MEANINGFULNESS AND INVARIANCE

Few disavow the principle that scientific propositions should be meaningful in the sense of asserting something that is verifiable or falsifiable about the qualitative or empirical situation under discussion. What makes this principle tricky to apply in practice is that much of what is said is formulated not as simple assertions about qualitative or empirical events – such as a certain object sinks when placed in water – but as laws formulated in rather abstract, often mathematical, terms. It is not always apparent exactly what class of qualitative observations corresponds to such (often numerical) laws. Theories of meaningfulness are methods for investigating such matters, and invariance concepts are their primary tools.

The problem of meaningfulness, which has been around since the inception of mathematical science in ancient times, has proved to be difficult and subtle; even today it has not been fully resolved. This entry surveys some of the current ideas about it, and illustrates, through examples, some of its uses. The presentation requires some elementary technical concepts of measurement theory (such as representation, scale type, etc.), which are explained in Measurement, Theory of.

1. Concepts of Meaningfulness

Some Notation and Definitions. The operation of functional composition is denoted $*$. The Cartesian product of $T_1, \ldots, T_n$ is denoted $\prod_i^n T_i$.

A *scale* $\mathcal{S}$ is a set of functions from a qualitative domain, a set X endowed with one or more relations, into the real numbers. Elements of $\mathcal{S}$ are called *representations.* An example is the usual physical scale to measure length. Two of its representations are the *foot representation* and the *centimeter representation.* $\mathcal{S}$ is said to be

- a *ratio scale* if and only if for each ϕ in $\mathcal{S}$,

$$\mathcal{S} = \{r\phi \,|\, r > 0\},$$

- an *interval scale* if and only if for each ϕ in $\mathcal{S}$,

$$\mathcal{S} = \{r\phi + s \,|\, r > 0,\ s \text{ a real}\},$$

- an *ordinal scale* if and only if for each ϕ in $\mathcal{S}$, the range of ϕ is a (possibly infinite) interval of reals and

$$\mathcal{S} = \{f * \phi \,|\, f \text{ is a strictly monotonic function from the range of } \phi \text{ onto itself}\}.$$

Intuitive Formulation of Meaningfulness and Some Examples. The following example, taken from Suppes and Zinnes (1963), nicely illustrates part of the problem in a very elementary way. Which of the following four sentences are meaningful?

(1) Stendhal weighed 150 on 2 September 1839.

(2) The ratio of Stendhal's weight to Jane Austen's on 3 July 1814 was 1.42.

(3) The ratio of the maximum temperature today to the maximum temperature yesterday is 1.10.

(4) The ratio of the difference between today's and yesterday's maximum temperature to the difference between today's and tomorrow's maximum temperature will be 0.95.

Suppose that weight is measured in terms of the ratio scale $\mathcal{W}$ (which includes among its representations the pound and kilogram representations and all those obtained by just a change of unit), and that temperature is measured by the interval scale $\mathcal{T}$, which for this example includes the Fahrenheit and Celsius representations.[1] Then Statement (2) is meaningful, because with respect to each representation in $\mathcal{W}$ it says the same thing, i.e., its truth value is the same no matter which representation in $\mathcal{W}$ is used to measure weight. That is not true for Statement (1), because (1) is true for exactly one representation in $\mathcal{W}$ and false for all of the rest. Thus we say that (1) is 'meaningless'. Similarly, (4) is meaningful with respect to $\mathcal{T}$ but (3) is not.

The somewhat intuitive concept of meaningfulness suggested by these examples is usually stated as follows: Suppose a qualitative or empirical attribute is measured by a representation from a scale of representations $\mathcal{S}$. Then a numerical statement involving values of the

representation is said to be *quantitatively meaningful* if and only if its truth (or falsity) is constant no matter which representation in $\mathcal{S}$ is used to assign numbers to the attribute. There are obvious formal difficulties with this definition, for example the concept of 'numerical statement' is not a precise one. More seriously, it is unclear under what conditions this is the 'right definition' of meaningfulness, for it does not always lead to correct results in some well-understood and non-controversial situations. (See the discussion involving situations where the measurement scale consists of a single representation for an example.) Nevertheless, it is the concept most frequently employed in the literature, and invoking it often provides insight into the correct way of handling a quantitative situation – as the following still elementary but somewhat less obvious example shows.

