A Foundation for Support Theory Based on a Non-Boolean Event Space

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Abstract

A new foundation is presented for the theory of subjective judgments of probability known in the psychological literature as "Support Theory." It is based on new complementation operations that, unlike those of classical probability theory (set-theoretic complementation) and classical logic (negation), need not satisfy the principles of the Law Of The Excluded Middle and the Law of Double Complementation. Interrelationships between the new complementation operations and the Kahneman and Tversky judgmental heuristic of availability are described.

1 Introduction

Subjective evaluations of degrees of belief are essential in human decision making. Numerous experimental studies have been conducted eliciting numerical judgments of probability, and many interesting phenomena have been uncovered. Amos Tversky and colleagues proposed a cognitive theory to explain some of the more prominent regularities revealed in these studies. This theory, known today as *Support Theory,* has a foundational base in the articles of Tversky and Koehler (1994) and Rottenstreich and Tversky (1997), and incorporates features of cognitive processing, particularly Kahneman's and Tversky's seminal research on judgmental heuristics (as, for example, described in Tversky and Kahneman, 1974). Tversky and Koehler (1994) write,

> The support associated with a given [description] is interpreted as a measure of the strength of evidence in favor of this [description] to the judge. The support may be based on objective data (e.g., frequency of homicide in the relevant data) or on a subjective impression mediated by judgmental heuristics, such as representativeness, availability, or anchoring and adjustment (Kahneman, Slovic, and Tversky, 1982). For example, the hypothesis "Bill is an accountant" may be evaluated by the degree to which Bill's personality matches the stereotype of an accountant, and the prediction "An oil spill along the eastern coast before the end of next year" be assessed by the ease with which similar accidents come to mind. Support may also reflect reasons or arguments recruited by the judge in favor of the hypothesis in question (e.g., if the defendant were guilty, he would not have reported the crime). *(pg. 549)*

How particular heuristically based processes differentially affect probability judgments is the focus of much recent research. This article focuses on a particular heuristic of Kahneman and Tversky, the *availability heuristic.* Tversky and Kahneman (1974) describe it as follows:

> The are situations in which people assess the frequency of a class or the probability of an event by the ease with which instances or occurrences can be brought to mind. For example, one may assess

> the risk of heart attack among middle-aged people by recalling occurrences among one's acquaintances. Similarly, one may evaluate the probability that a given business venture will fail by imagining various difficulties it could encounter. This judgmental heuristic is called availability. Availability is a useful clue for assessing frequency or probability, because instances of large classes are usually recalled better and faster than instances of less frequent classes. However, availability is affected by factors other than frequency and probability. *(pg. 1127)*

Support Theory has an empirical base of results showing that different descriptions of the same event often produce different subjective probability estimates. It explains this in terms of subjective evaluations of supporting evidence. It assumes that events are evaluated in terms of subjective evidence invoked by their descriptions, and that the observed numerical probability judgments are the result of the combining of such evaluations of support in a manner that is consistent with a particular equation (Equation 1 described later). The processes of evaluation are assumed to employ heuristics like those described in various seminal articles by Kahneman and Tversky, and because of this, are subject to the kinds of biases introduced by such heuristics.

This article focuses on support theory phenomena involving the availability heuristic. This focus is formulated in terms of concepts different from those commonly employed by support theorists. In particular, (*i*) it makes a sharp distinction between semantic interpretations of descriptions as part of natural language processing and cognitive interpretations of descriptions as part of a probabilistic judgment, and (*ii*) in modeling judgments of probability it employs two kinds of complementation operations that do not have counterparts in the natural language semantics.

One of the the two complementation operations mentioned in (*ii*) is used to construct cognitive events that are employed in the computation of the estimated probability. The other is used to formulate a structural difference between recall and recognition memory. Both operations have structural (= algebraic) features that differ significantly from the kind of complementation operations considered by cognitive psychologists and support theorists, that is, both have structural features different from the complementation operation of the algebra of events (set theoretic complementation) or of classical logic (negation). In particular, neither need to satisfy the Law of the Excluded Middle[1] and neither need to satisfy the Law of Double Complementation [2]

One of the complementation operations has the formal properties of the negation operation employed in a non-classical logic called the *Intuitionistic Propositional Calculus.* This logic was invented by the mathematician Brouwer for his alternative foundation of mathematics, in which mathematical objects

[1]For event spaces, the Law of the Excluded Middle states that the union of an event and its complement must be the sure event.

[2]For event spaces, the Law of Double Complementation states that the complement of the complement of an event is identical to the event.

are taken to be constructions of the human mind. Heyting (1930) formalized it, and since 1930 it has been a much studied subject by logicians. It has a variety of applications, including ones in artificial intelligence. Its relationship to the notion of "mathematical construction" (Kolmogorov, 1932) makes it a natural candidate for describing structural properties of "mental processing." This article uses it to model the differing roles of recall and recognition in judgments of probability involving the availability heuristic.

The article proceeds as follows: Section 2 presents a summary of the basic concepts of traditional support theory. Section 3 presents a new foundation for support theory phenomena where events are modeled by open sets from a topological space, instead of by sets from a boolean algebra. This shift in modeling allows for the introduction of new mathematical concepts that are useful for modeling the structure of the presumed mental processing used in making probability judgments. Section 4 is a brief discussion of the foundation presented in Section 3, and Section 5 provides a more detailed discussion of mathematical properties of the topological event space used in the foundation. Section 6 explains in terms of the algebraic concepts developed in Section 5 the intuition for various ideas and assumptions employed in the foundation. And Section 7 briefly summarizes what has been accomplished.

2 Traditional Support Theory

Tversky and Koehler (1994) and Rottenstreich and Tversky (1997) presented an empirically based theory of human probability judgments that form the foundation for current support theory. Narens (2007) presented a radically different approach based on an event space of open sets. This article follows and extends part of Narens' approach to make explicit the role of a new kind of event complementation operator in judgments of probability using the availability heuristic.

Support theory tries to explain a variety of empirical phenomena. One of the most prominent is where the subjective probability of an event dramatically increases when it is divided into mutually disjoint subevents and the subjective probabilities of the subevents are added together. The following example of Fox and Birke (2002) illustrates this.

Example: Jones vs. Clinton 200 practicing attorneys were recruited (median reported experience: 17 years) at a national meeting of the American Bar Association (in November 1997). 98% of them reported that they knew at least "a little" about the sexual harassment allegation made by Paula Jones against President Clinton. At the time that the survey, the case could have been disposed of by either

(A) judicial verdict or

(B) an outcome other than a judicial verdict.

Furthermore, outcomes other than a judicial verdict (B) included

(B1) settlement;

(B2) dismissal as a result of judicial action;

(A) judicial verdict	.20
(B) not verdict	.75
Binary partition total	.95
(A) judicial verdict	.20
(B1) settlement	.85
(B2) dismissal	.25
(B3) immunity	.00
(B4) withdrawal	.19
Five fold partition total	1.49

Table 1: Median Judged Probabilities for All Events in Study

(B3) legislative grant of immunity to Clinton; and

(B4) withdrawal of the claim by Jones.

