Philosophy of Set Theory

LPS 247
Fall 2016 - Winter 2017

The mathematical theory of sets is both a foundation (in some sense) for classical mathematics and a branch of mathematics in its own right. Both its foundational role and its particular mathematical features -- the centrality of axiomatization and the prevalence of independence phenomena -- raise philosophical questions that have been debated since the birth of the subject at the end of the 19th century: what is a ‘foundation’ supposed to do?, what makes one system of axioms ‘better’ than another?, how can problems independent of the axioms, like Cantor’s famous Continuum Hypothesis, be addressed? Our goal in this course is to survey these matters, from their beginnings to contemporary debates. Some familiarity with the mathematical fundamentals of the theory (say at the level of Herbert Enderton’s *Elements of Set Theory*) and with the central results of intermediate logic (completeness, compactness, Löwenheim-Skolem, non-standard models, Gödel’s incompleteness theorems) will be presupposed. The rest will be sketched in readings and/or in class.

The default requirement for those taking the course for a grade (other than S/U, which involves only reading and attending) is three short papers (750-1250 words) due at the beginning of class in the 4th week, 7th week, and 10th week. These papers should isolate one localized philosophical, conceptual or methodological point within one of the readings and offer some analysis and/or critique. The thesis and its defense needn’t be earth-shattering in any way; this is really just an exercise in finding a topic of the right size and crafting a thesis and defense to match. I encourage writers to email me their topic and thesis, or even better, a draft introductory paragraph, for discussion well before the due date.

Other options are open to negotiation.

I assume everyone has access to a copy of

Van Heijenoort, *From Frege to Gödel*.

The rest of the assigned readings are available from the syllabus on the course EEE web site. (This is limited to those registered for the course. If you'd like to participate without registering, alternative access can be arranged.)

Please come to the first meeting prepared to discuss the reading in Topic 1.

Topics

The subject begins with Cantor and Dedekind in the late 19th century.

1. Origins: Cantor and Dedekind

Burgess, *Rigor and Structure*, pp. 67-74.

Ewald, *From Kant to Hilbert*, pp. 838-839.

Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers, pp*. 85-89, 110-115.

Hallett, *Cantorian Set Theory and Limitation of Size*, pp. 49-58.

Hallett's 'principle (a)' is 'any potential infinity presupposes a corresponding actual infinity', and 'principle (b)' is 'the transfinite is on a par with the finite and mathematically is to be treated as far as possible like the finite' (p. 7).

Cantor, 'Letter to Dedekind, 1899', with introductory note by van Heijenoort.

Hallett, *Cantorian Set Theory*, pp. 72-74, 119-128.

When Hallett says that 'number classes' are 'interior representations' of cardinal size, he means that they are representative members of the cardinal equivalence class, as opposed to the equivalence class itself (a 'whole class representation') (p. 60).

Dedekind, 'Continuity and irrational numbers', with introductory note by Ewald.

Extra reading

Cantor, chapter 19 of Ewald (selections from his writings and his correspondence with Dedekind).

Dedekind, 'Was sind und was sollen die Zahlen?', with introductory note by Ewald.

Here Dedekind presents a set-theoretic characterization of the natural numbers via the 'Peano' axioms. (He calls sets 'systems'.)

Frege, *Grundlagen*, §§29-44.

Hallett, *Cantorian Set Theory*, pp. 128-142.

Frege and Hallett discuss .‘units’ or ‘ones’.

Avigad, ‘Methodology and metaphysics in Dedekind’s theory of ideals’.

Dedekind also made early use of sets in his theory of ideals. Avigad traces the development of this line of thought from the 1870s to the 1890s.

Moore, ‘Early history of the GCH: 1878-1938’.

As is well-known, Russell wrote to Frege, expounding his paradox, just as the culmination of Frege’s great logicist project was being prepared for publication.

2. The paradoxes

Russell, ‘Letter to Frege, 1902’, with introductory note by van Heijenoort.

Frege, ‘Letter to Russell, 1902’, with introductory note by van Heijenoort.

Moore, ‘The roots of Russell’s paradox’.

Ebbinghaus, *Ernst Zermelo*, pp. 41-47.