Consider a situation where M persons rate N objects (e.g. M judges judging N contestants in a sporting event). For simplicity, assume that person i rates objects according to the ratio scale of representations $\mathcal{R}_i$. The problem is to find

[1] The Kelvin scale for temperature, which assumes an absolute zero temperature, is different from $\mathcal{T}$.

an ordering on the N objects that aggregates the judgements of the judges in a reasonable way. It can be shown that their judgements cannot be coordinated in such a way that, for R_i in $\mathcal{R}_i$ and R_j in $\mathcal{R}_j$ that for some object a, the assertion $R_i(a) = R_j(a)$ is justified philosophically. The difficulties underlying such a coordination are essentially those that arise in attempting to compare individual utility functions. The latter problem – 'the interpersonal comparison of utilities' – has been much discussed in the literature, including discussions by Narens and Luce (1983) and Sen (1979). It is generally agreed that there are great, if not insurmountable, difficulties in carrying out such comparisons. Any rule that does not involve coordination among the raters can be formulated as follows: First, let F be a function that assigns to an object the value $F(r_1, \ldots, r_M)$ whenever person i assigns the number r_i to the object. Second, assume that object a is ranked just as high as b if and only if the value assigned by F to a is at least as great as that assigned by F to b. In practice F is often taken to be the arithmetic mean of the ratings $r_1, \ldots, r_M$ (e.g. Pickering et al., 1973). Observe, however, that arithmetic means for this kind of rating situation, in general, produce a non-quantatively meaningful ranking of objects, as illustrated by the following special case: Suppose $M = 2$ and, for $i = 1, 2$, R_i is person's i representation that is being used for generating ratings, and

$$R_1(a) = 2,\ R_1(b) = 3,\ R_2(a) = 3,\ \text{and}\ R_2(b) = 1\,.$$

Then the arithmetical mean of the ratings for a, 2.5, is greater than that for b, 2, and thus a is ranked above b. However, meaningfulness requires the same order if any other representations of persons 1 and 2 rating scales are used, for example, $10R_1$ and $2R_2$. But for this choice of representations, the arithmetic mean of a, 13, is less than that of b, 16, and thus b is ranked higher than a.

It is easy to check that the geometrical mean,

$$F(r_1, \ldots, r_M)(x) = [r_1 \cdots r_M]^{\frac{1}{M}}\,,$$

gives rise to a quantitatively meaningful, non-coordinated rule for ranking objects. It can be shown under plausible conditions that all other meaningful, non-coordinated rules give rise to the same ranking as that given by the geometric mean (Aczél and Roberts, 1989).

Many other applications of quantitative meaningfulness have been given by various researchers. In particular, Roberts (1985) provides a wide range of social science examples. In some contexts, quantitative meaningfulness presents certain technical difficulties that require a some modification in its definition (e.g., see Roberts and Franke, 1976; Falmagne and Narens, 1983).

Meaningfulness and Statistics. Another area of importance to social scientists in which invariance notions are thought to be relevant is applying statistics to numerical data. The role of measurement considerations in statistics and of invariance under admissible scale transformations was first emphasized by Stevens (1946, 1951); this view quickly became popularized in numerous textbooks, and it produced extensive debates in the literature. Continued disagreement exists, mainly created by confusion arising from the following two simple facts:

- Measurement scales are characterized by groups of admissible transformations of the real numbers.
- Statistical distributions exhibit certain invariances under appropriate transformation groups, often the same groups (especially the affine transformations), as those that arise from measurement considerations.

Because of these facts, some scientists have concluded that the suitability of a statistical test is determined, in part, by whether or not the measurement and distribution groups are the same. Thus, it is said that one may be able to apply a test, such as a t-test, that rests on the Gaussian distribution to ratio or interval scale data, but surely not to ordinal data, because the Gaussian distribution is invariant under the group of positive affine transformations, $x \to rx+s$, r, s real, $r > 0 -$ which arises in both the ratio and interval case, but not in the ordinal one. Neither half of the assertion is correct: first, a significance test should be applied only when its distributional assumptions are met, and they may very well hold for some particular representation of ordinal data. And, second, a specific distributional assumption may well not be met by data arising from a particular scale of measurement. For example, reaction times, being times, are measured on a physical ratio scale, but they are rarely well approximated by a Gaussian distribution.