Each attorney was randomly assigned to judge the probability of one of these six events. The results are given in Table 1. Note that the binary partition is logically equivalent to the five fold partition and that the five fold partition yields a substantial increase in probability over the binary partition.

As in the Jones versus Clinton example, several support theory experiments provided professionals with decisions similar to those they routinely make as part of their professional activities. Those experiments also revealed participants making dramatic overestimations. For example, Fox, Rogers, and Tversky (1996) had professional option traders judge the probability that the closing price of Microsoft stock would fall within a particular interval on a specific future date. They found that when four disjoint intervals that spanned the set of possible prices were presented for separate evaluations, the sums of the assigned probabilities were typically about 1.50. However, when binary partitions were presented, the sums of the assigned probabilities were about .98. Redelmeier, Koehler, Liberman, and Tversky (1995) presented a scenario involving a diagnosis, a physical examination, and a medical history to a group of 52 expert physicians a Tel Aviv University. Each physician was asked to evaluate one of the following four outcomes: (1) dying during this admission, (2) surviving this admission but dying within one year, (3) living for more than one year but less than ten years, and (4) surviving for more than ten years. The average judgments added to 164% (95% confidence interval: 134% to 194%). Several other examples presented to various kinds of professionals yielded similar results of overestimation. Numerous studies involving college students also yielded similar results.

The experimental methodology of a typical support theory experiment is based on presenting different descriptions of the same event and obtaining probability judgments for each description. Some experiments are between-participant experiments, where each participant judges only one of the two descriptions, and others are within-participant experiments, where each participant judges both descriptions with intervening judgments occurring between them. The theoretical part of support theory consists of accounting for observed deviations of the probability estimates from what would be expected from a normative model based on classical probability theory. It assumes participants make their judgments based on cognitive heuristics, for example, those described in Tversky & Kahneman (1974), and that the appropriate varyings of descriptions of the same event can manipulate the heuristics employed by participants.

The basic units in support theory are descriptions of events (called "hypotheses" by support theorists). In experimental paradigms, descriptions are presented to participants for probabilistic evaluation. It is assume that participants evaluate the descriptions in terms of a "support function," s, which is a ratio scale into the positive reals. Support theory assumes that the value of $s(\alpha)$ for a description α generally involves the use of judgmental heuristics. Most experiments are designed to elicit a judged (conditional) probability of descriptions of the form, "α occurring rather than γ occurring." Support theory articles generally write this probability as $P(\alpha, \gamma)$, with the assumption that the participant understands that the logical conjunction of α and γ describes and impossibility. The theory assumes $P(\alpha, \gamma)$ is determined by the equation,

$$P(\alpha, \gamma) = \frac{s(\alpha)}{s(\alpha) + s(\gamma)} \, . \tag{1}$$

(A notable exception to this is the extension of support theory by Idson, Krantz, Osherson, and Bonini, 2001, which uses a different equation.)

Support theory makes a distinction between "implicit" and "explicit" disjunctions. A description is said to be *null* if and only if it describes the null event, $\varnothing$. Descriptions of the form "α or γ," where α and γ are nonnull and the description "α and γ" is null, are called *explicit disjunctions.* Throughout this article, $\vee$ stands for the word "or." Thus the explicit disjunction α or γ will often written as $\alpha \vee \gamma$. A description is called *implicit* (or an *implicit disjunction*) if and only if it is nonnull and is not an explicit disjunction. An explicit disjunction δ and an implicit disjunction γ may describe the same event—that is, in the terminology of Tversky and Koehler, have the same *extension,* in symbols, $\delta' = \gamma'$—but have different support assign to them. Tversky and Koehler (1994) provides the following illustration:

> For example, suppose A is "Ann majors in a natural science," B is "Ann majors in biological science," and C is "Ann majors in a physical science." The explicit disjunction, $B \vee C$ ("Ann majors in in either a biological or physical science"), has the same extension as

A (i.e., $A' = (B \vee C)' = (B' \cup C')$), but A is an implicit disjunction because it is not an explicit disjunction. *(*pg. 548)

In their generalization of the support theory of Tversky and Koehler (1994), Rottenstreich and Tversky (1997) distinguishes two ways in which support and explicit disjunction relate. Suppose α is implicit, $\delta \vee \gamma$ is explicit, and α and $\delta \vee \gamma$ describe the same event. Rottenstreich and Tversky assume the following two conditions linking support to implicit descriptions and explicit disjunctions:

(1) *implicit subadditivity:* $s(\alpha) \leq s(\delta \vee \gamma)$.

(2) *explicit subadditivity:* $s(\delta \vee \gamma) \leq s(\delta) + s(\gamma)$.

A direct consequence of (1) and (2) is

(3) $s(\alpha) \leq s(\delta) + s(\gamma)$.

Instead of (2), Tversky and Koehler assumed *additivity,* $s(\delta \vee \gamma) = s(\delta) + s(\gamma)$, which along with (1) yields (3). Rottenstreich and Tversky presented examples where additivity failed but explicit subadditivity (2) held.

There is much empirical evidence in the literature that show (2) and (3) with strict inequality $<$ instead of $\leq$ to be a sizable and robust phenomena. However, the empirical evidence for (1) with strict inequality is much weaker. (See Sloman, Rottenstreich, Wisniewski, Hadjichristidis, and Fox, 2003, for a discussion of the issue.)

In support theory an explicit disjunction that has the same extension as an implicit disjunction α is called an *unpacking* of α. Of course, an implicit disjunction may have many unpackings. The following empirical finding has been much observed in the support theory literature:

Subadditive unpacking: A partition $\mathcal{P}_1 = (\kappa_1, \ldots, \alpha, \ldots, \kappa_n)$ with $n \geq 2$ elements is judged to have probability p_1, and when α is replaced by an unpacking $\delta \vee \gamma$ of it to yield the partition $\mathcal{P}_2 = (\kappa_1, \ldots, \delta \vee \gamma, \ldots, \kappa_n)$ and a judged probability p_2 for $\mathcal{P}_2$, then $p_1 < p_2$.

Tversky's and Koehler's theory implies subadditive unpacking is due to implicit subadditivity, because it assumes additivity. In Rottenstreich's and Koehler's theory, subadditive unpacking can be due to either implicit or explicit subadditivity.

