Endogenous formation of alliances in conflicts

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Abstract

This chapter surveys formal models of endogenous alliance formation in political science and economics. Using a unified model of conflict, we obtain general results on the effect of group sizes on conflict and on the size of stable alliances. We also discuss recent work on endogenous sharing rules and dynamic alliance formation in nested conflicts.

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1 Introduction

In many situations of conflict, agents pool resources and efforts to fight their adversaries. Alliances have been formed to win wars, political elections or rent-seeking contests. The history of warfare is a succession of alliances forming and collapsing over time, from the alliances between warlords in Ancient China to the intricate systems of alliances in the multipolar world of XIXth century Europe. The formation of coalitions of countries supporting the United States during the two Gulf Wars shows that alliance formation remains an essential aspect of international relations.

The formation of alliances has been studied both by political scientists and economists. The contribution of political scientists has centered around the following set of questions: Why are formal alliances concluded among states? Do alliances inhibit or foster wars? When do countries comply to the alliance treaty? Who forms alliances with whom? In order to answer these questions, scholars of international relations rely on diverse methods, ranging from formal game-theoretical models to case studies and empirical studies based on large historical data sets on wars and alliances. By contrast, economists have mainly focussed on theoretical models of conflict based on Tullock (1967)'s seminal specification of rent-seeking contests. By extending the model in various directions, economists have analyzed under which circumstances universal peace can be sustained, which alliances are endogenously formed by rational agents, how agents inside an alliance share the prize they obtain, and how successive nested contests are resolved.

In this chapter, we follow two objectives. We first aim at providing a comprehensive survey of the theoretical literature on alliance formation, arising both from political science and economics, in order to draw comparisons and contrasts between the different models of alliance formation. By lack of space, we will not be able to offer an exhaustive account of the empirical literature on alliance formation in international relations, and will only briefly mention references that the interested reader may consult in order to access the wealth of case studies and empirical work done by scholars in this area. Our second objective is to provide new results on alliance formation in contests by considering a unified model of conflict which encompasses most of the literature on public and private good contests. Within this unified model, we obtain new comparative statics results on group sizes and conflict (Section 3) and new results on endogenous alliance formation (Section 4), which generalize existing results and connect results dispersed in the literature.

Because the theoretical literatures on alliance formation in political science and economics focus on different questions and use different basic models, we have chosen to present them separately, and start by discussing formal political models of alliance formation. In the neo-realist view of international relations, alliances are one of the modes of organization of states in a world of anarchy where treaties cannot be enforced, and states follow independent foreign policies (Waltz (1979) and Snyder (1997)). Alliances formed in the multipolar world of the late XIXth century differ from the alliances formed in the bipolar world following World War II because membership in alliances which was totally open in the multipolar world (anybody can ally with anybody) becomes restricted in the bipolar world (where countries only have one choice, whether to ally or not). However, in both cases, alliances are formed to restore the balance of power, and stable alliance structures must entail alliances of similar sizes. The main trade-off faced by allies is the tension between abandonment (the fact that allies may fail to fulfill their commitments) and entrapment (the moral hazard problem due to the fact that countries, emboldened by the formation of the alliance, may choose to start conflict or take risky actions at the expense of their allies). We illustrate this neo-realist perspective on alliance formation and balance of power by discussing the game-theoretic model investigated by Niou and Ordeshook in a series of papers in the late 1980's.[1]

The institutionalist (or neo-liberal) view of alliance formation differs from the neo-realist view in important respects. First, institutionalists do not consider the world of international relations as a world of anarchy, but instead pay attention to the institutions and organizations put in place in the arena of international relations. In the case of alliance formation, this implies an emphasis on the details of international treaties, including the exact synergies among allies beyond the general principles of balance of power. The fact that international treaties are concluded after long negotiations, that alliances are costly to maintain indicates the importance and relevance of alliances in international relations. Morrow (1994) explains the formation of alliances as a signaling device, where countries wish to signal their willingness to fight alongside their allies if attacked by a challenger. Smith (1995) and (1998) develops a similar explanation, but views alliance formation as a commitment device, where countries tie their hands (because they will have to pay a "honor cost" if they don't abide by their agreement to help their allies) in order to deter the challenger from attacking. Fearon (1997) brings together both analyses, and shows that commitment is a more effective tool than signaling in a general model of deterrence.

Moving to the economics literature on alliance formation, we first consider models of collective action. Following the seminal paper by Olson and Zeckhauser (1966), an important strand of the literature, reviewed by Sandler (1993) and Hartley and Sandler (2001) has focused on *the public good problem* associated to the alliance. In an alliance, parties have an incentive to free-ride on the efforts and resources of other members of the alliance, and small parties have a tendency to rely on the efforts of larger parties. As a consequence, the total effort of the alliance is suboptimal – the classical public good problem – and large parties bear a disproportionate share of the cost of the alliance – the exploitation problem faced for example by the United States in NATO and other international organizations.

This free-riding problem had already been emphasized in Olson (1965)'s earlier

[1] Niou and Ordeshook (1991) do not consider their analysis as a "neo-realist" approach, and rightly point out that their theoretical model can be used to study both the neo-realist view and the liberal view on international relations . However, because the focus of their model is on the balance of power, we have chosen to present it as an illustration of the neo-realist view.

theory of collective action. While Olson and Zeckhauser (1966) focus on the free-riding problem in a single alliance faced with an exogenous threat, subsequent work has considered conflict across different groups, each of them being plagued by the free-riding problem. In this context, Olson (1965) postulated that small groups should be more effective and more likely to win conflicts, as the free-riding problem they face is less acute than in larger groups. This "paradox of group size" has generated a sizeable literature in political science and economics, which attempts to formalize Olson (1965)'s ideas and delineate conditions under which small groups are more effective in conflict.

Tullock (1967)'s formalization of rent-seeking contests provides the framework in which Olson (1965)'s ideas can be expressed. However, the structure of the probabilistic model of conflict of Tullock (1967) is unexpectedly complex, and the comparative statics results needed to establish whether small groups are more effective, remain elusive. Early work by Chamberlin (1974), McGuire (1974), Sandler (1992) and others established that, in some specific contexts, small groups are more effective when the conflict involves a rival private good, whereas large groups are more effective when agents fight over a nonrival public good.

Tullock (1980), Katz and Tokatlidu (1996), Baik and Lee (1997) and Wärneryd (1998) analyze two-group contests with different specifications of the prizes, and obtain partial comparative statics results on the sizes of the groups, suggesting again that small groups are more effective, and that total expenditures on conflict are higher when the two groups are more symmetric. Esteban and Ray (1999) and (2001) revisit Olson (1965)'s arguments in a more general model of conflict, emphasizing the complexity of the noncooperative game of conflict played among groups of different sizes. Esteban and Ray (2001) in particular show that the "common wisdom" according to which small groups are always more effective in conflicts over private goods does not necessarily hold when one considers general conflict technologies.

The comparative statics analysis of the effect of group sizes in conflict is a necessary building block to understand the formation of alliances. Using recent advances in the theory of coalition formation, Skaperdas (1998), Tan and Wang (1997), Esteban and Sakovics (2003), Garfinkel (2004), Bloch, Sanchez-Pages and Soubeyran (2006) and Sanchez-Pages (2007) attempt to endogenize the formation of alliances in model of contests. While each of the papers focuses on a different specification and different procedures of coalition formation, two general intuitions emerge from this strand of the literature.

First, as shown by Skaperdas (1998), Tan and Wang (1997)and Esteban and Sakovics (2003) in three-player models, *parties have no incentive to form a two-player alliance against the third* unless the formation of the alliance generates synergies which enhance the winning probability of the alliance. This negative result, labeled by Konrad (2009) "the paradox of alliance formation", relies on a very basic intuition. Inside an alliance, agents have less incentive to provide effort for two reasons: first, they only obtain a fraction of the prize instead of its full value because they still have to share the prize with the other alliance member (possibly having to fight in a second contest with her), and second the choice of effort inside the alliance is subject to free-riding. If the "paradox of alliance formation" indeed holds, economists need to explain why alliances are actually formed among states or parties in conflicts.

The second message of the literature is that *the grand coalition plays a special role.* In the grand coalition, the prize is peacefully shared among alliance members and no resources are wasted on conflict. There is a significant, qualitative gap between the payoff of players in the grand coalition and any other coalition structure. In a model where all coalitions may possibly form, Bloch, Sanchez-Pages and Soubeyran (2006) and Sanchez-Pages (2007) observe that the special role of the grand coalition implies that it will often emerge as the equilibrium of the process of alliance formation. This result poses another challenge to economists, who need to explain why universal peace does not always prevail, and why smaller, competing alliances form.

