Supporting Information Appendix to "Difference-Form Persuasion Contests" by Stergios Skaperdas, Amjad Toukan and Samarth Vaidya[1]

I. Impact of changes in evidence potency parameter α on equilibrium resource spending and welfare when assumptions 2 and 6 hold and the persuasion function is given by (17)

Hence using (6), (17) and the parametrization in assumptions 2 and 6, the first order conditions with respect to R_1 and R_2 are:

$$\tfrac{v+\omega}{2}\left[\alpha + \Delta(1-2h^2)\right]h'^1 - 1 = 0 \tag{I.1}$$

$$\tfrac{v-\omega}{2}\left[\alpha - \Delta + 2\Delta(h^1)\right]h'^2 - 1 = 0 \tag{I.2}$$

Using total differentiation and Cramer's rule, we get,

$$\frac{dR_1^*}{d\alpha} = \frac{\begin{vmatrix} -h'^1 & -2\Delta h'^1 h'^2 \\ -h'^2 & \dfrac{2h''^2}{(v-\omega)h'^2} \end{vmatrix}}{\begin{vmatrix} \dfrac{2h''^1}{(v+\omega)h'^1} & -2\Delta h'^1 h'^2 \\ 2\Delta h'^1 h'^2 & \dfrac{2h''^2}{(v-\omega)h'^2} \end{vmatrix}} \tag{I.3}$$

$$\frac{dR_2^*}{d\alpha} = \frac{\begin{vmatrix} \dfrac{2h''^1}{(v+\omega)h'^1} & -h'^1 \\ 2\Delta h'^1 h'^2 & -h'^2 \end{vmatrix}}{\begin{vmatrix} \dfrac{2h''^1}{(v+\omega)h'^1} & -2\Delta h'^1 h'^2 \\ 2\Delta h'^1 h'^2 & \dfrac{2h''^2}{(v-\omega)h'^2} \end{vmatrix}} \tag{I.4}$$

[1] Throughout this appendix, we abbreviate $h(R_i) = h^i, h'(R_i) = h'^i$ and $h''(R_i) = h''^i$ for $i = 1, 2$ for ease of exposition.

Let $D = \begin{vmatrix} \dfrac{2h''^1}{(v+\omega)h'^1} & -2\Delta h'^1 h'^2 \\ 2\Delta h'^1 h'^2 & \dfrac{2h''^2}{(v-\omega)h'^2} \end{vmatrix}$. From Assumption 1, it is clear that $D > 0$. Hence the signs of $\frac{dR_1^*}{d\alpha}$ and $\frac{dR_2^*}{d\alpha}$ are determined by the respective numerator terms in (I.3) and (I.4). The value of the numerator in (I.3) is $\frac{-2h'^1 h''^2}{(v-\omega)h'^2} - 2\Delta h'^1 (h'^2)^2$. Observe that the first term in this expression is positive while the second term is negative. Hence the numerator cannot be signed unambiguously so that the sign of $\frac{dR_1^*}{d\alpha}$ is indeterminate. The numerator in (I.4) is given by $-\frac{2h'^2 h''^1}{(v+\omega)h'^1} + 2\Delta (h'^1)^2 h'^2$ which is strictly positive. Hence it follows that $\frac{dR_2^*}{d\alpha} > 0$.

Intuitively, an increase in α tends to increase the net marginal benefit from resource spending for both players and has a tendency to increase both R_1 and R_2. However while an increase in R_1 induces R_2 to increase even further, the opposite is true for R_1 as R_2 increases. Hence while $\frac{dR_2^*}{d\alpha} > 0$, the sign of $\frac{dR_1^*}{d\alpha}$ turns out to be ambiguous.