In support theory, participants are presented with a description β that establishes the context for the probabilistic judgment of the description α. In such a situation, α is called the *focal* description. Support theory studies are almost always designed so that the context β contains a description γ, called the *alternative* description, such that it is clear to the participant that β implies that the intersection of the extensions of α and γ is null and that either α or γ must occur. In other words, a binary partition (α, γ) of β is presented to the participant who is asked to judge the probability of α given β, $\alpha \,|\, \beta$. Throughout this article, such a situation is described as, *the participant is asked to judge* $\alpha|\beta$, *"the probability of* α *given* β*."* In addition, in within-participant designs the participant is asked to judge $\gamma|\beta$, and in between-participants designs, $\gamma|\beta$ is judged only by other participants.

For a binary partition (α, γ) of β, both $\alpha|\beta$ and $\gamma|\beta$ are presented. For a ternary partition (θ, σ, τ) of β, all three of $\theta|\beta$, $\sigma|\beta$, and $\tau|\beta$ are presented, where, of course,

$$(\text{the extension of } \theta, \text{ the extension of } \sigma, \text{ the extension of } \tau)$$

is a partition of the extension of β. (Similarly for n-ary partitions for $n > 3$).

3 A Foundation for Support Theory

This section presents a modified and simplified account of the theoretical foundation of support theory presented in Chapter 10 of Narens (2007). The simplification involves only considering judgments based on frequency and the availability heuristic. The modification involves a more detailed account of the role of event complementation operators. The account differs in a number of ways from the traditional support theory formulations. For the purposes of this article, the two most important differences are: (i) the account's use of separate representations for linguistic and cognitive descriptions, and (ii) its use of event spaces consisting of open sets for cognitively representing descriptions.

3.1 Semantic and cognitive representations

In the foundation for support theory presented here, there are two kinds of representations: semantic and cognitive. It is assumed that the descriptions to be presented to a participant for probabilistic evaluation are propositions (sentences) in English. It is further assumed that the set $\mathbf{P}$ of descriptions is closed under the logical operations of disjunction, denoted by "or" or $\vee$, conjunction, denoted by "and" or $\wedge$, and negation, denoted by "not" or $\neg$. $\mathbf{P}$ is assumed to have a *natural semantics,* which is idealized as a function **sem** from $\mathbf{P}$ into a boolean algebra of events $\langle \mathcal{S}, \cup, \cap, - \rangle$ such that for all α and β in $\mathbf{P}$,

$$\mathbf{sem}(\alpha) = \mathbf{sem}(\beta) \ \text{ iff } \ \alpha \text{ and } \beta \text{ are logically equivalent in the natural semantics,}$$

and

$$\begin{aligned} \mathbf{sem}(\alpha \vee \beta) &= \mathbf{sem}(\alpha) \cup \mathbf{sem}(\beta)\,, \\ \mathbf{sem}(\alpha \wedge \beta) &= \mathbf{sem}(\alpha) \cap \mathbf{sem}(\beta)\,, \\ \text{and } \ \mathbf{sem}(\neg\, \alpha) &= -\, \mathbf{sem}(\alpha)\,. \end{aligned}$$

In other words, in the natural semantics $\mathbf{P}$ is interpreted as a boolean algebra of events, with $\vee$ interpreted as union, $\wedge$ as intersection, and $\neg$ as complementation. The natural semantics, as presented here, is not designed to capture the ideas presented by individual descriptions. Instead, they describe the logical relationships among the ideas presented by descriptions—what is sometimes called the "logical form" of the descriptions. For the purposes of this article, this is all that is necessary to assume about the natural semantics.[3]

sem(α) is often called one of the following: (i) the *semantic interpretation of* α, (ii) the *semantic representation of* α, or (iii) the *semantic extension* of α.

Convention 1 Throughout this section, let **P** be, as in the notation just above, a set of descriptions and **sem** be the natural semantics for **P**.

An important concept in support theory is "unpacking." It is defined through the use of the natural semantics as follows:

Definition 1 Let α, δ, γ, and θ be propositional descriptions in **P**. Then the following definitions hold:

- α and δ are said to be *semantically disjoint* if and only if $\mathbf{sem}(\alpha) \cap \mathbf{sem}(\delta) = \varnothing$.
- α is said to be *(semantically) null* if and only if $\mathbf{sem}(\alpha) = \varnothing$.
- $\gamma = \alpha \vee \delta$ is said to be an *explicit disjunction* if and only if α and δ are semantically disjoint and nonnull.
- $\alpha \vee \delta$ is said to be an *unpacking of* θ if and only if $\alpha \vee \delta$ is an explicit disjunction and $\mathbf{sem}(\alpha \vee \delta) = \mathbf{sem}(\theta)$.

In making a probability judgment about a propositional description α, it is assumed that the participant creates a representation of α as part of the probability estimation process. This representation, which is called the *cognitive representation of* α*,* is different from the participant's semantic representation of α. Cognitive representations are also called *cognitive interpretations* or *cognitive extensions.* The foundation models them as open sets from a topology. This form of modeling is possible, because only simple kinds of relationships among cognitive representations are needed, and these are isomorphic to elementary topological relationships among open sets within a topology.

A principal empirical result of support theory is that an unpacking of a proposition usually has a higher judged probability than the proposition. In terms of the foundation's concepts, part of the reason for this is that while the semantic representation of a proposition is the same as semantic representation of its unpacking (because a proposition and its unpacking are logically equivalent in the natural language semantics), the cognitive representation of a proposition usually differs from the cognitive representation of its unpacking.

Convention 2 Throughout this article, $\mathbf{c}(\alpha)$ stands for the cognitive representation of the description α. Also throughout this article it is assumed that $\mathcal{U}$ is a topology with universal set Ω.

[3]The logical form of descriptions and propositional logical relationships among them, for example, logical equivalence, are determined by additional features of the natural language semantics. Because these features play no other role in this article, only their existence needs to be assumed.

The relationships between the semantic and cognitive representations depend in part on the heuristics employed in making the probability judgment. Differing heuristics will usually require differing relationships. The foundation assumes that only the empty set is common to the semantic and cognitive representations, that is, for all α in $\mathbf{P}$,

$$\mathbf{c}(\alpha) = \mathbf{sem}(\alpha) \text{ iff } \mathbf{sem}(\alpha) = \varnothing\,.$$

In some support theory situations $\mathbf{c}$ and $\mathbf{sem}$ are so unrelated that there are descriptions α and γ such that

$$\mathbf{sem}(\alpha) \subset \mathbf{sem}(\gamma) \text{ and } \mathbf{c}(\gamma) \subset \mathbf{c}(\alpha)\,.$$

In other situations, the ranges of $\mathbf{c}$ and $\mathbf{sem}$ display greater similarity in terms of set-theoretic relationships.

3.2 Clear instances

It is assumed that participants are asked to make conditional probability estimates. These estimates are for *conditional descriptions* of the form $\alpha \mid \beta$ ("α is true given β is true"), where

$$\mathbf{sem}(\alpha) \subset \mathbf{sem}(\beta)\,.$$

Most of the support theory literature concerns probability estimations of conditional descriptions of the form $\alpha \mid \alpha \vee \delta$, where $\alpha \vee \delta$ is an explicit disjunction.