What is true, however, is that any proposition (hypothesis) that one plans to put to statistical test or to use in estimation had better, itself, be quantitatively meaningful with respect to the scale used for the measurements. In general, it is not quantitatively meaningful to assert that two means are equal when the quantities are measured by an ordinal scale, because equality of means is not invariant under strictly increasing transformations. Thus, no matter what distribution holds and no matter what test is performed, the result may not be quantitatively meaningful, because the hypothesis is not. In particular, if an hypothesis is about the measurement structure itself, for example that the representation is additive over a concatenation operation, then it is essential that (i) the hypothesis be invariant under the symmetries[2] of the structure and therefore invariant under the scale used to measure the structure,[3] and (ii) the hypotheses of the statistical test be met without going outside the transformations of the measurement representation. See Luce, Krantz, Suppes, and Tversky (1990) for a more detailed discussion of this issue.

2. Concepts of Invariance

Measurement laws are quantitative laws based primarily on interrelationships of scales of measurement. They have in common with quantitative meaningfulness that they are derived through considerations of admissible transformations of the measurements of relevant variables. In the view of Falmagne

[2]Symmetries are isomorphisms of the structure onto itself (see Qualitative Meaningfulness below).

[3]Because it is assumed that scales of measurement are structure preserving functions from a qualitative structure onto a quantitative one, (i) immediately follows.

and Narens (1983) they arise in an empirical situation "that is governed by an empirical law of which we know little of its mathematical form and a little of its invariance properties, but a lot about the structure of the admissible transformations of its variables, and use this information to greatly delimit the possible equations that express the law." (p. 298). They are generalizations of the kind of laws that have a long-standing tradition in physics, where they are known as laws derived according principles of "dimensional analysis." These principles involve the assertion that laws of nature are in a deep sense invariant under changes of unit, which correspond to invariance under symmetries. Thus, knowledge of the scale type of the relevant variables – a strong presupposition – greatly limits the forms of laws.

Measurement Laws: Simplest Case. These principles were introduced into the behavioral sciences by Luce (1959), which was concerned with special cases of "possible psychophysical laws." He abstracted features of dimensional analysis, the latter only employing ratio scale transformations of its variables, to the more general situation of the known the measurement scale types of the time, but to the more specialized situation of a single function of a single variable. Luce (1964) extended the 1959 formulation to include a few important cases of a single function of many variables.

Luce (1959) considered the case where the independent variable x and the dependent variable y were related by a law, $y = f(x)$, where f was some continuous function. He assumed that this law was invariant under admissible transformations of measurements, that is, for each admissible transformation ϕ of the independent variable, there was an admissible transformation ψ of the dependent variable such that for all x and y,

$$y = f(x) \text{ iff } \psi(y) = f(\phi(x))\,. \tag{1}$$

The following is an example of a use of Luce's theory. Suppose x is an objective variable measured by a ratio scale, e.g., a physical variable such as the intensity of light or the weight of gold, and y is the subjective evaluation of x, e.g., the subjective brightness of light, the subjective value of gold, and f is the law linking x and y. Suppose x and y are both measured on ratio scales and f is continuous. Suppose further that f satisfies Equation 1. Under these conditions, Luce shows that there are real numbers r and a, a depending on ψ, such that

$$f(x) = ax^r\,. \tag{2}$$

His method of proof was to show that Equation 1 implied that f satisfied the functional equation $h(s)f(t) = f(st)$ for some continuous function h and all positive s and t, and that this functional equation had Equation 2 as its only solution.

For most applications, such as the above brightness and subjective value examples, the scale for the dependent variable is known and continuity is a reasonable idealized approximation. Sometimes theory will specify the measurement scale for the dependent variable. However, often the scale for the dependent variable is unknown, and in many cases, unobservable, as, for example, when it is subjective. In such situations, the measurement scale for the dependent variable has to be hypothesized or derived from theory. It can be hypothesized to be one of several theoretically reasonable types of measurement scales, and then methods similar to the one used to derive Equation 2 can be used to arrive at a measurement law for each type of hypothesized scale. The set of resultant measurement laws provides a clear set of quantitative hypotheses for empirical testing. Quite often such hypotheses turn out to be a good place to begin a scientific investigation.

