SUPPLEMENTARY APPENDIX

TO "RISK AVERSION IN CONTESTS"

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In this Supplement we prove the results on Propositions not proven in the main text or the Appendix; we state rigorously the conditions for existence of pure-strategy equilibrium in sections II and III and prove that for section II; and we include a few additional comments.

Before proving Proposition 2, we derive some other needed results.

LEMMA A2: Let $c>b>0$, then

(a) $$\frac{c}{1-e^{-cT}} > \frac{b}{1-e^{-bT}}$$

(b) $$\frac{ce^{-cT}}{1-e^{-cT}} < \frac{be^{-bT}}{1-e^{-bT}}$$

PROOF: (a) We just need to show that the function $f(x) = \frac{x}{1-e^{-xT}}$ is increasing. Note that $f'(x) = (1-e^{-Tx}-Txe^{-Tx})/(1-e^{-Tx})^2 = (e^{Tx}-1-Tx)/(1-e^{-Tx})^2e^{Tx} \geq 0$ as required.

(b) Similarly, let $g(x) = \frac{xe^{-xT}}{1-e^{-xT}}$. Then, it can be shown that $g'(x) = (e^{xT}-1-Te^{xT})/(e^{xT}-1)^2 < 0$. ■

PROOF OF PROPOSITION 2(a): First, let $\gamma=\beta$, and consider (x_1^*, x_2^*) such that $\pi_1^1(x_1^*, x_2^*) = \pi_2^2(x_1^*, x_2^*)=0$. Defining $y_i=x_i+\delta$, letting (by (1)) $p=y_1^m/(y_1^m+y_2^m)$, and substituting into (A3) with equality and its analogue for π_2^2, we obtain the following unique (and symmetric) solution:

(A5) $$x_1^* = x_2^* = \frac{m(1-e^{-\beta T})}{2\beta(1+e^{-\beta T})} - \delta \text{ [>0 for } \delta \text{ sufficiently small.]}$$

It can be shown that,

$$-\frac{p_{11}}{p_1} = \frac{1}{y_1} + \frac{m\,(y_1^m - y_2^m)}{(y_1^m + y_2^m)\,y_1} \quad \text{and} \quad \frac{p_{22}}{p_2} = \frac{1}{y_2} + \frac{m(y_2^m - y_1^m)}{(y_1^m + y_2^m)y_2}$$

By the proof of Proposition 1a, $\pi_{11}^1(x_1^*, x_2^*)<0$ and $\pi_{22}^2(x_1^*, x_2^*)<0$ are, respectively, equivalent to $-p_{11}^*/p_1^* > \beta$ and $p_{22}^*/p_2^* > \gamma=\beta$.
By (A5), both of these equations are equivalent to

$$\frac{1}{y_1^*} > \beta \iff 2(1+e^{-\beta T}) > m(1-e^{-\beta T}) \text{ [where } y_1^*=x_1^*+\delta]$$

which is always true when $m\le1$. Thus, x_1^* and x_2^* in (A5) are local best responses to each other. For (x_1^*,x_2^*) to be an equilibrium, the strategies need to be global best responses. Suppose an $\hat{x}_1$, other than x_1^*, which is a global best response to x_2^*. There are two possible cases. (i) $\hat{x}_1\in(0,Y)$. Then, we would have $\pi_1^1(\hat{x}_1,x_2^*) = 0$. Since $\beta=\gamma$ and the payoff functions are symmetric with $x_1^*=x_2^*$, we must also have $\pi_2^2(x_2^*,\hat{x}_1)=0$. But the solution to the system $\pi_2^2(x_2^*,\hat{x}_1)=0$ and $\pi_1^1(\hat{x}_1,x_2^*) = 0$ has the unique (symmetric) solution described in (A5). Consequently, if $\hat{x}_1$ were at the interior it must coincide with $x_1^*=x_2^*$. (ii) Either $\hat{x}_1=0$ or $\hat{x}_1=Y$. If that were the case, and since x_1^* is a local maximum, there must be a point $x_1'\in(0,1)$ to the left (if $\hat{x}_1=0$) of x_1^* or to its right (if $\hat{x}_1=Y$) so that $\pi^1(x_1,x_2^*)$ attains at least a local minimum, and thus having $\pi_1^1(x_1',x_2^*) = 0$. By an argument similar to that under case (i), just above, x_1' must also equal x_1^*, which is a local maximum. Consequently, we have a contradiction for this case as well and, therefore, x_1^* is a global best response to x_2^*. By an identical argument, x_2^* is a global best response to x_1^* and, thus, (x_1^*,x_2^*) is an equilibrium.