Now, $\frac{d(R_1^* + R_2^*)}{d\alpha} = \frac{-\frac{2h'^1 h''^2}{(v-\omega)h'^2} - \frac{2h'^2 h''^1}{(v+\omega)h'^1} - 2\Delta h'^1 h'^2 (h'^2 - h'^1)}{D}$. In the numerator of the above expression, while the first two terms are positive, the sign of the last term depends on whether $h'^1 > h'^2$. If $R_1^* < R_2^*$, then $h'^1 > h'^2$ so that all three terms are positive leading to aggregate resource expenditures increasing with α. However, when $R_1^* > R_2^*$, $h'^1 < h'^2$ so that the last term in the numerator is negative. Given this, the sign of $\frac{d(R_1^* + R_2^*)}{d\alpha}$ is indeterminate.

Using the definition of *U*, (6), (17) and the parametrizations in assumptions 2 and 6 we get

$$U = v + \left[(\alpha+\Delta)h^1 - (\alpha-\Delta)h^2 - 2\Delta h^1 h^2\right]\omega - R_1 - R_2 \tag{I.5}$$

Hence at (R_1^*, R_2^*), $\frac{dU}{d\alpha} = \frac{\partial U}{\partial \alpha} + \left[\frac{\partial U}{\partial R_1} - 1\right]\frac{dR_1^*}{d\alpha} + \left[\frac{\partial U}{\partial R_2} - 1\right]\frac{dR_2^*}{d\alpha}$.

Now,

$$\frac{\partial U}{\partial \alpha} = (h^1 - h^2)\omega$$

$$\frac{\partial U}{\partial R_1} = \omega\left[\alpha + \Delta(1 - 2h^2)\right]h'^1$$

$$\frac{\partial U}{\partial R_2} = -\omega\left[\alpha - \Delta + 2\Delta h^1\right]h'^2$$

Using (I.1) and (I.2), it follows that at (R_1^*, R_2^*), $\frac{\partial U}{\partial R_1} = \frac{2\omega}{v+\omega}$ and $\frac{\partial U}{\partial R_2} = -\frac{2\omega}{v-\omega}$. Hence,

$$\frac{dU^*}{d\alpha} = (h^{*1} - h^{*2})\omega + \left[\frac{\omega - v}{v+\omega}\right]\frac{dR_1^*}{d\alpha} - \left[\frac{v+\omega}{v-\omega}\right]\frac{dR_2^*}{d\alpha}.$$

The sign of the first component $(h^{*1} - h^{*2})\omega$ is in general indeterminate as it depends on whether $R_1^* > R_2^*$ and ω is positive or negative. While $\frac{\omega - v}{v+\omega} < 0$, the sign of $\frac{dR_1^*}{d\alpha}$ is ambiguous as shown above. The only component that is unambiguously negative is $-\left[\frac{v+\omega}{v-\omega}\right]\frac{dR_2^*}{d\alpha}$ as $\frac{v+\omega}{v-\omega} > 0$ and $\frac{dR_2^*}{d\alpha} > 0$. Hence in general, the sign of $\frac{dU^*}{d\alpha}$ is indeterminate.[2]

II. Impact of changes in evidence asymmetry parameter Δ on equilibrium resource spending and welfare when the persuasion function is given by (17) and assumptions 2 and 6 hold.

Using the first-order conditions given by (I.1) and (I.2) and applying total differentiation and Cramer's rule, we get,

$$\frac{dR_1^*}{d\Delta} = \frac{\begin{vmatrix} -h'^1(1-2h^2) & -2\Delta h'^1 h'^2 \\ -h'^2(2h^1 - 1) & \dfrac{2h''^2}{(v-\omega)h'^2} \end{vmatrix}}{D} \qquad \text{(II.1)}$$

[2] If $\omega > 0$ and $R_1^* > R_2^*$, then $(h^{*1} - h^{*2})\omega > 0$ so that an increase in α tends to contribute towards an increase in aggregate welfare by increasing the win probability of the player with the higher stake. However since it also leads to an increase in at least Player 2's resource expenses that by itself contributes towards reducing aggregate welfare. Further, since the sign of $\frac{dR_1^*}{d\alpha}$ is indeterminate, we cannot be sure of the overall impact of an increase in α on aggregate welfare.