Ebbinghaus traces discussion of paradoxes in Germany, before Russell.

Frege made an unsuccessful attempt to circumvent the paradoxes in a hasty appendix to volume II of his *Basic Laws*, published in 1903 (see the extra reading). With time for reflection, Russell eventually resorted to the theory of ramified types, the cornerstone of *Principia Mathematica*, of which this gives a glimpse:

Russell, ‘Mathematical logic as based on the theory of types’, pp. 150-168, with introductory note by Quine.

(Notice Quine’s introductory remark that the requirement of predicativity leads to systems like Weyl’s that are ‘inadequate for classical analysis’. More recent predicativists, most conspicuously Feferman (see the extra reading), have worked hard to reproduce large parts of classical analysis.

Zermelo took a different turn which opened the road to the mathematical theory of sets as we know it today, but before we leave the Frege-Russell line of development behind, it’s worth quick look at Russell’s intriguing explanation of how such complex theories can still provide a kind of ‘foundation’ for more familiar mathematics:

Russell, ‘The regressive method of discovering the premises of mathematics’.

Extra reading

Frege, *Basic Laws of Arithmetic*, volume II, pp. 253-265.

Quine, ‘On Frege’s way out’.

Quine shows that Frege’s attempted fix doesn’t work.

Urquhart, Alasdair, ‘The theory of types’.

A helpful exposition of the ramified theory and its development.

Maddy, *Naturalism in Mathematics*, pp. 3-15.

Feferman, ‘Weyl vindicated: *Das Continuum* seventy years later’.

As we’ve seen, Cantor was uneasy about his fundamental well-ordering principle. In a dramatic scene at the 1904 International Congress of Mathematicians in Heidelberg, König presented an argument that CH is false because the continuum cannot be well-ordered. Cantor was in the audience and much disturbed by this, but an error in the purported proof was revealed to the congress the very next day, by young Ernst Zermelo (see Moore [1982], pp. 86-88, Ebbinghaus [2007], pp. 50-53). This episode peaked Zermelo’s interest in the well-ordering principle, which he then proved a few weeks later, introducing the Axiom of Choice along the way.

3. Zermelo I: ZC

Zermelo, ‘Proof that every set can be well-ordered’ (1904), with introductory note by van Heijenoort).

Much consternation ensued. Poincaré, for example, objected to the proof on predicativist grounds (see extra reading). In contrast, the French analysts rejected Choice:

Baire, Borel, Hadamard, and Lebesgue, ‘Five letters on set theory’, with introductory note by Ewald (pp. 1075-1086).

This inspired Zermelo to restate his proof and to formulate all his assumptions explicitly. (Apparently Zermelo was also concerned to clarify the equivalence of ‘finite’ -- equinumerous with an initial segment of ω -- and ‘Dedekind finite’ -- not equinumerous with a proper subset --, which also depends on Choice. See Ebbinghaus [2007], pp. 61-65.) Thus ZC:

Zermelo, ‘A new proof of the possibility of a well-ordering’ (1908), with introductory note by van Heijenoort.

Zermelo, ‘Investigations in the foundations of set theory I’ (1908), with introductory note by van Heijenoort.

Notice the novel form of Zermelo’s defense of his axioms. This is the first occurrence of what are now known as ‘intrinsic’ and ‘extrinsic’ justifications for set-theoretic axioms (though we saw a foreshadowing in Russell last week). From here on, we’ll examine various axiom defenses and try to catalog the different types that fall in each of these general categories.

It turned out that Zermelo was right in both aspects of his defense of choice. First, it had been used unconsciously by set theorists from Cantor (as we’ve seen) to the very French analysts who criticized it (see Moore in the extra reading). Second, it was not only vital in many of those uses, but as gradually became obvious, it was also essential in many old and new areas of set theory, analysis, abstract algebra, topology and logic. Moore’s book details much of this, with a particularly pointed example in the history of van der Warden’s influential textbook, *Modern Algebra* (p. 232).

Extra reading

Poincaré, ‘The logic of infinity’.

Moore, ‘The origins of Zermelo’s axiomatization of set theory’.