Now, under (1) π_1^1 and π_2^2 are continuous functions of β, γ, x_1, and x_2 (see A3). We can then use the implicit function theorem and show that the best response functions are continuous in β and γ. Since we have just found an interior equilibrium for $\beta=\gamma$, this continuity property of the best response functions imply the existence of an interior equilibrium for $\gamma>\beta$ when γ is sufficiently close to β.

PROOF OF PROPOSITION 2(b): At the interior equilibrium (x_1^*,x_2^*) we have $\pi_1^{1*}=0$ and $\pi_2^{2*}=0$, which imply, respectively, (see A4)

(A6a) $$p_1=B(pe^{-\beta T}+ 1-p) \qquad \text{where } B=\beta/(1-e^{-\beta T})$$

(A6b) $$-p_2=\Gamma[(1-p)e^{-\gamma T}+p] \qquad \text{where } \Gamma=\gamma/(1-e^{-\gamma T})$$

Dividing these two equations and then dividing both the numerator and the denumerator by p, yields:

(A7) $$-\frac{p_1}{p_2}=\frac{B}{\Gamma}\frac{(e^{-\beta T}+q)}{(qe^{-\gamma T}+1)}$$

where $q\equiv(1-p)/p$ [$=(y_2^*/y_1^*)^m$ where $y_i^*=x_i^*+\delta_i$]. Suppose, contrary to what we would like to prove, that the more risk averse agent puts more effort in equilibrium ($y_2^*>y_1^*$). Then, we would have $q=(y_2^*/y_1^*)^m>1$. Under (1), we also have $p_1/(-p_2)=y_2^*/y_1^*\geq(y_2^*/y_1^*)^m=q$ since $m\leq1$. Since $p_1/(-p_2)$ is the left-hand-side of (A7), we then have

$$q\leq\frac{B(e^{-\beta T}+q)}{\Gamma(qe^{-\gamma T}+1)}$$

which is equivalent to:

(A8) $$\Gamma e^{-\gamma T}q^2+(\Gamma-B)q-Be^{-\beta T}\leq0$$

Next, note that the positive solution to

(A9) $\Gamma e^{-\gamma T} z^2 + (\Gamma - B)z - Be^{-\beta T} = 0$ is

$$z = \frac{-(\Gamma-B) + \text{sqrt}\{(\Gamma-B)^2 + 4B\Gamma e^{-(\beta+\gamma)T}\}}{2\Gamma e^{-\gamma T}}$$

By Lemma A2, (a), $\Gamma > B$. Therefore

$$z < [4B\Gamma e^{-(\beta+\gamma)T}]^{1/2} / 2\Gamma e^{-\gamma T} = [Be^{-\beta T}/\Gamma e^{-\gamma T}]^{1/2} < 1$$

where the last ionequality follows by Lemma A2, (b). Now, (A8) and (A9) imply:

$$\Gamma\ e^{-\gamma T}(q^2 - z^2) + (\Gamma - B)(q - z) \le 0$$

Since, by lemma A2(a), $\Gamma > B$ and $\Gamma e^{-\gamma T} > 0$, we must have $q < z$. Since $z < 1$, we then have $q < 1$ which contradicts our initial supposition of $q > 1$. Thus, $x_1^* > x_2^*$ ($y_1^* > y_2^*$) and the less risk averse agent puts more effort in equilibrium. ■

PROOF OF PROPOSITION 3(a): It is straightforward to show that under the CSF in (2) we have

(A10) $p_{11} = -p_{22} = \{k^2 \exp[k(x_1+x_2)][\exp(kx_2) - \exp(kx_1)]\} / [\exp(kx_1) + \exp(kx_2)]^3$

which is negative if and only if $x_1 < x_2$. Since p_{11} and p_{22} have opposite signs, at least one of the conditions in Propostion 1 would always fail. Consequently, by Proposition 1, an interior pure-strategy equilibrium cannot exist for (2).