$$\frac{dR_2^*}{d\Delta} = \frac{\begin{vmatrix} \dfrac{2h''^1}{(v+\omega)h'^1} & -h'^1(1-2h^2) \\ 2\Delta h'^1 h'^2 & -h'^2(2h^1-1) \end{vmatrix}}{D} \tag{II.2}$$

Since $D > 0$, the sign of $\frac{dR_1^*}{d\Delta}$ is determined by the numerator in (II.1). The value of the numerator in (II.1) is $\frac{-2h'^1(1-2h^2)h''^2}{(v-\omega)h'^2} - 2\Delta h'^1(h'^2)^2(2h^1-1)$ whose sign depends among other things on the levels of h^1 and h^2 and therefore is generally indeterminate. Accordingly, the sign of $\frac{dR_1^*}{d\Delta}$ is indeterminate. Analogously, the sign of the numerator for $\frac{dR_2^*}{d\Delta}$ also depends on the the levels of h^1 and h^2 and is generally indeterminate and therefore so is the sign of $\frac{dR_2^*}{d\Delta}$.

Interestingly, when $h(.)$ is given by (7) and $\psi > \frac{1}{2}$, it is always the case that $h^i > \frac{1}{2}$, for any R_i, $i = 1, 2$. In this case, $(1-2h^2) < 0$ while $(2h^1-1) > 0$ so that the numerator of $\frac{dR_1^*}{d\Delta}$ is negative and hence $\frac{dR_1^*}{d\Delta} < 0$. This result can also occur more generally if both R_1^* and R_2^* are sufficiently high so that $h^i > \frac{1}{2}$, $i = 1, 2$. However in these circumstances, the sign of the numerator for $\frac{dR_2^*}{d\Delta}$ is indeterminate. Accordingly, the sign of $\frac{d(R_1^* + R_2^*)}{d\Delta}$ is indeterminate.

When both R_1^* and R_2^* are sufficiently low so that $h^i < \frac{1}{2}$, $i = 1, 2$ while $\frac{dR_1^*}{d\Delta} > 0$, the sign of $\frac{dR_2^*}{d\Delta}$ remains indeterminate. Hence the sign of $\frac{d(R_1^* + R_2^*)}{d\Delta}$ is indeterminate.

When there is considerable asymmetry in equilibrium resource spending where R_1^* is sufficiently higher than R_2^* so that $h^1 > \frac{1}{2}$ while $h^2 < \frac{1}{2}$, we observe that $\frac{dR_2^*}{d\Delta} > 0$. However, it is not possible to sign $\frac{dR_1^*}{d\Delta}$ unambiguously and so the impact on aggregate resources remains ambiguous. Analogously, when R_1^* is sufficiently lower than R_2^* so that

$h^1 < \frac{1}{2}$ and $h^2 > \frac{1}{2}$ we find that $\frac{dR_2^*}{d\Delta} < 0$. However, as before, it is not possible to sign $\frac{dR_1^*}{d\Delta}$ unambiguously so that the impact on aggregate resources remains ambiguous.

To examine the impact on aggregate welfare, observe that from (I.1), (I.2) and (I.5) it follows that,

$$\frac{dU^*}{d\Delta} = (h^{*1} + h^{*2} - 2h^{*1}h^{*2})\omega + \left[\frac{\omega - v}{v+\omega}\right]\frac{dR_1^*}{d\Delta} - \left[\frac{v+\omega}{v-\omega}\right]\frac{dR_2^*}{d\Delta} \tag{II.3}$$

Since by definition, $0 \le h^{*i} < 1$, for $i = 1, 2$, it follows that $h^{*1} + h^{*2} - 2h^{*1}h^{*2} > 0$. Also from Assumption 2, $\frac{\omega - v}{v+\omega} < 0$ and $\frac{v+\omega}{v-\omega} > 0$. However since we cannot sign $\frac{dR_i^*}{d\Delta}$ for $i = 1, 2$ unambiguously, the sign of the expression in (II.3) is in general indeterminate. Note that the first component in (II.3) suggests that an increase in Δ by itself tends to increase welfare when Player 1 has a higher stake ($\omega > 0$) as it contributes to increasing her win probability. However, the consequent changes induced to R_1^* and R_2^* potentially confound this effect when at least one of the players increases her expenditure.