The economic literature on alliance formation has developed along two other themes. First, following Nitzan (1991a), attention has been focused on *sharing rules within the alliance.* Nitzan (1991a) proposes a sharing rule which is a convex combination of the egalitarian rule (a "soft" rule which does not give high incentives to group members) and the relative effort rule (a "hard rule" which, as in a rat race, induces group members to exert high effort). Nitzan (1991a) and (1991b) (amended by Davis and Reilly (1999) and Ueda (2002)) computes the equilibrium level of conflict under different rules. Lee (1995), Baik and Shogren (1995), Baik and Lee (1997), Noh (1999) and (2002) endogenize the sharing rule and the sizes of groups. One of the most interesting outcomes of their analyses is that, in a contest among two groups of equal size, both groups have an incentive to choose the hard relative effort rule, even though they would both be better off with the soft egalitarian rule – a classical instance of the prisoner's dilemma. This result, is reminiscent of models of strategic delegation in games: both groups have an incentive to choose a hard rule (or a tough delegate), in order to increase their winning probability, even though this will result in large wasteful expenditures on conflict.

The second theme is the distinction between static and dynamic group contests. In a static group contest, the prize is shared among alliance members immediately; in a dynamic group contest (or "nested contest"), members of an alliance award the prize by a succession of contests, until only two contestants remain in the end. The dynamic model of contest has only received attention in recent years – Skaperdas (1998), Tan and Wang (1997), Wärneryd (1998), Esteban and Sakovics (2003), Garfinkel (2004) and Kovenock and Konrad (2008) propose different specifications of this model – and has only been studied in very restrictive contexts.

The rest of the chapter is organized as follows. The next Section discusses political models of alliance formation. Section 3 investigates the effect of group sizes on conflict. Section 4 studies general models of endogenous alliance formation based on recent advances in game theory. Section 5 provides a detailed analysis of sharing rules inside the alliance. Section 6 is devoted to models of dynamic contests and alliance formation. Finally, we conclude and provide directions for future research in the last Section.

2 Political models of alliance formation

An extensive literature in political science studies the formation of alliances among states.[2] Scholars of international relations recognize that states can enter into a large spectrum of agreements, from tacit alignment and informal entente to formal, explicit alliances. The political science literature on alliance formation has focussed on the last type of agreements, proposing theoretical models of alliance formation and compliance, introducing detailed case studies of alliance formation, and analyzing the impact of alliance formation on war using large data sets of conflicts.

A typology of alliances distinguishes between alliances formed in a multipolar world (like the late XIXth, early XXth century), where countries could freely choose their allies, and alliances formed in a bipolar world (like the late XXth century, following the second world war), where countries can only choose whether or not to join a fixed alliance. Political scientists have also noted the wide variety of alliance treaties, ranging from offensive alliances (where countries agree to fight together) to defensive alliances (where countries only commit to intervene when one country is attacked) and nonaggression and neutrality pacts (where countries commit to nonintervention). Another important distinction draws a line between symmetric alliances concluded among countries of similar size with similar military capabilities, and asymmetric alliances grouping one large power with smaller countries.

Empirical studies have analyzed the relation between the formation of alliances, the occurrence of war and the compliance of allies. A first strand of the literature, using the "Correlates of War" database assembled by Singer and Small (1967) has concluded that the formation of offensive alliances increases the risk of war whereas the formation of defensive alliances reduces the occurrence of war and the occurrence of war (Siverson and King, 1980). Surprisingly, the same data set shows that the rate of compliance of countries to alliances is particularly low – according to Sabrosky (1980), only 27 % of the allies fulfill their commitment. This last conclusion has recently been questioned by Leeds et al. (2000) who have assembled a new data set on alliances, the Alliance Treaties Obligations and Provisions (ATOP), which contains detailed information about the content of the alliance treaties. According to the new strand of studies using the ATOP data set (Leeds et al. (2000), Leeds (2003)), the real rate of compliance is closer to 75%, if one considers the exact provisions of the alliance treaties, which make intervention of an ally conditional on specific circumstances.

Theoretical studies of alliance formation reflect the debate among theorists of international relations between neorealists and institutionalists. In the neorealist view (see for example Walt (1979) and Snyder (1997)), the international system

[2] See Morrow (2000) for an excellent account of the literature

of states is anarchic, and alliances among countries cannot be enforced. Formal alliances are concluded to restore the "balance of power" among states, and allies face two antagonistic risks: the risk of abandonment (when an ally becomes more distant and fails to fulfill his commitment) and the risk of entrapment (when an ally, emboldened by the formation of the alliance, takes excessive risks and engages in excessive conflict). By contrast, institutionalists (see for example Morrow (2000) and Fearon (1997)) stress that international institutions and alliances play an effective role, that alliances may be formed among asymmetric countries do not necessarily aim at restoring the balance of power, but could be seen as signals or commitment devices chosen by the allies. We will now give a more extensive account of the formal, game-theoretical models of political alliance formation.

2.1 Alliance formation and the balance of power

In a series of papers, Niou and Ordeshook (1986) and (1991) introduce a formal, game-theoretic model of the balance of power. They describe a state of the political system as a vector $\mathbf{s} = (S, r_1, ..., r_S)$ where S denotes the set of countries, and r_s the resources of country s. Countries can acquire by force the resources of the other countries if they prevail by possessing more than one half of the total resources. Hence, the situation can be represented by a cooperative game where $v(S) = R$ if and only if $\sum_s r_s \geq \frac{R}{2}$.

Niou and Ordeshook (1986) introduce two definitions of stability of international political systems. They introduce a series of axioms on preferences and actions, asserting that country leaders have a preference over peaceful negotiations, prefer an increase in resources only if it does not threaten the existence of the country (i.e. if there is no state that could eventually be reached where the country is eliminated). Under these assumptions, countries will never seek to obtain more than half of the resources of the system, and a country will be deemed "predominant" if it owns exactly $\frac{R}{2}$ resources. *Resource Stability* corresponds to the usual definition of core stability: a state $\mathbf{s}$ of the system is stable if and only if there does not exist a coalition of countries which could strictly gain by precipitating a new state $\mathbf{s}'$. Given the structure of the cooperative game (which is equivalent to a "majority" game where the core is empty if no player has the majority of votes), the only stable states are states where one of the countries is "predominant" and owns exactly one half of the resources of the system. The second notion of stability, *system stability*, is based on a process of threats and counter-threats which corresponds to the cooperative notion of the bargaining set. A system of countries is called system stable if any threat to eliminate one of the countries in the system can be defeated by a counter-threat. In a remarkable result, Niou and Ordeshook (1986) show that a system of countries is system stable if and only if it contains only *essential countries*, namely countries which belong to some minimal winning coalition.

Niou and Ordeshook (1991) extends the analysis by moving from a cooperative notion of stability to a noncooperative description of the negotiation/war process.

In the extended model, countries are recognized to make threats (form coalitions and ask for a transfer of resources), which can be followed by counter-threats, ad infinitum. Focussing on the stationary perfect equilibria of this noncooperative game, Niou and Ordeshook (1991) characterizes the set of "primary" threats which are shown to be immune to further deviations. They obtain a result which extends the analysis of system stability: a country will survive in a stationary perfect equilibrium of the negotiation/war process if and only if it is essential.

2.2 Alliance formation as a signal or commitment device

Morrow (1994), Smith (1995) and (1998) and Fearon (1997) analyze the formation of alliances as a signalling or commitment device in a three-country game theoretic model of conflict. There are three actors in the model: a challenger, A, a target B and an ally C. The interaction is described as a multistage game.

1. Initially, countries B and C choose whether to form an alliance
2. Country A then decides whether to attack country B
3. Country B then decides whether or not to resist
4. Country C then decides whether or not to support his ally.

In order to make the behavior of the players probabilistic, it is assumed that each of the countries faces a random war cost, w_A, w_B, w_C which for simplicity are distributed uniformly over $[0, 1]$. Let 0 denote the value of the worst outcome for each player (the status quo for player A and the defeat for players B and C) and 1 the value of the best outcome for the players (victory for country A and status quo for countries B and C). It is also assumed that the probability that A wins the conflict against B alone, p_A is higher than q_A, A's winning probability when it fights against both countries. Finally, both countries B and C must pay a fixed cost F to sustain the alliance (either in terms of loss of autonomy, or joint maneuvers, etc..). This extensive form game is pictured in Figure 1.