In the notation $\alpha \mid \beta$, α is called the *focal description* and β the *conditioning description.*

Convention 3 Throughout this article, $\mathbb{P}(\alpha \mid \beta)$ stands for the participant's probability estimation of the conditional description $\alpha \mid \beta$. The situation under consideration involves participants making a few probability judgments with varying focal descriptions and a common conditioning description, β.

Throughout this section it is assumed that Ω—the universal set of the topology $\mathcal{U}$—is the set of *clear instances of* β; that is, it is assumed that Ω is the set of all instances i such that if the item "i is a clear instance of β," were presented to the participant for judgment on a Yes-No recognition test, then the participant would respond, "Yes." The concept of "clear instance" is discussed in more detail later.

3.3 Recall complement

The foundation models $\mathbf{c}(\alpha)$ as an open set from $\mathcal{U}$ that is a proper subset of Ω—in mathematical notation,

$$\mathbf{c}(\alpha) \in \mathcal{U} \text{ and } \mathbf{c}(\alpha) \subset \Omega = \mathbf{c}(\beta)\,.$$

In making $\mathbb{P}(\alpha \mid \beta)$, it is assumed that participants use of information presented to them, or their own knowledge, to create the *recall complement* of $\mathbf{c}(\alpha)$

with respect Ω. The recall complement of $\mathbf{c}(\alpha)$ with respect Ω is denoted by $\dot{-}\,\mathbf{c}(\alpha)$. It is assumed that

$$\dot{-}\,\mathbf{c}(\alpha) \in \mathcal{U} \quad \text{and} \quad \dot{-}\,\mathbf{c}(\alpha) \cap \mathbf{c}(\alpha) = \varnothing \,.$$

It is not assumed that $\dot{-}$ *is the set-theoretic complement with respect to* Ω*,* that is, it not assumed that $\dot{-}\,\mathbf{c}(\alpha) \cup \mathbf{c}(\alpha) = \Omega$. The operation $\dot{-}$ is called the *operation of recall complementation.*

It should be noted that in many cases the recall complement of $\mathbf{c}(\alpha)$ does not correspond to a description in $\mathbf{P}$. In particular, it is not assumed that $\mathbf{c}(\neg\,\alpha)$ is the recall complement of $\mathbf{c}(\alpha)$.

3.4 Support functions

It is assumed that making the judgment $\mathbb{P}(\alpha\,|\,\beta)$ the participant measures the *support for* α given β, $S^{+}(\alpha)$, measures the *support against* α given β, $S^{-}(\alpha)$, and estimates $\mathbb{P}(\alpha\,|\,\beta)$ in a manner consistent with the formula,

$$\mathbb{P}(\alpha\,|\,\beta) = \frac{S^{+}(\alpha)}{S^{+}(\alpha) + S^{-}(\alpha)} \,.$$

Throughout this article it is assumed that $S^{+}(\alpha)$ is completely determined by $\mathbf{c}(\alpha)$, and that $S^{-}(\alpha)$ is completely determined by $\dot{-}\,\mathbf{c}(\alpha)$. Because $\mathbf{c}(\alpha)$ and $\dot{-}\,\mathbf{c}(\alpha)$ are disjoint, this is equivalent to the existence of a function $\mathbf{S}^{+}$, called the *cognitive support function (for evaluating* $\mathbb{P}(\alpha\,|\,\beta)$*)*, such that

$$\mathbf{S}^{+}(\mathbf{c}(\alpha)) = S^{+}(\alpha) \;\; \text{and} \;\; \mathbf{S}^{+}(\dot{-}\,\mathbf{c}(\alpha)) = S^{-}(\alpha) \,.$$

Any natural language semantic information involved in the judging of $\mathbb{P}(\alpha\,|\,\beta)$ is assumed to be incorporated into the cognitive support function $\mathbf{S}^{+}$ and the cognitive representations $\mathbf{c}(\alpha)$ and $\dot{-}\,\mathbf{c}(\alpha)$. Thus,

$$\mathbb{P}(\alpha\,|\,\beta) = \frac{\mathbf{S}^{+}(\mathbf{c}(\alpha))}{\mathbf{S}^{+}(\mathbf{c}(\alpha)) + \mathbf{S}^{+}(\dot{-}\,\mathbf{c}(\alpha))} \,.$$

3.5 Unpacking

Most support theory experimental paradigms involve unpacking. Let α and β be such that $\mathbf{sem}(\alpha) \subset \mathbf{sem}(\beta)$ and $\gamma \vee \delta$ is an unpacking of α. The following two patterns of results are observed across most studies.

(1) *Implicit subadditivity:* $\mathbb{P}(\alpha\,|\,\beta) \leq \mathbb{P}(\gamma \vee \delta\,|\,\beta)$, and sometimes $\mathbb{P}(\alpha\,|\,\beta) < \mathbb{P}(\gamma \vee \delta\,|\,\beta)$.

(2) *Explicit subadditivity:* $\mathbb{P}(\gamma \vee \delta\,|\,\beta) \leq \mathbb{P}(\gamma\,|\,\beta) + \mathbb{P}(\delta\,|\,\beta)$, and often $\mathbb{P}(\gamma \vee \delta\,|\,\beta) < \mathbb{P}(\gamma\,|\,\beta) + \mathbb{P}(\delta\,|\,\beta)$.

A consequence of (1) and (2) is *subadditivity*, $\mathbb{P}(\alpha \,|\, \beta) \leq \mathbb{P}(\gamma \,|\, \beta) + \mathbb{P}(\delta \,|\, \beta)$. Subadditivity comes in two forms: *additivity*, $\mathbb{P}(\alpha \,|\, \beta) = \mathbb{P}(\gamma \,|\, \beta) + \mathbb{P}(\delta \,|\, \beta)$, and *strict subadditivity*, $\mathbb{P}(\alpha \,|\, \beta) < \mathbb{P}(\gamma \,|\, \beta) + \mathbb{P}(\delta \,|\, \beta)$.

Note that in paradigms designed to test implicit additivity, the participant judges both the propositional description and its unpacking; whereas, in paradigms designed to test for additivity and strict subadditivity, the participant judges the propositional description α but does not judge its unpacking $\gamma \vee \delta$. In the latter, the participant instead makes separate probability judgments of γ and δ. When separate probability judgments are made for γ and δ, subadditivity results by *the experimenter* adding the participant's judgments of γ and δ. In such situations, the sum $\mathbb{P}(\gamma \,|\, \beta) + \mathbb{P}(\delta \,|\, \beta)$ does not correspond to a judgment of a propositional description made by the participant.