III. Impact of changes in prize asymmetry on equilibrium resource spending and welfare when the persuasion function is given by (17) and Assumption 6 holds.

With asymmetric stakes we need to consider two possibilities. One possibility is that Player 1, who has the evidence advantage, also has the higher stake so that as per Assumption 2, $\omega > 0$ (Case 1). The other possibility is that Player 1 has a lower stake so that $\omega < 0$ (Case 2). We explore both the cases.

Case 1

Totally differentiating the first order conditions (as given by (I.1) and (I.2)) and applying Cramer's rule, we get:

$$\frac{dR_1^*}{d\omega} = \frac{\begin{vmatrix} -\frac{2}{v+\omega} & -2(v+\omega)\Delta h'^1 h'^2 \\ \frac{2}{v-\omega} & \frac{2h''^2}{h'^2} \end{vmatrix}}{\begin{vmatrix} \frac{2h''^1}{h'^1} & -2(v+\omega)\Delta h'^1 h'^2 \\ 2(v-\omega)\Delta h'^1 h'^2 & \frac{2h''^2}{h'^2} \end{vmatrix}} \tag{III.1}$$

$$\frac{dR_2^*}{d\omega} = \frac{\begin{vmatrix} \frac{2h''^1}{h'^1} & -\frac{2}{v+\omega} \\ 2(v-\omega)\Delta h'^1 h'^2 & \frac{2}{v-\omega} \end{vmatrix}}{\begin{vmatrix} \frac{2h''^1}{h'^1} & -2(v+\omega)\Delta h'^1 h'^2 \\ 2(v-\omega)\Delta h'^1 h'^2 & \frac{2h''^2}{h'^2} \end{vmatrix}} \qquad \text{(III.2)}$$

Let $\tilde{D} = \begin{vmatrix} \frac{2h''^1}{h'^1} & -2(v+\omega)\Delta h'^1 h'^2 \\ 2(v-\omega)\Delta h'^1 h'^2 & \frac{2h''^2}{h'^2} \end{vmatrix}$. From Assumption 1, it is clear that $\tilde{D} > 0$.

Hence $\frac{dR_1^*}{d\omega} > 0$ as the numerator in (III.1) is positive. Similarly, the sign of $\frac{dR_2^*}{d\omega}$ is determined by the sign of the numerator in (III.2). The value of this numerator is $\frac{4h''^1}{(v-\omega)h'^1} + \frac{4(v-\omega)\Delta h'^1 h'^2}{(v+\omega)}$. Notice that the first component is negative while the second component is positive. Hence the numerator cannot be signed unambiguously. As a result, the sign of $\frac{dR_2^*}{d\omega}$ is indeterminate. Given this, we cannot unequivocally sign $\frac{d(R_1^*+R_2^*)}{d\omega}$.

To examine the impact of a change in ω on aggregate welfare observe that from (I.1), (I.2) and (I.5) it follows that,

$$\frac{dU^*}{d\omega} = ((\alpha+\Delta)h^{*1} - (\alpha-\Delta)h^{*2} - 2\Delta h^{*1}h^{*2}) + \left[\frac{\omega - v}{v+\omega}\right]\frac{dR_1^*}{d\omega} - \left[\frac{v+\omega}{v-\omega}\right]\frac{dR_2^*}{d\omega} \qquad \text{(III.3)}$$

Notice that $(\alpha+\Delta)h^{*1} - (\alpha-\Delta)h^{*2} - 2\Delta h^{*1}h^{*2} = p_1^* - p_2^*$. Hence as long as Player 1 is the favorite in equilibrium ($p_1^* > p_2^*$), an increase in ω tends to increase aggregate welfare. However, the second component in (III.3) is always negative while the last component cannot be signed. Hence the overall sign of $\frac{dU^*}{d\omega}$ is ambiguous. Similarly, when $p_1^* < p_2^*$, the combined effect of the first two components in (III.3) is negative but the sign of the last term cannot be determined a priori. Hence the impact of an increase in ω on aggregate welfare cannot be determined unequivocally.