In Morrow (1994)'s analysis, country C uses the formation of the alliance to signal its war cost and hence its willingness to support his ally. Morrow (1994) characterizes the separating and pooling equilibria of the game. He shows that when its cost of war is low, country C has an incentive to offer an alliance to country B, signaling that both will fight if country A attacks, and ultimately deterring the challenger. If the cost of war is high, country C has no incentive to play this strategy, as it will never choose to support country B if it is attacked, country B will never resist and country A will not be deterred. This argument shows that separating equilibria exist, where low cost countries C choose to form an alliance which deters country A from attacking and high cost countries do not offer an alliance, and let country B be attacked without supporting it.

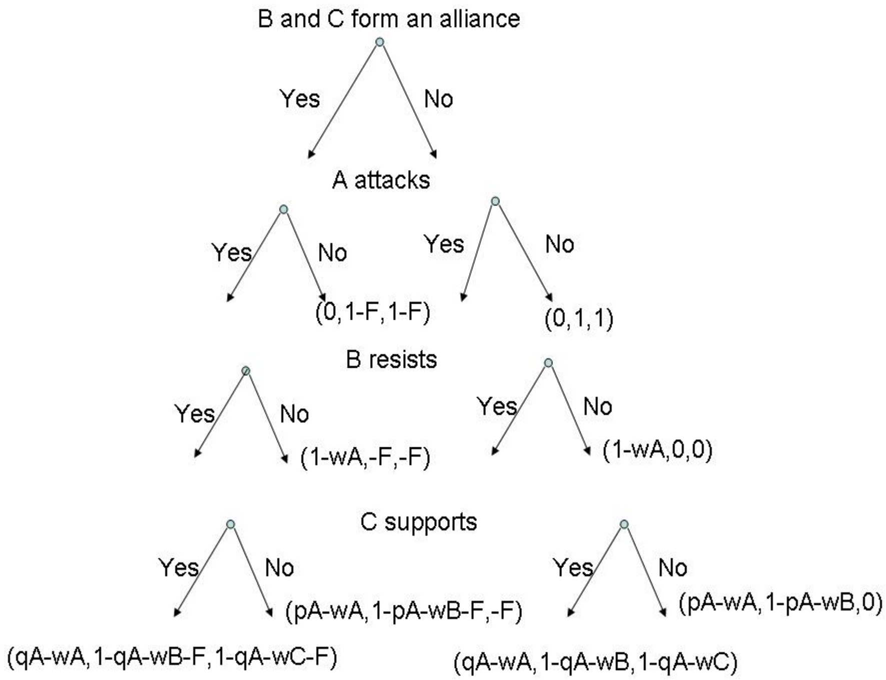

Figure 1: Political models of alliance formation with three countries

Smith (1995) and (1998) suggests another role for alliances. In his model, country C learns its war cost after the alliance has formed, so that alliance formation cannot be a signal of country C's willingness to support its ally. Instead, the formation of an alliance is a commitment device, as a country which signs an alliance but does not support his ally if attacked will have to pay a "honor" cost. By forming an alliance with country B, country C this ties his hands, making compliance to the alliance more likely, and hence decreasing country A's incentive to attack. The deterrence impact of the alliance is due to the fact that the formation of the alliance changes the payoffs of the subsequent game, by adding a cost of noncompliance.

Fearon (1997) compares the two models in a unified framework. He views the formation of an alliance as a way to increase audience costs (the domestic cost if the country backs down from the alliance) and shows that countries will on average prefer to deter attacks by tying their hands than by sending costly signals. His analysis thus encompasses both Morrow (1994) and Smith (1995)'s models and vindicates the use of alliances as commitment devices.

3 Alliances, collective action and group sizes

3.1 Alliances and collective action

Olson (1965)'s seminal study of collective action serves as the starting point for an economic analysis of alliance formation as the creation of a public good (collective security) by a set of countries. In the bipolar world following the end of World War II, NATO and the Warsaw pact are prototypical examples of alliances where one large country provides security to a set of smaller countries. Olson and Zeckhauser (1966) apply the analysis of collective action to military alliances. They analyze the formation of alliances as coalitions providing pure public goods to their members. As in any model of public good provision, they find that the level of defense expenditures in an alliance is suboptimal, and that larger countries provide a larger amount of public good (in proportion) than smaller countries. This last result is tested using military expenditures in NATO, and the data support the conclusion that large countries (in particular the United States) pay a disproportionate amount of the collective expenditures in military alliances. This analysis raises an important set of questions, relating the amount of resources spent in conflict on the size of alliances. Are smaller groups spending more or less than larger groups in conflict? In order to analyze this issue, we now consider a specific model of conflict.

3.2 A model of conflict

We now introduce a simple model to study the effect of group sizes on conflict, based on Tullock (1967)'s well-known specification of probabilistic contests. Consider n identical agents grouped in m alliances $A_1, ..., A_m$. We denote by a_j the size of group A_j. Each agent $i = 1, 2, ...n$ chooses an investment s_i in conflict. We let $s(A_j) = \sum_{i \in A_j}$ denote the total investments of alliance A_j and $s(N)$ the sum of investments of all agents. The probability that alliance A_j wins the contest is given by the logistic formula:[3]

$$p_j = \frac{s(A_j)}{s(N)}. \tag{1}$$

If alliance A wins the conflict, the prize is shared among its members according to a fixed sharing rule. We suppose that the sharing rule inside each alliance A is independent of the investments of the players in the inter-alliance conflict, $\mathbf{s} = (s_1, ..., s_n)$, and treats all agents symmetrically.

Assumption 1 *The value of the prize to a member of an alliance A is a function $V(a)$ which only depends on the size of alliance a.*

Under this assumption, the utility of an agent in group A_j is given by:

[3]See Skaperdas (1996) for an axiomatization of this contest function

$$U_i = V(a_j)p_j - c(s_i) \tag{2}$$

where $c(\cdot)$ is an increasing, convex cost function. This general specification of sharing rules encompasses most cases studied in the literature. If V is a nondecreasing function, the prize is a nonrival *public good* shared by all alliance members;[4] if V is nonincreasing, the prize is a *private good*. Furthermore, this specification can be interpreted as a reduced form of complex models of conflict where agents coordinate their strategies, or participate in multistage contests. To understand this point consider the following two examples, which describe two polar cases of the model.

Example 1 *Public good with full coordination.*

The average utility of an alliance member can be expressed as:

$$U_i = \max_s \frac{a_j s}{s(N)} V - c(s),$$

Assuming that the optimal investment s is interior, it is characterized by the first order condition:

$$a_j V \frac{\sum_{k \neq j} s(A_k)}{s(N)^2} - c'(s) = 0.$$

a formula yielding the same investment level as if each alliance member received a share of the prize $V(a_j) = V a_j$.

Example 2 *Multistage contest.*

Consider by contrast another model where agents in a group obtain a private prize, which is reallocated according to a subsequent intra-alliance contest where agents have a linear investment technology as in Katz and Tokatlidu (1996) and Garfinkel (2004). In the second stage contest, each agent gets a utility:

$$\begin{aligned} U_i &= \max_{s_i} \frac{s_i}{s(A_j)} V - s_i, \\ &= \frac{1}{a_j} - \frac{a_j - 1}{a_j^2}, \\ &= \frac{1}{a_j^2}. \end{aligned}$$

Hence, in the first stage inter-alliance contest, an agent in an alliance of size a_j anticipates to receive a share of the prize $V(a_j) = V/a_j^2$.

[4]See Katz, Nitzan and Rosenberg (1990) for a model of group competition with public goods.

The two sharing rules $V(a) = Va$ and $V(a) = \frac{V}{a^2}$ represent the two polar cases for reasonable sharing rules. Other usual sharing rules include the public good function $V(a) = V$ and the private good egalitarian sharing rule $V(a) = \frac{V}{a}$.