Though many have characterized Zermelo as reacting to the paradoxes, Moore argues persuasively that he was primarily interested in bolstering his proof of the well-ordering principle. Ebbinghaus also emphasizes Zermelo’s interest in furthering Hilbert’s foundational project:

Ebbinghaus, *Ernst Zermelo*, pp. 76-79.

(Notice how Hilbert saved Zermelo from considerable grief by encouraging him not to hold off publication of ‘Investigations …’ until he had a consistency proof.)

Moore, *Zermelo’s Axiom of Choice*, pp. 92-103.

Here Moore details the French constructivist reactions and traces their unconscious use of choice.

By the 1920s, Frankel and Skolem had noticed that Zermelo’s system didn’t allow for the formation of $\{\omega, \mathcal{P}\omega, \mathcal{P}\mathcal{P}\omega, \mathcal{P}\mathcal{P}\mathcal{P}\omega, \ldots\}$. ($V_{\omega+\omega}$ is a natural model of ZC. Notice this means there are uncountable sets that can be well-ordered but have no corresponding ordinal.) The development of this required new axiom whose formulation was intertwined with the question of what to make of Zermelo's 'definite properties' in the Axiom of Separation.

4. To ZFC and NGB

Hallett, *Cantorian Set Theory*, pp. 280-286.

Hallett sketches the history of the axiom of replacement.

Skolem, 'Some remarks on axiomatized set theory' (1922), with introductory note by van Heijenoort.

Here Skolem proposes that 'definite' be taken to mean 'first-order definable', which leads to the so-called 'Skolem paradox'.

von Neumann, 'On the introduction of transfinite numbers' (1923), with introductory note by van Heijenoort.

von Neumann, 'An axiomatization of set theory' (1925), with introductory note by van Heijenoort.

Von Neumann's alternative to ZFC involves both of sets and classes (or rather, characteristic functions of sets and classes), but the classes are predicative (as opposed to the impredicative classes of Morse-Kelly set theory). Though he states the theory in terms of functions, Heijenoort remarks that it 'is easily translated into a more customary language' (p. 394). Saving you the trouble, here is a version that's come down to us in modern dress, as modified by Bernays and Gödel:

Drake and Singh, *Intermediate Set Theory*, pp. 193-199.

Notice that von Neumann's characteristic axiom IV.2 has been replaced by the ZFC axioms it implies: Separation, Replacement and Choice. In any case, VNB (or NGB or NBG as it's sometimes called) is a conservative extension of ZF, which is why Fraenkel, Bar-Hillel, and Levy think 'they are, essentially, different formulations of the same theory' (extra reading, p. 119).

Hallett, *Cantorian Set Theory*, pp. 47-48, 240-241, 286-298.

Hallett returns to his central theme of limitation of size.

Extra reading

Ebbinghaus, *Ernst Zermelo*, pp. 135-138.

Fraenkel, Bar-Hillel, and Levy, *Foundations of Set* Theory, pp. 119-146.

Kanamori, 'In praise of replacement'.

For those intrigued by classes:

Maddy, ‘Proper classes’ and ‘A theory of classes’.

Zermelo’s ill health led to his forced retirement in 1916, but he eventually recovered and returned to set theory in the mid-20s.

5. Zermelo II: the iterative conception

Zermelo, ‘On boundary numbers and domains of sets: new investigations in the foundations of set theory’ (1930), with introductory note by Kanamori.

Hallett, ‘Introduction to Zermelo [1930]’.

Zermelo, ‘On the set-theoretic model’ (1930), with introductory note by Kanamori.

Finally, here’s an overview of some of the arguments given over the years for the axioms of ZFC:

Maddy, *Naturalism*, pp. 36-62.

Extra reading

Ebbinghaus, *Ernst Zermelo*, pp. 186-196.

Tait, ‘Zermelo’s conception of set theory and reflection principles’.

Kanamori, ‘The empty set, the singleton, and the ordered pair’.

Moore, ‘Early history of the GCH: 1878-1938’.

In addition to the iterative conception, Zermelo’s rich paper introduces (at least) three other influential themes: large cardinals (boundary numbers), second-order axiomatization, and potentialism vs actualism. We’ll return to all of these below (topics 7, 9, and 12, respectively), but first a look at Gödel’s seminal contributions in the 1930s, only a few years after these from Zermelo.