3.6 An example involving causes of death

Rottenstreich and Tversky (1997) conducted the following experiment involving availability and implicit and explicit subadditivity. 165 Standford undergraduate economic students were given questionnaires. Each student was presented with two cases for evaluation, with Case 2 being presented a few weeks after Case 1. In both cases, each student was informed of the following:

> Each year in the United States, approximately 2 million people (or 1% of the population) die from a variety of causes. In this questionnaire you will be asked to estimate the probability that a randomly selected death is due to one cause rather than another. Obviously, you are not expected to know the exact figures, but everyone has some idea about the prevalence of various causes of death. To give you a feel for the numbers involved, note that 1.5% of deaths each year are attributable to suicide.

Let

β = death, α = homicide,

α_s = homicide by a stranger, α_a = homicide by an acquaintance,

α_d = daytime homicide, α_n = nighttime homicide,

$\alpha_s \vee \alpha_a$ = homicide by a stranger or homicide by an acquaintance,

$\alpha_d \vee \alpha_n$ = homicide during the daytime or homicide during the nighttime.

For both Case 1 and Case 2, the participants were randomly divided into three groups of approximately equal size. Each group made the following judgments:

Case 1

- judge $\alpha \,|\, \beta$

- judge $(\alpha_s \vee \alpha_a) \,|\, \beta$
- judge both $\alpha_s \,|\, \beta$ and $\alpha_a \,|\, \beta$

Case 2

- judge $\alpha \,|\, \beta$
- judge $(\alpha_d \vee \alpha_n) \,|\, \beta$
- judge both $\alpha_d \,|\, \beta$ and $\alpha_n \,|\, \beta$

Rottenstreich and Tversky predicted that $\alpha_s \vee \alpha_a$ was "more likely to bring to mind additional possibilities than $\alpha_d \vee \alpha_n$." They reasoned,

> Homicide by an acquaintance suggests domestic violence or a partner's quarrel, whereas homicide by a stranger suggests armed robbery or drive-by shooting. In contrast, daytime homicide and nighttime homicide are less likely to bring to mind disparate acts and hence are more readily repacked as ["homicide"]. Consequently, we expect more implicit subadditivity in Case 1,
>
> $$\text{i.e., } \mathbb{P}(\alpha_s \vee \alpha_a \,|\, \beta) - \mathbb{P}(\alpha \,|\, \beta) > \mathbb{P}(\alpha_d \vee \alpha_n \,|\, \beta) - \mathbb{P}(\alpha \,|\, \beta)\,,$$
>
> due to enhanced availability, and more explicit subadditivity in Case 2,
>
> $$\begin{aligned}\text{i.e.,}\quad & \mathbb{P}(\alpha_d \,|\, \beta) + \mathbb{P}(\alpha_n \,|\, \beta) - \mathbb{P}(\alpha_d \vee \alpha_n \,|\, \beta) \;> \\ & \mathbb{P}(\alpha_s \,|\, \beta) + \mathbb{P}(\alpha_a \,|\, \beta) - \mathbb{P}(\alpha_s \vee \alpha_a \,|\, \beta)\,,\end{aligned}$$
>
> due to repacking of the explicit disjunction.

They found that their predictions held: With $\mathbb{P}$ standing for the median probability judgment, they found:

Case 1: $\mathbb{P}(\alpha \,|\, \beta) = .20$ $\quad\mathbb{P}(\alpha_s \vee \alpha_a) = .25$ $\quad\mathbb{P}(\alpha_s) = .15$ $\quad\mathbb{P}(\alpha_a) = .15$
Case 2: $\mathbb{P}(\alpha \,|\, \beta) = .20$ $\quad\mathbb{P}(\alpha_d \vee \alpha_n) = .20$ $\quad\mathbb{P}(\alpha_d) = .10$ $\quad\mathbb{P}(\alpha_n) = .21\,.$

3.7 Simplified account involving availability

The following is a simplified account of probability judgments based on availability and frequency. It is designed to illustrate one of the several uses of modeling cognitive representations as open sets and how the availability heuristic fits into a framework involving event spaces consisting of open sets. It is a variant and a specialization of an account in Chapter 10 of Narens (2007) with some additional and some changed concepts.

3.7.1 Some definitions, conventions, and assumptions

Convention 4 Throughout this section the following is assumed, unless explicitly stated otherwise:

1. β is a description,

2. Ω is the universe of the topology $\mathcal{U}$,

3. Ω is the set of all instances i that a participant would respond "Yes" to in a separate experiment if asked, "Is i a clear instance of β?"

4. θ is an arbitrary description such that

$$\varnothing \subset \mathbf{sem}(\theta) \subset \mathbf{sem}(\beta) \text{ and } \varnothing \subset \mathbf{c}(\theta) \subset \Omega = \mathbf{c}(\beta)\,.$$

5. The participant measures the support for a description in terms of the clear instances of the description that comes to mind, and measures the support against the description in terms of the clear instances that come to mind that violate it.

The set of clear instances of θ are divided into two kinds. The first is $\mathbf{c}(\theta)$ = the set of clear instances of θ that come to mind of the participant in making the probability judgment $\mathbb{P}(\theta \,|\, \beta)$. The second—called the *recognition extension of* θ—consists of all instances i in $\mathbf{c}(\beta)$ that a participant would respond "Yes" to in a separate experiment if asked, "Is i a clear instance of θ?" In other words, the first kind consists of instances that are recalled by the participant in judging $\mathbb{P}(\theta \,|\, \beta)$, and the second kind consists of clear instances of β that the participant would judge to be clear instances of θ in a Yes-No recognition experiment.

Clear instances of θ—that is, the elements of $\mathbf{c}(\theta)$—are called *realized clear instances* of θ. Another kind of clear instance i of θ is where i is a clear instance of θ that is not recalled in the probabilistic judging of θ. Such i are called *unrealized clear instances* of θ. The idea behind this terminology is that the cognitive representation of an unpacking $\gamma \vee \delta$ of α will often produce clear instances of α in either the judging of $\mathbb{P}(\gamma \,|\, \beta)$ or the judging of $\mathbb{P}(\delta \,|\, \beta)$ that were not realized in the judging of $\mathbb{P}(\alpha \,|\, \beta)$.

The recall complement of θ, $\dot{-}\mathbf{c}(\theta)$, need not correspond to a description in $\mathbf{P}$. This because it is a mental construction use to evaluate the support against θ, $S^{-}(\theta)$, in the production of $\mathbb{P}(\theta \,|\, \beta)$ and is not necessarily a cognitive representation of a description. Clear instances of Ω that clearly do not belong to θ also divide into two kinds. The first is the recall complement of θ and consists of those that are in $\dot{-}\mathbf{c}(\theta)$. The second, called the *recognition complement of* θ, consists of those that the participant would judge to be clear instances of $\neg\, \theta$ in a Yes-No recognition experiment. It is assumed for a "Yes" response in such an experiment that the instance under consideration is a clear instance of $\neg\, \theta$, and for a "No" response that it is not a clear instance of $\neg\, \theta$. It is assumed that the recognition complement of θ is an open set in $\mathcal{U}$.