3.3 Two-group conflict

We first analyze a situation where two groups A_1 and A_2 of sizes a_1 and a_2 compete in the contest.[5] By a slight abuse of notation we let S_1 and S_2 represent the total investments in conflict of the two groups and S the total level of conflict. In order to compute the equilibrium level of conflict, we derive the first-order conditions, differentiating equation (2) with respect to s_i in the two groups, to obtain the system of best response functions:

$$V(a_1)S_2 - c'(\frac{S_1}{a_1})S^2 = 0, \tag{3}$$

$$V(a_2)S_1 - c'(\frac{S_2}{a_2})S^2 = 0. \tag{4}$$

Figure 2 graphs the best response choices of group conflict levels, $\phi_1(S_2)$ and $\phi_2(S_1)$. One complication in the analysis (which makes comparative statics difficult in the contest game) is that, as shown in the Appendix, the best response functions are *not monotonic*: the best response function $\phi_i(S_j)$ is *increasing in S_j when $S_i > S_j$ and decreasing in S_j when $S_j > S_i$.*

Figure 2 also illustrates why equilibrium in a two-group contest must be unique. If an equilibrium (S_1^*, S_2^*) with $S_1^* > S_2^*$ exists, as in the graph, there cannot be any other equilibrium in the region where $S_1 > S_2$, as the best response function $\phi_1(S_2)$ is increasing, and the best response function $\phi_2(S_1)$ is decreasing. Furthermore, if we let $\tilde{S_1}$ and $\tilde{S_2}$ denote the intersection points between the best response functions and the 45 degree line, the existence of an equilibrium with $S_1^* > S_2^*$ necessarily implies that $\tilde{S_1} > \tilde{S_2}$, precluding the existence of another equilibrium in the region where $S_1 < S_2$.

Using equations (3) and (4), we compute:

$$\frac{c'(s_1)s_1}{c'(s_2)s_2} = \frac{\frac{V(a_1)}{a_1}}{\frac{V(a_2)}{a_2}}. \tag{5}$$

As $c'(s)s$ is increasing we obtain our first result on the effectiveness of small groups:

Remark 1 *If $\frac{V(a)}{a}$ is non-increasing, individual investments are larger in smaller groups.*

[5]The two-group model was discussed in Tullock (1980)'s original paper and has been analyzed by Katz and Tokatlidu (1996), Baik and Lee (1997) and Wärneryd (1998)).

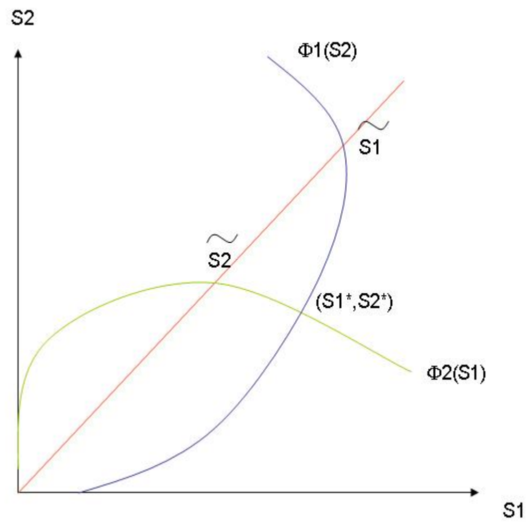

Figure 2: Best response functions in the two-group contest

We can similarly write

$$\frac{S_1}{S_2} = \frac{\frac{V(a_1)}{c'(s_1)}}{\frac{V(a_2)}{c'(s_2)}}. \tag{6}$$

implying that

Remark 2 *If $V(a)$ is non-decreasing, group investments are larger for larger groups.*

Remarks 1 and 2 allow us to conclude that small groups are effective when the sharing rule is strongly decreasing in a whereas large groups are effective when the sharing rule is non-decreasing in a. In order to obtain results in the intermediate cases, as in Esteban and Ray (2001), we consider the family of iso-elastic cost functions $c(s) = s^\alpha$ for $\alpha > 1$. For these cost functions, we obtain:

$$\frac{S_1 c(S_1)}{S_2 c(S_2)} = \frac{a_1^{-\alpha} V(a_1)}{a_2^{-\alpha} V(a_2)}. \tag{7}$$

so that

Remark 3 *If cost functions are iso-elastic and given by $c(s) = s^{\alpha}$, and $V(a)a^{-\alpha}$ is nondecreasing, group investments are larger for larger groups.*

Remark 3 shows that the effectiveness of large or small groups is related to the relative curvatures of the sharing rules and cost functions. If the sharing rule is not decreasing too fast with respect to the cost function, then large groups will be more effective. This comparison between the curvatures of sharing rules and cost functions also plays a role in the following general analysis of the effect of changes in group sizes. Consider a fixed society of n agents divided into two groups of sizes a and $n-a$, and let a vary between 1 and n. Using the techniques of comparative statics in games introduced by Dixit (1986), we compute in the Appendix exact conditions under which group investment is increasing and decreasing in a.

Proposition 1 *If $\inf_a \frac{aV'(a)}{V(a)} \geq \sup_s \frac{-sc''(s)}{c'(s)}$, then, for $a \geq \frac{n}{2}$, group investments of the large group, S_a are increasing in a. If $\sup_a \frac{aV'(a)}{V(a)} \leq \inf_s \frac{-sc''(s)}{c'(s)}$, then for $a \leq \frac{n}{2}$, group investments of the small group S_a are decreasing in a.*

Proposition 1 provides another illustration of the trade-off between the curvature of the cost and sharing rules. For iso-elastic costs, $c(s) = s^{\alpha}$, $\inf_s \frac{-sc''(s)}{c'(s)} = \sup_s \frac{-sc''(s)}{c'(s)} = 1-\alpha$, and group investments are increasing in a for large groups whenever $\frac{aV'(a)}{V(a)} \geq 1-\alpha$, while group investments of small groups are decreasing in a whenever $\frac{aV'(a)}{V(a)} \leq 1-\alpha$.

3.4 Conflict among multiple symmetric groups

With more than two groups, the analysis of conflict becomes much more complex. We report results in the case of conflict among $k \geq 2$ symmetric groups of the same size $a = \frac{n}{k}$. With k symmetric groups, the unique symmetric equilibrium is characterized by

$$V(\frac{n}{k})(k-1) - c'(\frac{Sk}{n})k^2 S = 0. \tag{8}$$

A direct computation shows that

$$\begin{aligned} \text{sign}\, \frac{\partial S}{\partial k} \quad &= \quad \text{sign}\, \{-\frac{n}{k^2}V'(\frac{n}{k})(k-1) + V(\frac{n}{k}) \\ &\quad -\frac{S}{n}c''(\frac{Sk}{n})k^2 S - 2kSc'(\frac{Sk}{n})S\}. \end{aligned}$$

leading to the following result:

Proposition 2 *In the case of public goods (when $V(a)$ is nondecreasing), an increase in k results in a decrease in the symmetric group investment S. In the case of private goods, when $\inf_a \frac{aV'(a)}{V(a)} \leq 1 - \sup_s \frac{sc''(s)}{c'(s)}$, an increase in k results in an increase in the symmetric group investment S.*

Proposition 2 illustrates again the tension between the sharing and cost rules in the comparative statics on group investments: with public goods, fragmentation unambiguously results in a decrease in group investment ; with private goods, if the curvature of the cost function with respect to the sharing rule is not too high, an increase in fragmentation yields an increase in group investment. For example, with an iso-elastic cost function $c(s) = s^{\alpha}$, group investment increases with k if $\frac{aV'(a)}{V(a)} \leq -\alpha$.

4 The size of alliances and endogenous alliance formation

When is the formation of a unique peaceful alliance stable? Are there limits to the size of stable alliances? Which alliances will be formed by agents acting in succession in conflicts? In order to answer these questions, we introduce in this section different models of endogenous formation of alliances, and characterize stable alliance structures. We first specialize the model of conflict by assuming that agents have a linear investment cost and that the sharing rule is a simple power function rule[6]:

Assumption 2 *Let $c(s) = s$ and $V(a) = a^{\rho}$, with $-2 \leq \rho \leq 1$.*

4.1 Models of alliance formation with externalities

Because agents' payoffs in an alliance depend on the entire alliance structure (including the alliances formed by other agents), the relevant models of endogenous alliance formation are the recent models of coalition formation with externalities discussed in Ray (2007). We will focus on the two models proposed by Hart and Kurz (1983) where agents simultaneously announce a coalition C_i to which they belong. In the γ version of the game, coalition C is formed if and only if $C_i = C$ for all $i \in C$. Unanimity is thus required for an alliance to form. This specification implies that if an agent contemplates defecting from an alliance, he anticipates that the alliance will break apart, and all its members act as single agents. In the δ version of the game, coalition C is formed if and only if there exists some coalition $D \supset C$ such that

[6]Bloch et al. (2006) consider an alternative specification for which closed form solutions can be computed: they suppose a quadratic cost $c(s) = \frac{1}{2}s^2$ and an egalitarian sharing rule, $V(a) = \frac{1}{a}$. Esteban and Ray (1999) also provide exact computations when $V(a) = v$ is independent of a, and $c(s) = s^{\alpha}$ is an iso-elastic cost function.