6. Gödel: the constructible universe

Drake and Singh, *Intermediate Set Theory*, pp. 128-154.

Gödel, ‘The consistency of the axiom of choice and of the GCH’ (1938).

Thus one worry about Choice was put to rest. (Moore poses the lingering questions -- ‘did the Axiom of Choice, which had been the subject of so much debate, render the system more likely to be contradictory?’ -- and concludes ‘Gödel found that the answer was no’ (Moore [1982], pp. 279-280).) As for CH, any resolution based on ZFC could only come from proving it true. And that isn’t all …

Kanamori, ‘The emergence of descriptive set theory’, pp. 241-254.

Maddy, ‘Believing the axioms’, pp. 490-500, especially 491-495.

Extra reading

Solovay, ‘Introduction to [Gödel 1938-40]’.

This piece, by one of the finest set theorists of our day, lays out the full story of L, including more recent developments.

7. Large cardinals

Small large cardinals:

Drake, *Set* Theory, p. 67, 107-125.

For the record, a cardinal κ is ‘regular’ if there is no sequence of length less than κ whose limit is κ; otherwise κ is singular. (So $\aleph_0$ is regular; $\aleph_\omega$ is singular.) Also, Drake’s $\aleph(\kappa)$ is what’s more often written κ^+, the next cardinal number after κ.

Kanamori, *The Higher Infinite*, pp. 16-21, 57-667.

Maddy, ‘Believing the axioms’, pp. 501-504.

Large large cardinals:

Jech, *Set Theory*, pp. 125-128, 285-292.

Maddy, ‘Believing the axioms’, pp. 505-508, 748-756.

In his piece in the extra reading to topic 6, Solovay reports that Gödel offered something like a *uniformity* argument for a measureable cardinal (p. 19). Though he too believes in MCs, Solovay finds this unconvincing.

Extra reading

Drake, *Set Theory*, chapter 6.

Jensen, ‘Inner models and large cardinals’.

Here a ground-breaking figure in the investigation of L considers large cardinals by way of L-like inner models.

Kamamori and Magidor, ‘The evolution of large cardinal axioms’.

Solovay, Reinhardt, and Kanamori, ‘Strong axioms of infinity and elementary embeddings’.

8. Gödel’s realism

Gödel, ‘Russell’s mathematical logic’.

Gödel, ‘Some basic theorems on the foundations of mathematics and their implications’ (aka the Gibbs lecture), pp. 311-323.

Gödel, ‘What is Cantor’s continuum hypothesis?’ (1967).

Extra reading

Parsons, ‘Russell’s mathematical logic’ (with postscript).

Boolos, ‘Introduction to Gödel [1951]’.

Moore, ‘Introduction to Gödel [1947/64]’.

Parsons, ‘Platonism and mathematical intuition in Kurt Gödel’s thought’.

Maddy, *Naturalism*, pp. 89-94.

On Gödel’s case against CH:

Maddy, ‘Believing the axioms’, pp. 495-497.

Moore, ‘Introduction to Gödel [1947/64]’, pp. 173-175.

Solovay, ‘Introduction to Gödel [1970]’.

Burgess, ‘Intuitions of three kinds in Gödel’s views on the continuum’.

On Gödel’s interest in Leibniz and especially Husserl:

Van Atten and Kennedy, ‘On the philosophical development of Kurt Gödel.

Whatever the fate of Gödel’s efforts to cast doubt on, or even disprove, the CH, he was right in his prediction that it couldn’t be proved from ZFC (assuming ZFC is consistent, of course). Cohen’s mid-

60s proof of this fact introduced a new method and changed the face of set theory.

9. Cohen: forcing

Weaver, *Forcing for Mathematicians*, pp. 25-52.

Kunen, *Set Theory* (1980), pp. 232-237.

Kunen, *Set Theory* (2013), pp. 279-282.

(A notational warning: Weaver’s symbol for ‘q is an *extension* of p’ (p. 30) is non-standard. See Drake and Singh in the extra reading (p. 155) for discussion.) Cohen’s first results on CH came in 1963; his comprehensive book appeared in 1966. By 1967, Levy and Solovay had used the new method to dash the hope of settling CH with large cardinal axioms: all known large cardinals are unaffected by the type of forcings used to influence the truth value of CH.