Measurement Laws: More Complex Cases. In a number of ways, Falmagne and Narens (1983) greatly generalized Luce's 1959 approach for deriving laws from measurement considerations. In particular:

- Instead of one independent variable and one dependent variable, they assumed n independent variables and one dependent variable. They formulated matters for two independent variables to simplify notation, but their approach easily extends to n independent variables.)
- They allowed for a general relationship R among the admissible transformations of the independent variables to hold; i.e., for the sets T_i of admissible transformations of the independent variables $x_1, \ldots, x_n$, R can be any nonempty subset of $\prod_i^n T_i$.
- They allowed for more general kinds of laws by allowing for a family $\mathcal{F}$ of functions to related the dependent variable with n independent variables. They interpret $\mathcal{F}$ as follows: Initially, representations $\varphi_1, \ldots, \varphi_n$ are used to measure the n independent variables, $x_1, \ldots, x_n$. These measurements determine a function $f(\varphi_1(x_1), \ldots, \varphi_n(x_n))$ that is the value of the dependent variable measured on an unknown scale when $x_1, \ldots, x_n$ are measured by $\varphi_1, \ldots, \varphi_n$. There are other equally valid ways of measuring *each* independent variable x_i. These are obtained by transforming φ_i by the elements of T_i. However, valid measurements for the *set* of independent variables may be additionally constrained by the empirical law relating the dependent variable to the independent variables. The additional constraint is captured by the relation R. Thus each other valid measurement of the independent variables is given by $\tau_1 * \varphi_1, \ldots, \tau_n * \varphi_n$ for some $\tau_1, \ldots, \tau_n$ such that $R(\tau_1, \ldots, \tau_n)$. The law giving the numerical value of the dependent variable, when the set of independent variables $x_1, \ldots, x_n$ are measured respectively by $\tau_1 * \varphi_1, \ldots, \tau_n * \varphi_n$, is given by

  $$f_{\tau_1,\ldots,\tau_n}(\tau_1 * \varphi_1(x_1), \ldots, \tau_n * \varphi_n(x_n))\,.$$

  In this way, it is the family of functions,

  $$\mathcal{F} = \{f_{\tau_1,\ldots,\tau_n}(\tau_1 * \varphi_1(x_1), \ldots, \tau_n * \varphi_n(x_n)) \,|\, R(\tau_1, \ldots, \tau_n)\,.\}$$

  that expresses the empirical law relating the dependent variable to the independent variables $x_1, \ldots, x_n$. Only in very restrictive cases will $\mathcal{F}$ consist of an single function.

Order Meaningfulness. In place of assuming the scale type of the dependent variable, they assume "order meaningfulness," that is, they assume the following: Using the just presented notation, suppose $\mathcal{F}$ is a family of functions that is a law relating the dependent variable with n independent variables and $f_{\sigma_1,\ldots,\sigma_n}$ and $f_{\tau_1,\ldots,\tau_n}$ are in $\mathcal{F}$. Then for all $x_1, \ldots, x_n$ and $u_1, \ldots, u_n$,

$$f_{\sigma_1,\ldots,\sigma_n}(\sigma_1 * \varphi_1(x_1), \ldots, \sigma_n * \varphi_n(x_n)) \leq f_{\sigma_1,\ldots,\sigma_n}(\sigma_1 * \varphi_1(u_1), \ldots, \sigma_n * \varphi_n(u_n))$$

if and only if

$$f_{\tau_1,\ldots,\tau_n}(\tau_1 * \varphi_1(x_1), \ldots, \tau_n * \varphi_n(x_n)) \leq f_{\tau_1,\ldots,\tau_n}(\tau_1 * \varphi_1(u_1), \ldots, \tau_n * \varphi_n(u_n)).$$

By considering families of functions rather than a single function for laws, Falmagne and Narens generalized the notion of "dimensional constants" that appear in many laws. Their generalization allows for the formulation of behavioral laws (Falmagne and Narens, 1983; Falmagne, 1985) and physical laws (Falmagne, 2004) that cannot be obtained by considering only a single function. Of course, Falmagne and Narens' theory also allows for the case of a single function, by allowing the family of functions to degenerate to a set consisting of a single function.

In many situations order meaningfulness is a testable condition, making it a preferable assumption to assuming a scale type for a dependent variable unless, of course, one already has a well developed theory for the dependent variable. In the Falmagne-Narens theory, the scale type of the dependent variable is not needed to obtain the law linking the independent and dependent variables.