$C_i = D$ for all $i \in C$. In this case, coalitions can be formed even if some prospective members reject the offer. Upon contemplating a defection, each agent anticipates that other alliance members stick together.

4.2 When is peace sustainable?

We first analyze the following question: when is the grand coalition $\{N\}$ stable? In order to answer these questions, we compute the equilibrium utility of an agent i in three alliance structures: the grand coalition, the alliance formed of single agents, $\{1\}, \{2\}, ...\{n\}$ and the alliance formed of one isolated agent and an alliance of the $n-1$ remaining agents:

$$
\begin{aligned}
U_i(\{N\}) &= n^{\rho}, \\
U_i(\{1|1|...|\}) &= n^{-2}, \\
U_i(\{i|N \setminus i\}) &= \frac{1}{(1+(n-1)^{\rho})^2}.
\end{aligned}
$$

Proposition 3 *The grand coalition, $\{N\}$ is always an equilibrium of the γ game of alliance formation. If $\rho \log n + 2\log((1+(n-1)^{\rho})) \geq 0$, the grand coalition is also an equilibrium of the δ game of alliance formation.*

The intuition underlying this proposition is easily grasped. Because we assume $\rho \geq -2$, players in the grand coalition can never do worse than if they fight as singletons: the peaceful grand coalition is thus always an equilibrium outcome if players anticipate that the coalition collapses into singletons after the departure of a member. If, on the other hand, players anticipate that other coalition members stick together after they leave, the grand coalition still remains an equilibrium if ρ is large enough. Players have no incentive to leave the grand coalition and start a conflict as a singleton facing an alliance of $n-1$ members if $\rho \geq \rho^*(n)$, where $\rho^*(n)$ is implicitly defined by: $\rho \log n + 2\log((1+(n-1)^{\rho})) \geq 0$. Notice that, as $2log2 > 0$, and $\log(n-1) - \log n < 0$, we have $-1 \leq \rho^*(n) \leq 0$. Furthermore, $\rho^{*'}(n) > 0$ and $\lim_{n \to \infty} \rho^*(n) = 0$, reflecting the fact that free-riding incentives are higher in larger societies, and the grand coalition can only remain stable if the sharing rule does not decrease too fast.

4.3 The size of stable alliances

Suppose now that agents form a single alliance of size a with all other players remaining isolated. Simple computations show that the utility of an independent player, U_i and of an alliance member, U_a are given by:

$$
\begin{aligned}
U_i &= \frac{1}{(1+k^\rho(n-k))^2}, \\
U_a &= \frac{k^{\rho-1}(k^\rho(n-k)-(n-k-1))(k^{\rho+1}(n-k)-n+2k)}{(k^\rho(n-k)+1)^2}
\end{aligned}
$$

if $k^\rho(n-k)-(n-k-1)\geq 0$ and $U_i = \frac{1}{(n-a)^2}, U_a = 0$ otherwise.

We first compare the utility of a member of an alliance with the utility of a player in the singleton structure, and note that U_a is strictly increasing in ρ, whenever $k^\rho(n-k)-(n-k-1)\geq 0$ to obtain the following Proposition:

Proposition 4 *For any n and any alliance size $2 \leq a \leq n-1$, there exists $\rho^*(a,n) \in (-1,0)$ such that the alliance structure $\{A|1|...|1\}$ is an equilibrium of the γ game of coalition formation if and only if $\rho \geq \rho^*(a,n)$. Furthermore, $\rho^*(a,n)$ is* not monotonic *in a.*

Proposition 4 characterizes the set of alliances which are γ stable. If $\rho < -1$, the only γ stable structure is the grand coalition. If $\rho > 0$, free-riding never occurs and any alliance can be sustained as an equilibrium of the γ game. For $\rho \in (-1,0)$, different alliance structures may turn out to be γ-stable. The fact that $\rho^*(a,n)$ is not monotonic in a is illustrated by the following example for $n = 10$. Notice that, in this example, when $\rho = -0.13$, alliances of sizes 2, 6, 7, 8, and 9 are stable, but not alliances of sizes 3, 4 and 5.

Example 3 *Let $n = 10$. The threshold values of ρ are given by the following table:*

a	$\rho^*(a)$
2	-0.1563
3	-0.1221
4	-0.1179
5	-0.1267
6	-0.1521
7	-0.1957
8	-0.3112
9	-1

In order to investigate which alliances are δ stable, we need to consider a player's incentive to leave an alliance in the alliance structure $\{A|1|..|1\}$ in order to be an independent in the alliance structure $\{A \setminus i|1|...|1\}$,

$$
\begin{aligned}
\Delta(\rho,a,n) \equiv &\frac{1}{(1+(k-1)^\rho(n-k+1))^2} \\
&-\frac{k^{\rho-1}(k^\rho(n-k)-(n-k-1))(k^{\rho+1}(n-k)-n+2k)}{(k^\rho(n-k)+1)^2}
\end{aligned}
$$

It is easy to check that for all a and n, Δ is decreasing in ρ (individual players have a stronger incentive to leave the alliance when the sharing rule becomes more convex) and that $\frac{1}{(1+(k-1)^{\rho}(n-k+1))^2} \geq \frac{1}{n^2}$ (individual players have a stronger incentive to leave the alliance when the other alliance members remain together). Hence, we obtain:

Proposition 5 *For any n and any alliance size $2 \leq a \leq n-1$, there exists $\tilde{\rho}(a,n) \in (-1,0)$ with $\tilde{\rho}(a,n) > \rho^*(a,n)$ such that the alliance structure $\{A|1|...|1\}$ is an equilibrium of the δ game of coalition formation if and only if $\rho \geq \tilde{\rho}(a,n)$. Furthermore, $\tilde{\rho}(a,n)$ is not monotonic in a.*

Hence, we find that an alliance is harder to sustain in the δ game of alliance formation than in the γ game, and that again, even if two alliances a and a' are stable, it is not necessarily true that all alliances of sizes between a and a' are stable.

4.4 Alliance formation with farsighted players

The open membership and γ and δ models of alliance formation assume that agents have limited foresight, and do not consider the long term consequences of their behavior. By contrast, the sequential game of coalition formation introduced in Bloch (1996) models players as farsighted agents who anticipate the consequences of their decisions on subsequent players. In this game, players are ordered according to an exogenous protocol. The first player announces a coalition that she wants to form. If all members agree, the coalition is formed and the next player (in the remaining set of players) gets to make an offer. If one member of the coalition rejects, she becomes the counter proposer next period. In this sequential game, it is natural to focus attention to the coalition structures generated by stationary perfect equilibria of the game.

As a finite game with perfect information, the sequential model of coalition formation admits a *unique* equilibrium alliance structure. However, this unique alliance structure is typically hard to compute, as it requires a complete characterization of optimal strategies in all the subgames generated by the decisions of the players. A complete characterization of equilibrium coalition structures can only be achieved in some special cases.[7]

In the Appendix, we compute the equilibrium alliance structure for three particular values of ρ: $\rho = 0, -1, -2$ and the following graph shows equilibrium alliance structures for different values of ρ.

Figure 3 displays some gaps – parameter regions where the equilibrium coalition structure cannot be determined.[8] Interestingly, it shows that the grand coalition is the equilibrium alliance structure both for high and low values of ρ. For high values of ρ, this is due to the fact that the grand coalition dominates any other alliance

[7]The following discussion is based on Sanchez-Pages (2007).

[8]However, we conjecture that $\{N\}$ remains the equilibrium structure for all $\rho \in (-2,-1)$.

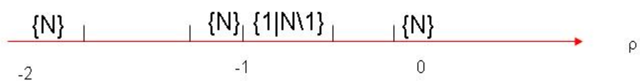

Figure 3: Equilibrium alliance structures in the sequential game

structure ; for low values of ρ, to the fact that any move by the first player will be followed by the formation of singletons, so that the first player effectively chooses between forming the grand coalition or the structure of singletons.[9]

Garfinkel (2004) studies the model with $V(a) = \frac{1}{a^2}$ using a different equilibrium concept, inspired by Chwe (1994)'s notion of farsighted dominance. She finds that any symmetric equilibrium structure – a collection of alliances of size $\frac{n}{k}$ – is stable, as well as coalition structures comprised of coalitions which are "almost symmetric". The intuition underlying this result is as follows. If any player deviates to form a singleton, this will trigger the formation of the coalition of singletons, where the deviator gets a payoff $\frac{1}{n^2} < \frac{1}{k(\frac{n}{k})^2} = \frac{k}{n^2}$. Hence, a deviation to leave a coalition is not profitable. Furthermore, in the model members of large alliances have lower payoffs. This implies that players have no incentive to deviate by joining an existing coalition which would be larger than the one she they leave, because in the symmetric alliance all alliances initially have the same size.