In light of these results, one option some have found appealing is to fall back on Zermelo’s second-order axiomatization:

Zermelo, ‘Report … about my research concerning the foundations of mathematics’ (1930), with introductory note by Kanamori, pp. 435-441.

Kreisel, ‘Two notes on the foundations of set theory’, p. 107 (and footnote 2).

In fact, Shapiro shows how to formulate the CH in second-order logic alone:

Shapiro, *Foundations without Foundationalism*, pp. 104-107.

Of course anyone worried that the power set operation isn’t determinate enough to ground a truth value for CH will also worry that the (full) second-order quantifiers aren’t determinate enough to ground the corresponding second-order validities. (See Weston’s paper in the extra reading.) On the other hand, anyone confident that (full) second-order logic is determinate will feel equally confident about CH -- illustrating once again that one person’s modus ponens is another person’s modus tollens.

Extra reading

Drake and Singh, *Intermediate Set Theory*, pp. 154-192.

Weston, ‘Kreisel, the CH and second order set theory’.

Maddy, ‘Believing the axioms’, pp. 498-499.

Here we find Cohen's view on CH: that it's very badly false. This was once a majority view amongst those with an opinion, but for some time now, $\aleph_1$ (CH) and $\aleph_2$ (Gödel's view, in the extra readings for topic 8, though now not for Gödel's reasons) have been the leading candidates for the size of the continuum.

Moore, 'The origins of forcing'.

10. Two more realisms: Quinean and Set-theoretic

Maddy, *Naturalism*, pp. 87-109.

Maddy, *Realism*, pp. 170-177.

Maddy, *Naturalism*, pp. 133-160.

The second passage from *Naturalism* challenges the indispensability arguments on which both Quinean and set-theoretic realism are based.

Extra reading

Benacerraf, 'Mathematical truth'.

Maddy, *Realism*, pp. 36-80.

These pages detail the set-theoretic realist's account of perception and intuition. It begins from the claim that we perceive a simple sets of physical objects when we see, e.g., three things, because a set is the bearer of that perceived number property. This claim -- that 'three' is a property of set -- depends in turn on an indispensability argument (see pp. 60-63).

Benacerraf, 'What numbers could not be'.

Maddy, *Realism*, pp. 81-106.

Here are Benacerraf's case against set-theoretic foundations and the set-theoretic realist's response.

---------- End of fall quarter, Beginning of winter quarter ----------

As we've seen, one of the leading roles set theory was designed to play is foundational, but there's considerable disagreement about what this comes to.

11. Set theory as a foundation

Maddy, *Defending the Axioms*, pp. 2-37.

Maddy, ‘Set-theoretic foundations’, pp. 1-35.

Extra reading

Ernst, ‘The prospects of unlimited category theory’.

This is the paper cited in ‘Set-theoretic foundations’, where the hopes for a theory capable of generating ‘the category of Xs’ for any mathematical type X are dashed.

Maddy, *Naturalism*, pp. 22-35.

This is an earlier treatment of set theory’s foundational role.

Koellner, ‘Independence and large cardinals’.

Here Koellner describes in some detail the role of the large cardinal hierarchy in assessment of consistency strength.

Though the iterative conception entered set-theoretic thinking with Zermelo [1930], it took a while longer to reach philosophers.

12. The iterative conception II

Boolos, ‘The iterative concept of set’ (1971).

Shoenfield, ‘Axioms of set theory’, pp. 321-327, 335-336.

In his introduction to this paper, Burgess remarks, that it ‘in the early 1970s [the iterative conception], though familiar to set theorists, was little known to philosophers, who tended to assume that the only intuitive conception of set was the naïve one, and that the generally accepted axioms were merely an *ad hoc* list’. (This assumption animates Quine’s *Set Theory and its Logic*.) One aim of Boolos’s paper, Burgess writes, is ‘to provide philosophers with an accessible account of the cumulative or iterative conception’.

Boolos, ‘Iteration again’ (1989), pp. 88-97.