It should be stressed again that it is not assumed that $\dot{-}\mathbf{c}(\theta)$ is $\mathbf{c}(\neg\, \theta)$. In fact, it is expected that in many cases that $\dot{-}\mathbf{c}(\theta) \neq \mathbf{c}(\neg\, \theta)$.

3.7.2 Simplifying assumptions

The support theory literature employs various kinds of cognitive heuristics and stimulus items that have cognitive characteristics that influence the judgings of probabilities. The formalizations of these often require additional cognitive and topological assumptions about cognitive representations and complementation operations that are particular to heuristics and stimulus items employed. For the portion of literature that is the focus of this article, the following simplifying assumptions are made:

- (The recognition extension of θ) $\cap$ (the recognition complement of θ) $= \varnothing$.
- $\mathbf{c}(\theta) \subseteq$ the recognition extension of θ.
- $\dot{-}\,\mathbf{c}(\theta) \subseteq$ the recognition complement of θ.

The above simplifying assumptions imply that

$$\mathbf{c}(\theta) \cap \dot{-}\,\mathbf{c}(\theta) = \varnothing . \tag{2}$$

Equation 2 is a reasonable extrapolation for most support theory experiments that rely on the traditional use of the availability heuristic. However, for some experiments *relying on other heuristics,* one would expect many examples of instances i, where i clearly belongs to $\mathbf{c}(\theta)$, when judging $\mathbf{S}^{+}(\theta)$, and clearly belongs to $\dot{-}\,\mathbf{c}(\theta)$, when judging $\mathbf{S}^{+}(\dot{-}\,\mathbf{c}(\theta))$.

The following additional simplifying assumptions are made, where $\gamma \vee \delta$ is an unpacking of α. The first is that

$$\mathbf{c}(\alpha) \subseteq \mathbf{c}(\gamma \vee \delta) \subseteq \mathbf{c}(\gamma) \cup \mathbf{c}(\delta) . \tag{3}$$

The intuition for Equation 3 is that the unpacking of α into $\gamma \vee \delta$ makes more clear instances of α available to the participant in a probability judging task, and the separate judgings of γ and δ, even makes more clear instances of α available to the participant. This naturally leads to the simplifying assumption,

$$\mathbf{S}^{+}(\mathbf{c}(\alpha)) \le \mathbf{S}^{+}(\mathbf{c}(\gamma \vee \delta)) \le \mathbf{S}^{+}(\mathbf{c}(\gamma)) + \mathbf{S}^{+}(\mathbf{c}(\delta)) . \tag{4}$$

Support theory experiments are usually designed with the intent of showing judgments that violate presumed normative rules of probability by selecting α, γ, δ, and β in manners so that

$$\text{strict subadditivity:} \quad \mathbb{P}(\alpha \,|\, \beta) < \mathbb{P}(\gamma \,|\, \beta) + \mathbb{P}(\delta \,|\, \beta) ,$$

is observed.[4] There are a number of factors that can contribute to the production of strict subadditivity. The ones most cited in the literature are Equation 4 and that in the computation of

$$\mathbb{P}(\theta \,|\, \beta) = \frac{\mathbf{S}^{+}(\mathbf{c}(\theta))}{\mathbf{S}^{+}(\mathbf{c}(\theta)) + \mathbf{S}^{+}(\dot{-}\,\mathbf{c}(\theta))}$$

for $\theta = \gamma, \delta$, more attention is used in the mental formation and analysis of $\mathbf{c}(\theta)$ than in $\dot{-}\,\mathbf{c}(\theta)$, yielding a bias that tends to raise of $\mathbf{S}^{+}(\mathbf{c}(\theta))$ relative to $\dot{-}\,\mathbf{S}^{+}(\mathbf{c}(\theta))$, which, through a simple mathematical calculation, produces a bias towards an increase in $\mathbb{P}(\theta \,|\, \beta)$ (e.g., see Brenner and Rottenstreich, 1999).

[4] For examples designed to violate $\mathbb{P}(\alpha \,|\, \beta) \le \mathbb{P}(\gamma \,|\, \beta) + \mathbb{P}(\delta \,|\, \beta)$ see Sloman, Rottenstreich, Wisniewski, Hadjichristidis, & Fox (2004).

4 Discussion of the Foundation

One of the key features of the foundation presented in this article is the sharp distinction between the semantic processing employed in the use of language and cognitive processing employed specifically for probability judgments. The lack of such a distinction has, in my view, generated some misunderstanding and controversies in the literature.

In the simplified form of the semantics presented here, the logical connectives "and", "or", and "not," which act on propositional descriptions to produce other propositional descriptions, are in the natural language semantics interpreted as, respectively, $\cap$, $\cup$ and $-$, which are operations on on sets. The foundation has not provided for how these logical connectives are to be interpreted in the cognitive representations used in making probability judgments. Obviously, the foundation would be greatly enhanced with the addition of such interpretations. But first a great deal of empirical research is needed to establish basic facts about them and their relationship to the recall complementation operator $\dot{-}$.

The foundation presented here was designed for the kinds of studies generally conducted by support theorists. Some probabilistic estimation tasks do not fall into this paradigm. For example, presenting a partition and asking the participant "to assign probabilities to each of the alternatives so that the probabilities add to 1." A central feature of the foundation is that the participant creates a complement $\dot{-}A$ of a cognitive event A and assigns probabilities through some comparison between A and $\dot{-}A$. What distinguishes the foundation from other approaches to support theory is that the support functions are not on events from a boolean algebra of events.

Others in the literature have generalized support theory's foundation by providing alternatives to the formula,

$$P(\alpha, \gamma) = \frac{s(\alpha)}{s(\alpha) + s(\gamma)} \,,$$

where $P(\alpha, \gamma)$ is the subjective probability of α rather than γ occurring for disjoint propositions α and γ. For example, Idson, Krantz, Osherson, & Bonini. (2001) use the formula,

$$P(\alpha, \gamma) = \lambda \frac{s(\alpha)}{s(\alpha) + s(\gamma)} + (1 - \lambda) \frac{s(\alpha)}{s(\alpha) + K},$$

where λ and K are positive constants that depend on the participant and the method of evaluation he or she employs. Like support theory, this generalization assumes an underlying boolean structure—the same kind of structure demanded by rationality and logic for ordinary propositions. To my knowledge, scientifically or philosophically based justifications for this assumption for situations involving the psychological processing of information have not been attempted in the literature. This article's approach is to retain boolean logic for natural language semantics and the support theory's principle that

$$\mathbb{P}(\alpha|\beta) = \frac{\text{the support for } \alpha|\beta}{\text{the support for } \alpha|\beta + \text{the support against } \alpha|\beta} \,,$$

but to allow the mental interpretations on which the support function acts to be part of a logical structure that naturally arises out of the judgmental heuristics employed. This allows for different kinds of heuristics to give rise to different kinds of logical structures.