[9]The grand coalition also emerges as the unique equilibrium alliance structure in the model of Bloch et al. (2006), where $V(a) = \frac{1}{a}$ and players incur a quadratic cost.

5 Alliance formation with endogenous sharing rules

5.1 Endogenous sharing rules

We have assumed that the sharing rule in each alliance is independent of the investments of the players. Alternatively, we could have considered situations where the effort of each player in the inter-alliance conflict affects the sharing of the prize among alliance members. Nitzan (1991a) proposed the following specification of a group contest with a sharing rule depending on investments in conflict. In his model, the share of the prize obtained by an agent i in alliance A is given by:

$$f_i(s_1, .., s_n) = (1-\alpha)\frac{s_i}{\sum_{j \in A} s_j} + \alpha \frac{1}{a}.$$

The sharing rule is a convex combination of the egalitarian sharing rule, and the relative effort rule which is proportional to every agent's investment in conflict. Under this sharing rule, agent i in alliance A receives an expected utility

$$U_i = (1-\alpha)\frac{s_i}{\sum_j s_j} + \alpha \frac{\sum_{j \in A} s_j}{a \sum_j s_j} - c(s_i).$$

When $\alpha = 0$, the contest is exactly identical to a contest where all agents are singletons and the formation of groups is irrelevant. In the other polar case, where $\alpha = 1$, the conflict is equivalent to a group conflict with sharing rule $V(a) = \frac{1}{a}$, as in Section 2. When a group of size a faces $n - a$ independent players, denoting by s_a and s_i the individual efforts of group members respectively, the best response maps are given by:

$$\begin{aligned}(1-\alpha)(a-1)s_1 + (n-a)s_i &= (as_a + (n-a)s_i)^2. \\ as_a &= (as_a + (n-a)s_i)^2.\end{aligned}$$

The best response map of group members, $\phi_a(s_i)$ is *decreasing in* α while the best response map of independents, $\phi_i(s_a)$ is independent of α. When α increases, the investment of group members decreases, while the investment of independent players goes up. Lower values of α correspond to a more aggressive behavior of the group, and results in higher investments for the alliance, and a higher probability to win the conflict. This result is easily understood. By increasing the weight placed on relative investments, the alliance induces agents to spend more on conflict, as in a "rat race" where an agent's payoff depends on his effort relative to the group. Hence, as noted by Nitzan (1991a), an increase in α *reduces* total investment in conflict.

Nitzan (1991a)'s analysis treats the sharing rule as exogenous. Baik and Shogren (1995), Lee (1995) and Baik and Lee (1997) consider the endogenous determination of the weight α which maximizes the expected payoff of the group under different alliance structures.[10] In Baik and Shogren (1995), a single group of size a faces a collection of singletons. In that case, in an interior equilibrium, the optimal sharing rule is given by:

$$\alpha^* = 1 - \frac{2an - 2a^2 - n}{2(a-1)(n-1)}.$$

We observe that α^* is decreasing in a. For larger groups, the "rat race" results in overspending on conflict, and the optimal investments are obtained when the group uses an egalitarian sharing rule. Given the optimal sharing rule, one can compute the equilibrium payoff of a member and a nonmember. The following diagram illustrates how members and nonmember's payoffs depend on a.

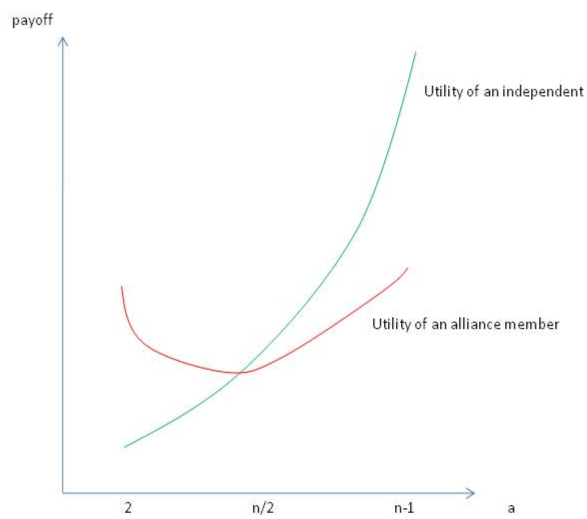

Figure 4: Payoff of independent and alliance members with optimal sharing rule

Lee (1995) and Baik and Lee (1995) analyze endogenous sharing rules chosen strategically by two groups in a two-stage model where groups simultaneously choose

[10]The initial analysis of Nitzan (1991a) and Baik and Shogren (1995) and Lee (1995) focussed on interior equilibria and ignored the possibility that some group optimally choose not to participate in the contest. Ueda (2002) corrects this error by characterizing *all equilibria* of Nitzan (1991a)'s contest model including corner equilibria.

the parameter α_i of the sharing rule in the first stage, and compete in an inter-alliance conflict in the second stage.[11] Consider the following simplified version of Lee (1995)'s setup where two groups of equal size $a_1 = a_2 = \frac{n}{2}$ choose between the relative effort ($\alpha = 0$) and egalitarian ($\alpha = 1$) rules. Simple computations of the equilibrium levels of conflict and utilities following any choice (α_1, α_2) of the two firms allow us to draw the following matrix to study the game played by the two groups in the initial stage.

	$\alpha_2 = 0,$	$\alpha_2 = 1$
$\alpha_1 = 0$	$(\frac{1}{n^2}, \frac{1}{n^2})$	$(\frac{4}{(n+2)^2}, \frac{8}{n(n+2)^2})$
$\alpha_1 = 1$	$(\frac{8}{n(n+2)^2}, \frac{4}{(n+2)^2}$	$(\frac{1}{2n}, \frac{1}{2n})$

This game has the structure of a prisoner's dilemma. Both groups have a dominant strategy: to choose the aggressive relative effort rule, $\alpha = 0$, even though they would both be better off selecting the softer egalitarian rule, $\alpha = 1$. Lee (1995) generalizes this intuition to any choice of $\alpha \in [0, 1]$ and observes that, if two alliances of equal size are formed, the only equilibrium is for both groups to choose the most aggressive sharing rule, setting α to 0. On the other hand, if the groups are of different sizes (and negative values of α are allowed), Baik and Lee (1997) show that the two groups are likely to adopt different sharing rules, with the smaller group adopting a more aggressive sharing strategy ($\alpha = 0$), while the larger group favors a more balanced sharing rule ($\alpha > 0$). Noh (1999) analyzes a model with endogenous sharing rules among heterogeneous players who are endowed with different resources that they can either spend on production or conflict.[12] He shows that, contrary to Lee (1995), this specification of the contest either leads the two groups to choose the egalitarian rule, or a small group to choose the relative effort rule while the large group selects $\alpha \in (0, 1)$. The contrast between Noh (1999)'s and Lee (1995)'s result stems from the fact that in Noh (1999)'s model, more aggressive behavior of the two groups reduces the common pool, resulting in lower utilities for the group members.

6 Dynamic alliance formation

6.1 Dynamic alliance formation with exogenous effort

The models analyzed so far assume that the prize is shared among alliance members once and for all according to a sharing rule $V(a)$ or $V(a, \mathbf{s})$. Even when players are assumed to fight inside the alliance to obtain the prize (and obtain for example $V(a) = \frac{1}{a^2}$ when the investment cost is linear), the intra-alliance contest is supposed

[11] Nitzan (1991b) initially observed that, when the two groups choose different sharing rules, interior pure strategy equilibria may not exist. Davis and Reilly (1999) later observed that an equilibrium always exists, but may involve some groups choosing zero investment and retiring from the contest.

[12] See also Noh (2002) for a model of alliance formation among three asymmetric agents.

to be final. There is no room for further sub-alliance formation inside the alliance, and the conflict ends after two stages.

This specification of conflict misses an important element of alliance formation, and introduces an asymmetry between a first stage conflict (where alliances can be formed) and a second-stage conflict (where sub-alliances cannot be formed). By contrast, models of continuing conflict (or "nested contests") analyze alliance formation and conflict as a dynamic process, resulting in the successive elimination of players, until the final contest is played between the last two players.