For the case where the family $\mathcal{F}$ consists of a single function f of n-independent variables, Aczél, Roberts, and Rosenbaum (1986) provided more general results. Through an insightful mathematical argument, they were able to characterize measurement laws using only measurability assumptions from real analysis about f instead of monotonicity or continuity assumptions. Aczél and Roberts (1989) use the general approach of Aczél, Roberts, and Rosenbaum (1986) to derive measurement laws of economic interest.

3. Relation Between Meaningfulness and Invariance

Quantitative meaningfulness lacks a serious account as to why it is a good concept of meaningfulness; that is, it lacks a sound theory as to why it should yield correct results. Formulating a serious account for it is difficult. One tack (Krantz, Luce, Suppes, and Tversky, 1971; Luce, 1978; Narens, 1981) is to observe that if meaningfulness expresses valid qualitative relationships, then it must correspond to something purely qualitative, and therefore it should have a purely qualitative description. A long tradition in mathematics for formulating qualitative relationships that belong naturally to some structure or concept goes back to at least nineteenth century geometry and was the center piece of the famous Erlanger Programme for geometry of Felix Klein. It was based on the idea that associated with each geometry was a set of transformations $\mathcal{T}$, and the relations and concepts belonging to the geometry were exactly those that were left invariant by all the transformations in $\mathcal{T}$. There are strong connections between (i) geometric techniques of establishing coordinate systems and measurement techniques for establishing scales, and (ii) the Erlanger Programme's concept of "geometric" and the measurement-theoretic concept "meaningfulness." To examine these connections, some definitions and conventions are needed.

Convention. Throughout the remainder of this entry, it is assumed that $\mathcal{X}$ is a qualitative structure, which consists of a qualitative set X as its domain and relations based on based on X (called the *primitives of* $\mathcal{X}$*);* $\mathcal{N}$ is a numerically based structure, that is, $\mathcal{N}$ is a structure that has a subset of the real numbers as its domain; and $\mathcal{S}$ is the measurement scale consisting of all isomorphisms from $\mathcal{X}$ onto $\mathcal{N}$. (See the entry *Measurement, Theory of* for a more detailed description of this kind of measurement scale.)

Qualitative Meaningfulness. An isomorphism of $\mathcal{X}$ onto itself is called a *symmetry* (or *automorphism*) of $\mathcal{X}$. It easily follows that if α is a symmetry of $\mathcal{X}$ and ϕ and ψ are elements of $\mathcal{S}$, then

- $\phi * \alpha$ is in $\mathcal{S}$
- $\phi^{-1} * \psi$ is a symmetry of $\mathcal{X}$,
- $\theta = \phi * \psi^{-1}$ is an admissible transformation of $\mathcal{S}$, i.e., $\theta * \eta$ is in $\mathcal{S}$ for each η in $\mathcal{S}$, and all admissible transformations can be obtained in the just mentioned manner by appropriate selections of ϕ and ψ.

An n-ary relation R on X is said to be *qualitatively meaningful* if and only if it is invariant under the symmetries of $\mathcal{X}$, that is, if and only if for each symmetry α of $\mathcal{X}$ and each $x_1, \ldots, x_n$ in X,

$$R(x_1, \ldots, x_n) \text{ iff } R(\alpha(x_1), \ldots, \alpha(x_n)).$$

Quantitative Meaningfulness. Although a relation T being "quantitatively meaningful" was previously defined, it is defined here again here to make explicit the role the scale $\mathcal{S}$ plays in qualitative meaningfulness: An n-ary relation T on N is said to be *quantitatively* $\mathcal{S}$*-meaningful* if and only if for each admissible transformation τ of $\mathcal{S}$ and each $r_1, \ldots, r_n$ in N,

$$T(r_1, \ldots, r_n) \text{ iff } T(\tau(r_1), \ldots, \tau(r_n)).$$

$\mathcal{S}$ can be used to interpret T as a relation U on X as follows: The n-ary relation U on X is said to be the $\mathcal{S}$*-interpretation* of T if and only if for all ϕ in $\mathcal{S}$ and all $r_1, \ldots, r_n$

$$T(r_1, \ldots, r_n) \text{ iff } U(\phi^{-1}(r_1), \ldots, \phi^{-1}(r_n)).$$

Basic Result. The above definitions and relationships between symmetries and admissible transformations immediately yield the following theorem relating qualitatively and quantitatively meaningful relations:

Theorem: A relation T is quantitatively $\mathcal{S}$-meaningful if and only if its $\mathcal{S}$-interpretation is qualitatively meaningful.

