization in terms of three qualitative structures based on the set of possible stimuli, A: a *physical structure* $\mathfrak{X} = \langle A, C_1, \ldots, C_m \rangle$ for characterizing the relevant physical aspects of the stimuli in A; a *psychological structure* $\mathfrak{Y} = \langle A, D_1, \ldots, D_n \rangle$ for characterizing the relevant behavior of the subject in regards to A; and a psychophysical structure $\mathfrak{Z} = \langle A, C_1, \ldots, C_m, D_1, \ldots, D_n \rangle$ that combines psychological and physical structures and characterizes the situation under consideration. In psychophysics, the physical structure is used to measure the stimuli in A through an element ϕ of a scale $\mathscr{S}$. Then the subject's behavior is usually described and analyzed in terms of the numerical ϕ-values of stimuli from $\mathfrak{X}$. Suppose S is a first-order relation on $\phi(A)$ that the scientist believes to be important. For purposes of theory, it is often helpful to interpret S as a first-order relation R on A. In this circumstance, $\phi^{-1}(S)$ is the obvious choice for R. Assume this choice.

Case 1. R is outside the scientific topic generated by $\mathfrak{Z}$. Then R is not completely determined by $\mathfrak{Z}$; i.e., $\mathfrak{Z}$ is inadequate for describing R. In this situation, we say "S (under the interpretation ϕ^{-1} has some nonpsychophysical content." It may be possible to remedy this by selecting a different structure of psychophysical primitives so that R belongs to the scientific topic determined by the new structure.

Case 2. R is in the scientific topic generated by $\mathfrak{Z}$. Then it is often of epistemological and theoretical interest to determine if R has purely psychologyical content—that is, if R is in the scientific topic generated by the qualitative psychological structure $\mathfrak{Y}$. Suppose R is not in the scientific topic generated by $\mathfrak{Y}$. In this case we say "S (under the interpretation ϕ^{-1}) has psychophysical content and some nonpsychological content." Narens and Mausfeld (1992) gave examples of well-known psychophysical situations where the psychologically important quantitative relations were treated in theoretical discussions as if they were purely psychological. However, Narens and Mausfeld showed that these relations had psychophysical content and some nonsychological content. They concluded that additional psychological primitives and additional experimental results were needed to rigorously establish that the relations had purely psychological content. In short, Narens and Mausfeld showed that the original psychophysical situations and experimental observations were inadequate for drawing the intended theoretical conclusions.

The topical content of R (and therefore S) may be evaluated by invariance and definitional methods. The definitional method consists of providing a scientific definition of R in terms of an appropriate structure and its primitives. This method is only useful for establishing that R has a particular kind of content; it is not useful for showing that R does not have a particular kind of content. Invariance methods can be employed for both purposes: For psychophysical content, one first forms the mathematical representing structure $\mathfrak{Z}' = \phi(\mathfrak{Z})$ and then uses regular mathematical methods to check whether S is invariant under the automorphisms of $\mathfrak{Z}'$. By Lemma 2, ϕ is an isomorphism of $\mathfrak{Z}$ onto $\mathfrak{Z}'$. Thus by isomorphism, $R = \phi^{-1}(S)$ is invariant under the automorphisms of $\mathfrak{Z}$ if and only if S is invariant under the automorphisms of $\mathfrak{Z}'$. Therefore by the equivalence of Statements 3 and 4 of Theorem 5,

R has psychophysical content iff it belongs to the scientific topic generated by $\mathfrak{Z}$
iff S is invariant under the automorphisms of $\mathfrak{Z}'$.

In particular, if S is not invariant under the automorphisms of $\mathfrak{Z}'$, then R has some nonpsychophysical content. An analogous evaluation for the psychological content of R arises by substituting the qualitative psychological structure $\mathfrak{Y}$ for $\mathfrak{Z}$ in the above procedure.

The theory of meaningfulness comprising Axioms 1 to 3 was designed to provide an epistemologically sound foundation for the Erlanger program's concept of geometrical content and various uses of invariance in science. A principal use of this theory—showing certain entities to be nonmeaningful—becomes valueless when the qualitative generating structure has the identity as its only automorphism, i.e., becomes valueless when all entities in V are meaningful. Other theories of meaningfulness that generalize the Erlanger program are developed Narens (2002), and some of these appear to deal effectively with the case of generating structures with trivial automorphisms. Their discussion, however, is outside the scope of this article.

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