In order to understand the dynamics of conflict and alliance formation, we focus on a model with three agents $i = 1, 2, 3$. Each agent has a fixed, exogenous endowment y_i. The probability of winning the contest is exogenously given by

$$p_i = \frac{y_i}{\sum_j y_j}.$$

At the initial stage, either the three players compete as singletons, or an alliance of two players is formed. If the alliance wins the contest, its two members subsequently compete to obtain the prize.[13] We compare the utility obtained by player 1 in the overall contest and when she forms an alliance with player 2. In the first case, she obtains

$$U_1 = V \frac{y_1}{y_1 + y_2 + y_3}$$

and in the second

$$U_1' = V \frac{y_1 + y_2}{y_1 + y_2 + y_3} \frac{y_1}{y_1 + y_2} = U_1.$$

It thus appears that players *have no incentive to form an alliance.* Konrad (2009) labels this result "the paradox of alliance formation".

In order to allow for synergies in the investments of the two alliance members, we consider an increasing mapping $f(y_i)$ such that the probability of winning the contest is given by:

$$p_i = \frac{f(y_i)}{\sum_j f(y_j)}.$$

In this model, an alliance between two players will be formed if and only if the mapping f is *superadditive*, namely $f(y_i + y_j) > f(y_i) + f(y_j)$ for all y_i, y_j. Two frequently used functional forms are the probit and logit specifications:

[13] With three players, the continuing conflict model is identical to the two-stage model with one stage of inter-alliance and one stage of intra-alliance conflict. The difference between the models would only appear with more than three players.

$$
\begin{aligned}
f_i(y_i) &= y_i^m, \\
f_i(y_i) &= e^{ky_i}.
\end{aligned}
$$

Skaperdas (1998) provides general conditions under which the alliance if formed by the two stronger or weaker players. In particular, he notes that for the probit specification, the alliance will always be formed among the two weakest players. For the logit model, the alliance will be formed among the weakest players if and only if:

$$
\begin{aligned}
H(y_1, y_2, y_3) &= e^{k(y_1+y_2+y_3)} - e^{k(y_1+y_2)} - e^{k(y_1+y_3)} - e^{k(y_2+y_3)} \\
&\geq 0.
\end{aligned}
$$

Tan and Wang (1997) independently analyze the formation of alliances among free players and obtain essentially the same result, showing that the two weak players form an alliance for the probit model and when $f(y) = \exp \gamma y - 1$. They note that the strong player may not have the highest probability of winning the contest (taking for example, $f(y) = y^2, y_1 = 0.6, y_2 = 0.5$ and $y_3 = 0.2$, the winning probabilities of the three players are $0.42, 0.5$ and 0.08 respectively.) Moving to the more complex case of four players with $y_1 > y_2 > y_3 > y_4$, Tan and Wang (1997) show that if $y_1 > y_3 + y_4$, the unique equilibrium coalition structure is $\{1|234\}$ followed by $\{2|34\}$ if the alliance 234 wins. Hence, at each stage, the weakest players form an alliance against the strongest player.[14] If $y_1 < y_3 + y_4$, the unique equilibrium coalition structure is of the form $\{ij|kl\}$, with two alliances of two players.

6.2 Dynamic alliance formation with endogenous effort

Consider now a situation where players endogenously choose their level of effort. The first result on hierarchical contest, due to Wärneryd (1998) shows that the total level of resources spent on conflict is lower in a hierarchical conflict than in a single, simultaneous conflict. Consider a conflict with linear costs among inhabitants of two regions, labeled A and B with sizes n_A and n_B. In a simultaneous conflict, the total expenditures on conflict are $\delta^S = V\frac{n_A+n_B-1}{n_A+n_B}$. Consider next a two-stage conflict where the two regions fight in the first stage, and members of the winning region subsequently fight to obtain the prize. The second stage payoff in the two regions are $\frac{1}{n_A^2}$ and $\frac{1}{n_B^2}$, so that the expected resources spent on conflict are

$$
\delta^H = V\left(\frac{n_A^2 + n_B^2}{(n_A^2 + n_B^2)^2} + \frac{n_A^2}{n_A^2 + n_B^2}\frac{n_B - 1}{n_B} + \frac{n_B^2}{n_A^2 + n_B^2}\frac{n_A - 1}{n_A}\right) < \delta^S.
$$

establishing that total rent dissipation is lower in a hierarchical contest. This result is easily explained. In a hierarchical contest, resources spent in the first-stage contest

[14] This result generalizes to the case of n players when $y_i > \sum_{k=i+1}^{n} y_k$ for all i.

are lower because players only obtain a fraction of the prize if they win. In the second-stage contest, fewer players are involved and the amount of resources spent on conflict is reduced accordingly.

Esteban and Sakovics (2003) analyze a similar model among three groups with a quadratic investment cost. They obtain a negative result: free-riding incentives are so strong that alliances of two players will never form. This new instance of "the paradox of alliance formation" illustrates the two forces which hinder the formation of coalitions in models of conflict. On the one hand, even if effort was exogenous, players would not have an incentive to form coalitions unless the effort levels of the two partners exhibit some synergies. On the other hand, when effort choice is endogenous, free-riding in the coalition lowers the incentives to provide effort, and reduces the winning probability of the alliance. Both effects concur in making the formation of an alliance unprofitable.

6.3 Dynamic alliance formation with budget constraints

Kovenock and Konrad (2008) reexamine the alliance formation puzzle, in a discriminating contest without noise, as in Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996). Let the three players have budget constraints b_1, b_2 and b_3, with $b_1 > b_2 = b_3$. In the all-pay auction, every player chooses an investment level $s_i \in [0, b_i]$ and player i wins with probability 1 if $s_i > \max\{s_j, s_k\}$, with probability 0 if $s_i < \max\{s_j, s_k\}$ and with probability $\frac{1}{m}$ if m players choose the same effort level s_i. In this context, there is an equilibrium where the strongest player, player 1 has an expected payoff $U_1 = b_1 - b_2$, while the weakest players have an expected payoff $U_2 = U_3 = 0$.

Next, consider a situation where players 2 and 3 initially pool their efforts (and choose a common effort level s_{23} in response to s_1), and then compete against each other in a second stage if they win the first stage contest. In this game, when $b_2 \in (\frac{b_1}{2}, \frac{1-b_1}{2})$, Kovenock and Konrad (2008) prove existence of a subgame perfect equilibrium with expected payoffs $U_1 = 0$, $U_2 = U_3 = \frac{1-b_1-2b_2}{2}$. Hence, contrary to Esteban and Sakovics (2003), the two weak players have an incentive to form an alliance and obtain a higher payoff than the strong player in a contest without noise.

In the same vein, Kovenock and Roberson (2007) analyze a model where a single player A faces two adversaries B and C on two different battlefields. By forming an alliance, players B and C can choose to make unilateral resource transfers. The analysis shows that there exists a range of parameters for which these transfers prior to a Colonel blotto game with asymmetric resources will be beneficial to both alliance members.

7 Conclusion

This paper has highlighted how recent advances in game theory (on coalition formation, all-pay auctions, comparative statics in games) have helped further our understanding of alliance formation in conflict. A number of questions remain open for future research.

First, the analysis of sharing rules has so far been restricted to convex combinations of the egalitarian and relative effort rules. While this specification of sharing rules as a one-parameter family is useful as a first step, one wonders whether the introduction of more general sharing rules wouldn't help our understanding of the interaction between group conflict and internal sharing rules. Second, the current analysis of dynamic contests has been limited to three players, and has assumed that players have access to the same resources in the first and second stage of the contest. Generalizing the results to contests with an arbitrary number of players is a challenging task, but it should be undertaken in the future. Furthermore, one should also consider nested contests where players make decisions on the way in which they share fixed resources among successive contests. Third, while the literature has focused on free-riding problems in alliances, it has not yet analyzed in detail incentive contracts for alliance members. As every alliance faces a moral hazard problem, allowing a principal to offer a contract in each alliance seems natural. This would lead to a study of the interaction between inter-alliance conflict and intra-alliance contracts, and how this interaction affects the endogenous choice of group sizes. Finally, the analysis of alliance formation so far has supposed that agents live in a world of complete information, and know perfectly each other's strength in conflict. In reality, different alliance members may have private information about their types (conflict technology and/or payoffs), leading to the study of alliance formation as a game of incomplete information.