The basic concepts used in the foundation and simplifying assumptions have a logical structure that is best explicated through topological concepts. This is done in detail in Section 6. The basic idea is that the use of open sets can capture the structural properties of the various complementations used in the foundation and the simplifying assumptions. Because algebras based on open sets from a topology are richer in structure than boolean algebras of sets, they provide a richer set of concepts for use in modeling than boolean algebras of sets. For example, Narens (2007) uses features of the boundary of open sets to model various kinds ambiguity that can be associated with events. This article's foundation uses the boundary of open sets to explicate the role of unrealized elements in describing the effect of unpacking on probability judgments. Different heuristics or even different kinds of uses of the same heuristic may require topologies with properties that are peculiar to them. For example, this article's use of the availability heuristic makes special assumptions about recall and recognition memory by assuming a generation-recognition model of recall.[5] It also does not allow for "ambiguous recall," that is, does not allow for some description α that $\mathbf{c}(\alpha) \cap (\dot{-}\mathbf{c}(\alpha)) \neq \varnothing$. Other heuristics or experimental situations may require more complicated models of memory and ambiguous recall, which in turn may require different topological assumptions to account for observed phenomena.

5 Event Spaces Based on Open Sets

5.1 Algebraic properties

The difference between boolean algebras of sets and event spaces based on open sets is due to the kind of complementation operation assumed: a boolean algebra of sets assumes set-theoretic complementation, denoted by $-$, whereas an event space based on open sets assume an operation called "pseudo complementation," denoted by $\ulcorner$ and defined in Definition 3 below.

The following definitions and theorems provide the algebraic concepts and properties of event spaces based on open sets and pseudo complementation.

Definition 2 A collection $\mathcal{V}$ is said to be a *topology with universe* X if and only if X is a nonempty set, $X \in \mathcal{V}$, $\varnothing \in \mathcal{V}$, for all A and B in $\mathcal{V}$, $A \cap B$ is in $\mathcal{V}$, and for all nonempty $\mathcal{W}$ such that $\mathcal{W} \subseteq \mathcal{V}$,

$$\bigcup W \text{ is in } \mathcal{V}\,. \tag{5}$$

[5] The generation-recognition model of memory states that recall is a two stage process: In the first stage, the participant generates alternatives to a recall probe; in the second, he or she selects (i.e., recognizes) the alternative(s) satisfying the recall probe. This model was designed to explain the important and often observed fact that for most kinds of items, recognition is easier than recall.

Note that it is immediate from Equation 5 that for all A and B in $\mathcal{V}$, $A \cup B$ is in $\mathcal{V}$.

Let E be an arbitrary subset of X and $\mathcal{V}$ be a topology with universe X. Then the following definitions hold:

- E is said to *open (in the topology $\mathcal{V}$)* if and only if $E \in \mathcal{V}$.
- E is said to be *closed (in the topology $\mathcal{V}$)* if and only if the set-theoretic complement of E with respect to X, $-E$, is open.

It immediately follows that X and $\varnothing$ are closed as well as open. In some cases $\mathcal{V}$ may have X and $\varnothing$ as the only elements that are both open and closed, while in other cases $\mathcal{V}$ may have additional elements that are both open and closed. The following definitions hold for all $E \subseteq X$:

- The *closure* of E, $\mathsf{cl}(E)$, is, the smallest closed set C such that $E \subseteq C$; that is,
$$\mathsf{cl}(E) = \bigcap \{B | B \text{ is closed and } E \subseteq B\}\,.$$
- The *boundary* of E, $\mathsf{bd}(E)$, consists of those elements of $\mathsf{cl}(\mathsf{E})$ that are not in E.
- The *interior* of E, $\mathsf{int}(E)$, is the largest open set D such that $D \subseteq E$; that is,
$$\mathsf{int}(E) = \bigcup \{F | F \text{ is open and } F \subseteq E\}\,.$$

It easily follows that the definition of "topology" implies the existence of the closure, interior, and boundary of E for all $E \subseteq X$.

Definition 3 $\mathfrak{X} = \langle \mathcal{X}, \cup, \cap, \ulcorner, X, \varnothing \rangle$ is said to be a pseudo complemented open set algebra of $\mathcal{V}$ if and only $\mathcal{V}$ is a topology, $\mathcal{X} \subseteq \mathcal{V}$, and *with respect to* $\mathcal{V}$,

$$\ulcorner\ A = \mathsf{int}(\mathsf{cl}(\,-A))\,,$$

for all A in $\mathcal{X}$. $\ulcorner$ is called the *pseudo complementation operator of* $\mathfrak{X}$.

$\mathfrak{X} = \langle \mathcal{X}, \cup, \cap, \ulcorner, X, \varnothing \rangle$ is said to be a *pseudo complemented open set algebra* if and only if for some $\mathcal{V}$, $\mathfrak{X}$ is a pseudo complemented open set algebra of $\mathcal{V}$.

Pseudo complemented open set algebras obviously exist, because

$$\mathfrak{V} = \langle \mathcal{V}, \cup, \cap, \ulcorner, X, \varnothing \rangle$$

is a pseudo complemented open set algebra, where $\mathcal{V}$ is a topology with universe X. In particular, if $\mathcal{V}$ is a topology where each open set is closed, then $\ulcorner = -$, and thus $\mathfrak{V}$ is a boolean algebra.

Theorem 1 *Suppose $\mathfrak{X} = \langle \mathcal{X}, \cup, \cap, \ulcorner, X, \varnothing \rangle$ is a pseudo complemented open set algebra. Then the following eight statements are true for all A and B in $\mathcal{X}$:*

1. $\neg X = \varnothing$ *and* $\neg \varnothing = X$.

2. *If* $A \cap B = \varnothing$*, then* $B \subseteq \neg A$.

3. $A \cap \neg A = \varnothing$.

4. *If* $B \subseteq A$*, then* $\neg A \subseteq \neg B$.

5. $A \subseteq \neg\neg A$.

6. $\neg A = \neg\neg\neg A$.

7. $\neg (A \cup B) = \neg A \cap \neg B$.

8. $\neg A \cup \neg B \subseteq \neg (A \cap B)$.

Proof. See Narens (2003) or Narens (2007).

The following theorem gives some fundamental properties of boolean algebras of sets that fail for some pseudo complemented open set algebras.

Theorem 2 *There exists a pseudo complemented open set algebra* $\mathfrak{X} = \langle \mathcal{X}, \cup, \cap, \neg, X, \varnothing \rangle$ *such that the following three statements are true about* $\mathfrak{X}$.

1. *For some* A *in* $\mathcal{X}$, $A \cup \neg A \neq X$.

2. *For some* A *in* $\mathcal{X}$, $\neg\neg A \neq A$.

3. *For some* A *and* B *in* $\mathcal{X}$, $\neg (A \cap B) \neq \neg A \cup \neg B$.