In this later treatment, Boolos incorporates ideas from Dana Scott to prove Regularity from surprisingly weak assumptions. (Shoenfield gives another version of the same argument.)

Linnebo, ‘The potential hierarchy of sets’.

Linnebo gives a recent update to Parsons’s modal approach to the hierarchy.

Extra reading

Shoenfield, 'Axioms of set theory'.

Boolos, 'Iteration again', pp. 97-104.

Paseau, 'Boolos on the justification of set theory'.

In the remainder of his 1989 paper, Boolos presents a second conception of set, a neo-Fregean version of limitation of size, and uses it to defend Replacement, Choice and Extensionality (the axioms he took not to follow from the iterative conception). Paseau disagrees.

Wang, 'The concept of set'.

Parsons, 'What is the iterative conception of set?'

We're seen that reflection is one of the most popular tools for intrinsic justifications, but it can mean different things to different people. Here is a contemporary disagreement.

13. Reflection

Koellner, 'On reflection principles'.

Welch, 'Obtaining Woodin's cardinals'.

Welch and Horstein, 'Reflecting on absolute infinity'.

Extra reading

Shapiro and Uzquiano, 'Frege meets Zermelo: a perspective on ineffability and reflection'.

After the independence results of Gödel, Cohen, and Levy/Solovay, there was some pessimism on the prospects for settling open questions unaffected by large cardinals, but a new type of axiom candidate revitalized this effort in the 1960s.

14. Determinacy

Maddy, 'Believing the axioms', pp. 736-748, 756-758.

Martin, 'Mathematical evidence'.

Koellner, 'Large cardinals and determinacy', pp. 38-48.

Extra reading

Koellner, 'Large cardinals and determinacy', the rest.

Koellner, ‘On the question of absolute undecidability’, pp. 201-209.

Welch, ‘Large cardinals, inner models, and determinacy’.

Turning at last to the contemporary scene, let’s look first at some recent philosophical thinking. Then, in the final three weeks, we’ll look at the three leading mathematical projects related to CH and new axioms.

Much as Gödel once inspired versions of realism, he’s now seen as the father of set-theoretic conceptualism.

15. Conceptualism

Martin, ‘Gödel’s conceptual realism’.

Parsons, ‘Analyticity for realists’.

Martin, ‘Completeness or incompleteness of basic mathematical concepts’, pp. 1-6, 15-23.

Extra reading

Martin, ‘Multiple universes of sets and indeterminate truth values’.

Martin, ‘Mathematical evidence’.

Parsons, ‘Quine and Gödel on analyticity’.

Parsons, ‘Platonism and mathematical intuition in Kurt Gödel’s thought’.

Burgess, ‘Intuitions of three kinds in Gödel’s views on the continuum’.

Mathematics in general presents a difficult challenge for some versions of naturalism -- often in some form of Benacerraf’s epistemological challenge -- but here’s a form of naturalism especially designed for set theory.

16. Naturalism I: methodology

Maddy, *Naturalism*, pp. 110-132, 172-176, 183-197.

Maddy, *Defending the Axioms*, pp. ix, 38-55.

Maddy, *Naturalism*, pp. 206-215.

Extra reading

Maddy, *Naturalism*, pp. 161-171, 177-182, 216-232.

The first two passages traces early hints of 'mathematical naturalism' in Wittgenstein and Quine. The third gets into the weeds of arguing against V=L. Here are some responses:

Löwe, 'A first glance at restrictiveness'.

Löwe, 'A second glance at restrictiveness'.

Hamkins, 'A multiverse perspective on V=L'.

Cummings, 'Some challenges for the philosophy of set theory'.

This entry in Koellner's EFI project advocates what looks to be a fairly naturalistic approach.

17. Naturalism II: metaphysics and epistemology

Maddy, *Defending the Axioms*, pp. 55-112, 123-137.

'Afterword to special issue of *PM*', pp. 245-248.

Extra reading

Maddy, *Naturalism*, pp. 200-205.

This passage emphasizes that the metaphysical and epistemological issues remain open after the methodological conclusions of *Naturalism* -- issues eventually addressed *Defending.*

Now the three leading contemporary programs. The first current response rejects the underlying notion that set theory should aim for a single preferred theory that will settle CH one way or the other.