PROOF OF PROPOSITION 3(b): We will prove that $x_1 = Y$ is a best response to $x_2 = Y$ for agent 1; exactly the same steps can be used to prove that Y is also a best response to Y for agent 2. First note that $\pi_1^1(0,0) > 0$ is equivalent to: (see (A3))

(A11) $$e^{-\beta Y}\left[\frac{k}{4}(1-e^{-\beta T}) - \frac{\beta}{2}(1+e^{-\beta T})\right] > 0$$

$$\iff \frac{k}{4} > \frac{\beta(1+e^{-\beta T})}{2(1-e^{-\beta T})}$$

Next note that

$$\pi_1^1(Y,Y)=e^{-\beta Y}\left[\frac{k\, e^{2kY}}{(2e^{kY})^2}(1-e^{-\beta T}) - \frac{\beta}{2}(1+e^{-\beta T})\right]$$

$$=e^{-\beta Y}\left[\frac{k}{4}(1-e^{-\beta T}) - \frac{\beta}{2}(1+e^{-\beta T})\right]$$

$$=\pi_1^1(0,0) > 0$$

Thus, $\pi_1^1(Y,Y)=\pi_1^1(0,0) > 0$. Since, by supposition, $\pi^1(Y,Y) \geq \pi^1(0,Y)$, $x_1=Y$ is a best response to Y if there does not exist $x_1^* \in(0,Y)$ such that $\pi^1(x_1^*,0) \geq \pi^1(Y,Y)$. Suppose such an x_1^* existed. Then, we would have $\pi_{11}^1(x_1^*,0) \leq 0$ and $\pi_1^1(x_1^*,0) = 0$, (using (A2′) with equality)

$$\pi_{11}^1(x_1^*,Y) = -e^{-\beta(Y-x_1^*)}(1-e^{-\beta T})(-p_{11}^* - p_1^*\beta)$$

Note, by (A10) and since $x_1^* < Y$, that $p_{11}^*>0$. Since $p_1^*>0$, we must have $\pi_{11}^1(x_1^*,Y)>0$ which is a contradiction. Hence, such $x_1^*\in(0,Y)$ cannot exist and Y is a best response to Y. ∎

To prove existence of pure-strategy equilibrium for the limited liability contest (payoff functions in (6)-(7)), we need the following condition (see also footnote #7):

(A12) $p_{11}p \leq p_1^2$ [and, by symmetry, $-p_{22}(1-p) \leq p_2^2$.]

PROPOSITION A1: Assume (A12). Then $\pi^1(1)$ is quasiconcave in x_i for i=1,2 and a pure-strategy equilibrium exists.

PROOF: As we did in the Proof of Proposition 1a, we use Lemma A1 and, to show quasi-concavity of each agent's payoff function in the agents's own strategy, we just need to establish that $\pi^1_{11}(1) \le 0$ if $\pi^1_1(1) \le 0$ and similarly for agent 2's payoff function.

Differentiation of 1's payoff function yields:

(A13) $\pi^1_1(1) = p_1 U_1 - pU'_1$

(A14) $\pi^1_{11}(1) = p_{11} U_1 - 2p_1 U'_1 - p\, U''_1$

If $\pi^1_1(1) \le 0$, then, by (A13), we have $U_1 p_1/p \le U'_1$. Using this inequality in (A14), we obtain $\pi^1_{11}(1) \le U'(\frac{pp_{11}}{p_1} - 2p_1) + pU_1''$. By (A12), $p_{11} p/p_1 \le p_1$ and thus the first term in the right-hand side of this inequality is negative, whereas the second term is also negative since $U_1'' < 0$ and $p>0$. Therefore, we have $\pi^1_{11}(1) \le 0$. ∎

PROPOSITION 5. *Consider two situations with limited liability and identical agents: One in which both agents have Von Neuman-Morgenstern utility function* U(•) *and one in which both agents have Von Neuman-Morgenstern utility function* k(U(•)). *Assume a unique interior symmetric equilibrium in both cases. Then, equilibrium effort will be higher when the two agents are more risk averse.*