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9 Appendix

Best-response functions in the two-group contest

$$\begin{aligned}
\phi_1'(S_2) &= \frac{V(a_1) - 2c'(\frac{S_1}{a_1})S}{(S(2c'(\frac{S_1}{a_1}) + c''(\frac{S_1}{a_1})\frac{S}{a_1})}, \\
&= \frac{c'(\frac{S_1}{a_1})(S_1 - S_2)}{S_2(2c'(\frac{S_1}{a_1}) + c''(\frac{S_1}{a_1})\frac{S}{a_1})}, \\
\phi_2'(S_1) &= \frac{c'(\frac{S_2}{a_2})(S_2 - S_1)}{S_1(2c'(\frac{S_2}{a_2}) + c''(\frac{S_2}{a_2})\frac{S}{a_2})}.
\end{aligned}$$

Proof of Proposition 1: Define

$$F[a, S_a, S_b] \equiv V(a)S_b - c'(\frac{S_a}{a})S^2, \tag{9}$$

$$G[a, S_a, S_b] \equiv V(n-a)S_a - c'(\frac{S_b}{n-a})S^2. \tag{10}$$

Next compute

$$
\begin{aligned}
\frac{\partial F}{\partial S_a} &= -\frac{1}{a}c''(\frac{S_a}{a})S^2 - 2Sc'(\frac{S_a}{a}) < 0, \\
\frac{\partial F}{\partial S_b} &= V(a) - 2c'(\frac{S_a}{a})S, \\
\frac{\partial F}{\partial a} &= V'(a)S_b + \frac{S_a}{a^2}c''(\frac{S_a}{a})S^2, \\
\frac{\partial G}{\partial S_a} &= V(n-a) - 2c'(\frac{S_b}{n-a})S, \\
\frac{\partial G}{\partial S_b} &= -\frac{1}{n-a}c''(\frac{S_b}{n-a})S^2 - 2Sc'(\frac{S_b}{n-a}) < 0, \\
\frac{\partial G}{\partial a} &= -V'(n-a) - \frac{S_b}{n-a}^2 c''(\frac{S_b}{n-a})S^2.
\end{aligned}
$$

If $S_a > S_b$, $\frac{\partial F}{\partial S_b} > 0$ and $\frac{\partial G}{\partial S_a} < 0$. Conversely, if $S_a < S_b$, $\frac{\partial F}{\partial S_b} < 0$ and $\frac{\partial G}{\partial S_a} > 0$. Hence, we have:

$$
\begin{vmatrix} \frac{\partial F}{\partial S_a} & \frac{\partial G}{\partial S_a} \\ \frac{\partial F}{\partial S_b} & \frac{\partial G}{\partial S_b} \end{vmatrix} > 0
$$

Furthermore, if $\inf_a \frac{aV'(a)}{V(a)} \geq \sup_s \frac{-sc''(s)}{s}$, $\frac{\partial F}{\partial a} > 0$ and $\frac{\partial G}{\partial a} < 0$. Recalling that $S_a = S_{n-a}$ when $a = n - a$, we conclude that, for $a \geq n - a$,

$$
\begin{vmatrix} \frac{\partial F}{\partial a} & \frac{\partial G}{\partial S_a} \\ \frac{\partial F}{\partial b} & \frac{\partial G}{\partial S_b} \end{vmatrix} > 0
$$

so that S_a is increasing in a. If, on the other hand, $\sup_a \frac{aV'(a)}{V(a)} \leq \inf_s \frac{-sc''(s)}{s}$, $\frac{\partial F}{\partial a} < 0$ and $\frac{\partial G}{\partial a} > 0$, and a similar argument shows that S_a is decreasing in a for $a \leq n - a$.

Equilibrium of the sequential game of alliance formation

First notice that, when costs are linear and agents heterogeneous, the solution may be at a corner. Following Hillman and Riley (1989) and Sanchez Pages (2007), we order the alliances by increasing size, $a_1 \leq a_2 \leq a_3 ... \leq a_m$. If $\rho > 0$, larger alliances are more likely to be active, and in equilibrium, alliances $A_\kappa, ..., A_m$ are active where:

$$
\kappa = \min\{k, a_k^\rho \geq (m-k-2) \sum_{j=k+1}^{m} a_j^{-\rho}\}.
$$

If $\rho < 0$, on the contrary, small alliances are more likely to be active and in equilibrium, alliances $A_1, ..., A_\kappa$ are active where:

$$\kappa = \max\{k, a_k^{\rho} \geq (k-2)\sum_{j=1}^{k-1} a_j^{-\rho}\}.$$

Suppose for example that $\rho = 0$, so that every agent obtains $V(a) = 1$ in any alliance. In this case, it is easy to see that, by forming the grand coalition, players achieve the payoff of 1, whereas in any other alliance structure they obtain a lower payoff, as they receive the prize of 1 with probability $p < 1$ and must furthermore expand resources on conflict. Hence the grand coalition is the unique subgame perfect equilibrium outcome of the sequential game of alliance formation.

If the payoff is equally shared among all coalition members, $\rho = 1$, the equilibrium utility is given by:

$$\begin{aligned} U_i &= \frac{V}{a_i n^2}(n - a_i(\kappa-1))(n-(\kappa-1)) \text{ if } a_i \leq \frac{n}{\kappa-1}, \\ U_i &= 0 \text{ otherwise.} \end{aligned}$$

We will show that, as long as $\kappa \geq 1$ and the number of active players satisfies $s < n-1$, the optimal strategy of every player is to announce the formation of a singleton. The result is obviously true for the last player, n. Consider a situation where $\kappa \geq 1$, $s < n-1$, and such that all players form singletons after any move by player $n-s+1$. The maximization problem of player $n-s+1$ can thus be written:

$$\max_a \frac{1}{a}(n - a(\kappa+s-a))(n-(\kappa+s-a)).$$

It is easy to check that, as long as $\kappa \geq 1$ and $s + \kappa < n$,

$$\begin{aligned} (n - a(\kappa+s-a)) &< n-(s+\kappa-1), \\ \frac{1}{a}(n-(\kappa+s-a)) &< n-(s+\kappa-1), \end{aligned}$$

establishing the result. This induction argument allows us to show that all players form singletons in any subgame following the formation of a coalition if $s < n-1$. If $s = n-1$ and $\kappa = 1$, player 2 is indifferent between forming a singleton or a coalition of size $n-1$. (She would strictly prefer to form a singleton if $\rho < -1$, and strictly prefer to form an alliance of size $n-1$ if $\rho > -1$.) We conclude that there are two subgame perfect equilibria: one where the first player forms a singleton and the second player a coalition of size $n-1$ and one where the first player forms the grand coalition.

Finally, consider the model with $\rho = -2$, where players inside a coalition of size a obtain the payoff $V(a) = \frac{1}{a^2}$, corresponding to the equilibrium payoff of a subsequent contest among members of the alliance. We will show again by induction that, after the formation of $\kappa \geq 1$ coalitions, every player's optimal strategy is to form a singleton. The result is obviously true for player n, so we consider the choice of player $n-s$ after κ coalitions are formed. By a simple computation, we check that, if to players form singletons, any other alliance of size $a \geq 2$ is inactive. We can also check that, among active alliances, smaller alliances invest more and have a higher probability of winning the contest.

If two players have formed singletons before $n-s$ moves, player $n-s$'s optimal strategy is thus also to form a singleton. If one player has already formed a singleton, player $n-s$ has the choice between two strategies: form a singleton (and get a payoff of $\frac{1}{(s+1)^2}$), or form a single alliance of size s, which wins the contest with probability $p < \frac{1}{2}$, resulting in a payoff smaller than $\frac{1}{2s^2}$. As $s \geq 2$, $2s^2 > (s+1)^2$, and player $n-s$'s optimal choice is to form a singleton. Finally, suppose that no singleton has formed before player $n-s$. If player $n-s$ forms an alliance of size $a < s-2$, it will be followed by two singletons, and be inactive. If it forms an alliance of size s, it will obtain a payoff smaller than $\frac{1}{s^2}$. If it forms an alliance of size $s-1 > 1$, it will be followed by a singleton and obtain a payoff smaller than $\frac{1}{2(s-1)^2}$. If it forms a singleton, it will obtain a payoff $\frac{1}{s^2}$. Hence, forming a singleton is an optimal strategy for player $n-s$. The first player in the game will be indifferent between forming a singleton or the grand coalition, as both will result in the same payoff of $\frac{1}{n^2}$. However, if $\rho > -2$, the grand coalition dominates the partition of singletons and the first player will choose to form the alliance N.