The above theorem shows that each quantitatively meaningful relation has, through measurement, a corresponding qualitatively meaningful relation. Luce (1978) used this idea to provide a qualitative theory for practice of dimensional analysis in physics: Luce produced a qualitative structure $\mathcal{X}$ for measuring physical attributes. He showed that under measurement, the quantitatively meaningful relationships among the attributes were the "dimensionally invariant functions" of dimensional analysis. It is a principle of dimensional analysis that physical laws are such dimensionally invariant functions. Thus, by the just mentioned theorem, it then follows from the principles of dimensional analysis that each physical law corresponds to a qualitatively meaningful relation of $\mathcal{X}$. (Measurement-theoretic foundations for dimensional analysis can be found in Krantz et al., 1971; Luce et al., 1990; and Narens, 2002.)

Qualitative meaningfulness is just the Erlanger concept of "geometric" applied to science. Mathematically, the two concepts are identical. The Erlanger Programme, as formulated by Klein and as used in mathematics, lacks a serious justification for assuming that the invariance of a relation under the symmetries of a geometry implies that the relation belongs to the geometry.

Scientific Definability. Narens (2002, 2006) sought to find a justification for Klein's assumption. He thought that a reasonable concept of a relation R belonging to a structure $\mathcal{X}$ was that R should somehow be definable in terms of the primitives of $\mathcal{X}$. But the usual concepts of "definable" used in logic failed to provide a match with the Erlanger's concept of "geometric." Narens developed a new definability concept to capture the Erlanger Programme's concept of "geometric." He called the new concept *scientific definability.*

Scientific definability assumes that the quantitative world is constructed from relationships based on real numbers and is completely separated from the qualitative situation under investigation, $\mathcal{X}$, which is conceptualized as a qualitative structure. Unlike definability concepts from logic, scientific definability allows the free use of concepts from the quantitative world for defining relationships based on the domain X of a qualitative structure $\mathcal{X}$. Narens shows that a relation on X is qualitatively meaningful if and only if it is scientifically defined in terms of $\mathcal{X}$.

There is one obvious case where the Erlanger Programme appears to produce a remarkably poor concept of "geometric." This where the geometry $\mathcal{X}$ has the identity function as its only symmetry, yielding that every relation on X is "geometric," and for measurement situations where the scale consists of a single representation, making each relation on the domain of the numerical representing structure quantitatively meaningful, and, thus by the above theorem, each relation on X qualitatively meaningful. There are many important examples of this case, for example the geometry of physical universe under Einstein's general theory of relativity.

Narens (2002) provides generalizations of "scientific definability" that appear to yield reasonable and productive concepts of "geometric" ("qualitatively meaningful") for situations where the geometry (qualitative structure) has the identity as its only symmetry. The main idea for the generalizations is the following: Instead of formulating meaningfulness in terms of a single qualitative structure, a family $\mathcal{F}$ of isomorphic qualitative structures is used. It is assumed that all the structures in $\mathcal{F}$ have the same domain called the *common domain* (of $\mathcal{F}$). A relation R on the common domain is said to be $\mathcal{F}$*-meaningful* if and only if there exist a structure $\mathcal{X}$ in $\mathcal{F}$, primitives $R_{j_1}, \ldots, R_{j_n}$ of $\mathcal{X}$, and a formula φ used for scientific definitions such that

(i) R has a scientific definition in terms $R_{j_1}, \ldots, R_{j_n}$ and φ, and

(ii) R has the same scientific definition for all $\mathcal{X}' = \langle X, R'_j \rangle_{j \in J}$ in $\mathcal{F}$; that is, R has the same scientific definition as in (i) but with $R_{j_1}, \ldots, R_{j_n}$ replaced by $R'_{j_1}, \ldots, R'_{j_n}$.

For the case where $\mathcal{F}$ consists of a single structure, $\mathcal{F}$-meaningfulness coincides with qualitative meaningfulness.

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See also Measurement, Theory of

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