PROOF: Denote by x^u the equilibrium level of effort under U(•) and by x^k the equilibrium effort under k(U(•)). Let $r^u(\cdot)$ and $r^k(\cdot)$ be the best response functions under the two different utility functions. Clearly, we have $r^i(x^i) = x^i$ for $i = u,k$. Since (x^i, x^i) is a unique equilibrium, (0,0) cannot be an equilibrium, and thus we must have $r^i(0) > 0$ for $i = u,k$. Consequently, we must have

(A15) $\quad r^i(x) > x$ for all $x \in [0, x^i)$ and $i = u,k$

Along agent 1's best response function we have

$$\pi^1_1(r^u(x), x; U) = p_1 U_1 - pU'_1 = 0 < p_1 k(U_1) - pk'(U_1)U'_1 = \pi^1_1(r^u(x), x; k(U))$$

where "U" and "k(U)" refer to the expected utility function used and with the inequality following by (9). This inequality then implies that $r^u(x) < r^k(x)$ for all x.

Now, contrary to what we want to prove, suppose $x^k \le x^u$. Then, by (9) and the inequality just derived, we have $x^k \le r^u(x^k) < r^k(x^k)$. But this contradicts $x^k = r^k(x^k)$. Thus, we must have $x^k > x^u$ as required. □

To obtain a pure-strategy equilibrium in the contest in which the prize is divisible, the following condition can be imposed on the sharing function $\alpha(p)$:

(A16) $\quad 0 < \alpha(p) < 1; \quad \alpha'(p) > 0; \quad \alpha''(p) \le 0$

A stronger version of (A12) is also needed:

(A17) $\quad p_{11} \le 0$ and $-p_{22} \le 0$ [Each agent's probability of winning is concave in the agent's own effort.]

PROPOSITION A2: *Consider the game with the payoff functions in* (12)-(13) *and assume* (A16)-(A17). *Then for each* $i=1,2$ $\pi^i(\alpha)$ *is concave in* x_i *and a pure-strategy equilibrium exists.*

(The proof is straightforward and is omitted.)

Note on sharing functions (section III):

For sharing rules other than the simple one we have discussed in the main text, there are two issues of interest (but which are not rigorously examined). First, there is the issue of optimality of equilibrium efforts, either from the agents' perspective or from those who enjoy the benefits of the efforts. From (13) it can be seen that, aside from the size of the prize T, the factors influencing the size of equilibrium effort are $\alpha'(1/2)$ and the derivatives of the CSF with respect to each agent's effort at a symmetric point. Under (A2') the higher is the level of effort, the lower are the derivatives of the CSF. Consequently, a higher $\alpha'(1/2)$ would imply a higher level of equilibrium effort by the contestants. Thus, other things being equal, sharing rules with lower derivatives would be preferable from the contestants' point of view. An extreme case would be to follow the rule $\alpha(p) = \bar{\alpha}$ where $\bar{\alpha}$ is a constant. Then, putting zero effort would be an equilibrium provided the constant $\bar{\alpha}$ is such that the threat points of both contestants (either $\pi^i(0,0)$ or $\pi^i(0,0;1)$) are smaller than their payoff when this sharing rule is used.

The second issue of interest is whether α is greater or less than 1/2 in equilibrium. Axiomatic bargaining solutions and other bargaining games tend to favor the less risk averse.[13] Our expectation is that sharing rules derived from either the Nash or the Kalai-Smorodinsky solutions would favor the less risk averse[14] (so that in equilibrium, given $p=1/2$, we would have $\alpha>1/2$).

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[13] For the canonical setting see Roth (1979). For exceptions see Osborne (1985) and Roth and Rothblum (1982).

[14] This expectation is partly based on the aforementioned results of bargaining models and partly on our experimentation with specific parameter values under (U) with π^i's as the disagreement points. We have not been

Supplementary reference:

Osborne, M.J. (1985) "The Role of Risk Aversion in a Simple Bargaining Model," in A.E.Roth (ed.) *Game-Theoretic Models of Bargaining*, Cambridge: Cambridge University Press.

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able to obtain analytical results even for this special case.