18. Multiverse

Hamkins, 'The set-theoretic multiverse', §§1-3, 7, 9.

Hamkins urges that set theory is not about a unique universe of sets, but about a generous array of set-theoretic universes. Woodin considers a more limited generic multiverse.

Woodin, 'The continuum hypothesis and the search for mathematical infinity', January 2015, minutes 9-16:50, (slides 10-12) https://www.youtube.com/watch?v=nVF4N1Ix5WI

Koellner, ‘The continuum hypothesis’, pp. 29-35.

Koellner spells out Woodin’s objection to the generic multiverse in technical detail.

Steel, ‘Gödel’s program’, pp. 153-171.

Steel proposes a different, though still generic, approach.

Maddy, ‘Set-theoretic foundations’, pp. 35-51.

Maddy registers skepticism all ‘round.

Extra reading

Woodin, ‘The continuum hypothesis, the generic-multiverse of sets, and the Ω conjecture’.

This is Woodin’s presentation of the mathematics of his case against the multiverse.

Koellner, ‘Hamkins on the multiverse’.

This is Koellner’s extended critique of Hamkins. Koellner takes Hamkins to be a ‘pluralist’, to believe that some set-theoretic statements lack determinate truth values because set theory is actually the study of a multiverse of set-theoretic universes. In contrast, the non-pluralist believes that the statements of set theory have determinate truth values because set theory is the study of a unique set-theoretic universe. It’s hard to know where to place the naturalist of topic 17 in this classificatory scheme: she denies that set theory is out to determine truth values, which sounds a bit pluralistic, but she also thinks (at least for now) that there are sound mathematical reasons to prefer a universe theory, which sounds a bit non-pluralistic. The problem is that both the pluralist and the non-pluralist shrug off those mathematical reasons as ‘merely practical’ -- focusing instead on an ontological debate about truth and existence -- while the naturalist takes them to be the fundamental constraints on which theory to adopt.

The second current response is the Harvard School, with a case against CH followed by a case for CH.

19. Woodin/Koellner

Koellner, ‘The continuum hypothesis’, pp. 1-25, 35-51.

Woodin, ‘The continuum hypothesis and the search for mathematical infinity’, January 2015, minutes 0-9, 16:50-55, (slides 1-9, 11-30), https://www.youtube.com/watch?v=nVF4N1Ix5WI

This is a relatively accessible version of Woodin's current views.

Extra reading

Koellner, 'On the question of absolute undecidability', pp. 209-214, 217-222.

Woodin, 'The continuum hypothesis I and II'.

Woodin, 'Strong axioms of infinity and the search for V'.

Woodin, 'Formulating the axiom V=Ult(L)', June 2016, https://www.youtube.com/watch?v=YZnV8Y6Vc7Q

Here Woodin speaks to an elite group of set theorists at a conference in honor of Magidor's 70th birthday.

The program aimed at V=Ult(L) is one live possibility for extending ZFC+LCs and settling CH positively. A third line of inquiry points in a different direction.

20. Forcing axioms

Blass, 'Combinatorial cardinal characteristics of the continuum', pp. 1-3.

Glancing through the rest of Blass's paper, you'll see these cardinals have names like $\mathfrak{c}$ (the size of the continuum), $\mathfrak{d}$, $\mathfrak{b}$, $\mathfrak{a}$, $\mathfrak{p}$, $\mathfrak{m}$, etc. Kunen refers to them in the first excerpt here:

Kunen, *Set Theory*, pp. pp. 171-176, 379-381, 387-388.

Magidor, 'Some set theories are more equal'.

Todorcevic, 'The power set of ω_1 and the continuum problem'.

This paper is especially difficult. An attempt will be made to sort it out in class.

Extra reading

Weaver, *Forcing for Mathematicians*, pp. 89-92.

Caicedo and Veličković, 'The bounded proper forcing axiom and well orderings of the reals', pp. 10001-10002.

Viale, 'Review'.

The introduction to Caicedo and Veličković’s paper and Viale’s review of work by J. T. Moore are examples that give the flavor of recent work on forcing axioms (including the connection with Woodin’s P-max).

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