Proof. Let X be the set of real numbers, $\mathcal{X}$ be the be usual topology on X determined by the usual ordering on X, C be the infinite open interval $(0, \infty)$, and D be the infinite open interval $(-\infty, 0)$. Then the reader can verify that Statement 1 follows by letting $A = C$, Statement 2 by letting $A = C \cup D$, and Statement 3 by letting $A = C$ and $B = D$.

The $\neg$ operator has the properties of the negation connective of intuitionistic logic. This logic was formalized in Heyting (1930) as a description of the logical principles the mathematician L. L. J. Brouwer used in his alternative form of mathematics. (For a complete, formal account of intuitionistic logic see Rasiowa and Sikorski, 1968.) Although Heyting designed his logic for Brouwer's mathematics, it was shown to have other applications. For example, Kolmogorov (1932) showed that it had the correct formal properties of a theory of mathematical constructions. Kolmogorov achieved this result by giving interpretations to the logical primitives that were different from Heyting's. Similarly, this article provides a new interpretation for the negation operator of intuitionistic logic as the operation of recall complementation.

Pseudo complemented event algebras share many features of intuitionistic logic. The principle difference is that intuitionistic logic is based primarily on an implication connective that is not part of pseudo complemented event algebra. Logical implications do not play a role in theory of probabilistic judgments presented here, because probabilities are computed directly in terms of the supports for a cognitive event and its recall complement.

6 Open set modeling

As previously discussed, event spaces that are pseudo complemented algebras of open sets provide a richer set of modeling concepts than are available for event spaces that are boolean algebras sets. The important topological modeling idea used in this article is that different roles can be given to the elements of an open set and its boundary. A related distinction cannot be made for boolean algebras, because notions that functions like "boundary" in the just-mentioned topological modeling are not formulable using only boolean concepts.

In the foundation presented in this article, the elements of an open set and its boundary are interpreted as memory instances of a description. The open set is interpreted as the set of recalled clear instances of the description. Its boundary is interpreted as either unrealized clear instances or various kinds of poor, vague, or ambiguous instances. A separation of boundary points into various topological kinds, allows for subtle distinctions to be made ambiguity, vagueness, and poorness of recalled and recognition instances of descriptions. For the simplified account presented in this article, only the distinction between realized and unrealized instances was needed.

In judging the probability of $\alpha \,|\, \beta$, the foundation assumes the existence of cognitive representations $\mathbf{c}(\alpha)$ and $\dot{-}\,\mathbf{c}(\alpha)$ that are open sets from a topology $\mathcal{U}$ with universe Ω. $\mathbf{c}(\alpha)$ and $\dot{-}\,\mathbf{c}(\alpha)$ are subjective, and, according to the foundation, are realized and judged by the participant in a way that matches the equation for $\mathbb{P}(\alpha \,|\, \beta)$ given in Equation 1. Ω, which is the recall extension of β, is, in general, not realized by the participant. Similarly, it follows from assumptions of the foundation that the recognition complement of α is the pseudo complement of $\mathbf{c}(\alpha)$ in the topology $\mathcal{U}$, that is, is the open set $\ulcorner\, \mathbf{c}(\alpha)$ in $\mathcal{U}$ that is the interior of the set-theoretic complement (with respect to Ω) of $\mathbf{c}(\alpha)$.

For the simplified situation considered in this article involving the availability heuristic and judgments based on frequency, the boundary of $\mathbf{c}(\alpha)$ is modeled so that it consists only of unrealized clear instances. It also follows from the simplified assumptions that

$$\text{(the recall extension of } \alpha) \cap \text{(the recognition complement of } \alpha) = \varnothing$$

and

$$\text{(the recall extension of } \alpha) \cup \text{(the recognition complement of } \alpha) = \Omega\,.$$

From these assumptions, the foundation, and the definition of pseudo complementation, it then follows that

$$\text{recall extension of } \alpha \;=\; \ulcorner\ulcorner\, \mathbf{c}(\alpha)\,, \tag{6}$$

$$(\ulcorner\, \mathbf{c}(\alpha)) \cup (\ulcorner\ulcorner\, \mathbf{c}(\alpha)) \;=\; \Omega\,, \tag{7}$$

$$\text{and} \quad \ulcorner\ulcorner\ulcorner\, \mathbf{c}(\alpha) \;=\; \ulcorner\, \mathbf{c}(\alpha)\,. \tag{8}$$

Note that Equation 8 is a pseudo complementation law (Statement 6 of Theorem 1) applied to $\mathbf{c}(\alpha)$. A similar law, $- - - \,\mathbf{c}(\alpha) = -\,\mathbf{c}(\alpha)$ also holds for set-theoretic complementation. Equation 7 is a special case of a pseudo complementation law (Statement 3 of Theorem 2 and Statement 5 of Theorem 1) which has the form of the Law of the Excluded Middle for pseudo complemented events. For most support theory applications, $\mathbf{c}(\alpha) \neq \ulcorner\ulcorner\, \mathbf{c}(\alpha)$, which violates the form of the set-theoretic Law of Double Complementation, $\mathbf{c}(\alpha) = - - \mathbf{c}(\alpha)$. Equations 6 and 7 follow from availability and the simplifying assumptions. In other support theory phenomena which employ other heuristics or simplifying assumptions, Equations 6 and 7 may fail.

The foundation is based on the premise that a pseudo complemented event algebra better models the structure of mental phenomena and behavior associated with subjective estimations of probabilities than boolean algebras of sets. From one point of view, this is hardly surprising: Pseudo complemented event algebras correspond to a major part of intuitionistic logic, a subject matter originally designed for a foundation of mathematics in which mathematical objects were construed to be mental constructions, whereas, boolean algebras of sets correspond to classical propositional logic, a subject matter designed for platonic objects. From another point of view it is obvious: Because pseudo complemented event algebras are more general than boolean algebras of sets, they allow for a richer base of modeling concepts. However, for the purposes of this article the reason may be put as follows: Pseudo complementation can be used to derive basic memory relationships (as described by, say, the generation-recognition model of memory) that are used in judgments involving the availability heuristic. More generally, one can view a heuristic as having a "logic" associated with it, with different heuristics generally having different logics. The logic associated with the availability heuristic is much more like a pseudo complemented event algebra than a boolean algebra of sets.

7 Conclusions

Typically, boolean algebras of sets have been used for the psychological modeling of event algebras involving subjective probability. There are other event algebras that have been studied for some time in mathematics and logic that may be more appropriate for this. In my view, the most appropriate are open sets from a topology (corresponding to intuitionistic logic) and closed subspaces of a hilbert space (quantum logic). To my knowledge, although quantum logic has been used in the modeling of psychological decision making, it not been used to model support theory phenomena. This article suggests that open sets from a topology provide a richer set of useful concepts for the understanding and modeling support theory phenomena than boolean algebras of sets.

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