Case 2

For ease of exposition, we modify Assumption 2 as follows:

Assumption 7: Let $v_1 = v - \varpi$, $v_2 = v + \varpi$ where $v > 0$ and $0 \le \varpi < v$.

Given the above parameterization, the first order conditions are given by,

$$\tfrac{v-\varpi}{2}\left[\alpha + \Delta(1-2h^2)\right]h'^1 - 1 = 0 \qquad \text{(III.4)}$$

$$\tfrac{v+\varpi}{2}\left[\alpha-\Delta+2\Delta(h^1)\right]h'^2-1=0 \qquad \text{(III.5)}$$

By totally differentiating the above conditions and applying Cramer's rule, we get,

$$\frac{dR_1^*}{d\varpi}=\frac{\begin{vmatrix}\frac{2}{v-\varpi} & -2(v-\varpi)\Delta h'^1h'^2\\ -\frac{2}{v+\varpi} & \frac{2h''^2}{h'^2}\end{vmatrix}}{\begin{vmatrix}\frac{2h''^1}{h'^1} & -2(v-\varpi)\Delta h'^1h'^2\\ 2(v+\varpi)\Delta h'^1h'^2 & \frac{2h''^2}{h'^2}\end{vmatrix}} \qquad \text{(III.6)}$$

$$\frac{dR_2^*}{d\varpi}=\frac{\begin{vmatrix}\frac{2h''^1}{h'^1} & \frac{2}{v-\varpi}\\ 2(v+\varpi)\Delta h'^1h'^2 & -\frac{2}{v+\varpi}\end{vmatrix}}{\begin{vmatrix}\frac{2h''^1}{h'^1} & -2(v-\varpi)\Delta h'^1h'^2\\ 2(v+\varpi)\Delta h'^1h'^2 & \frac{2h''^2}{h'^2}\end{vmatrix}} \qquad \text{(III.7)}$$

Notice that the denominator in both the above expressions is the same and it is positive. Hence the sign of both (III.6) and (III.7) is determined by the numerator term in the respective expressions. By inspection, it is apparent that the numerator in (III.6) is negative so that $\frac{dR_1^*}{d\varpi}<0$. The numerator in (III.7) is given by $-\frac{4h''^1}{(v+\varpi)h'^1}-\frac{4(v+\varpi)\Delta h'^1h'^2}{v-\varpi}$. It follows from monotonicity and strict concavity of $h(.)$ that while the first term is positive, the second term is negative. Hence the numerator cannot be signed unambiguously and therefore the sign of $\frac{dR_2^*}{d\varpi}$ is indeterminate. Accordingly, the sign of $\frac{d(R_1^*+R_2^*)}{d\varpi}$ is also indeterminate.

We now examine the impact of a change in ϖ on aggregate welfare. Since, $U=v-\left[(\alpha+\Delta)h^1-(\alpha-\Delta)h^2-2\Delta h^1h^2\right]\varpi-R_1-R_2$ and $U^*=U(R_1^*,R_2^*)$, it follows that:

$$\frac{dU^*}{d\varpi}=-((\alpha+\Delta)h^{*1}-(\alpha-\Delta)h^{*2}-2\Delta h^{*1}h^{*2})-\left[\frac{v+\varpi}{v-\varpi}\right]\frac{dR_1^*}{d\varpi}-\left[\frac{v-\varpi}{v+\varpi}\right]\frac{dR_2^*}{d\alpha} \qquad \text{(III.8)}$$

Firstly notice that the second component in the above expression is always positive since $\frac{v+\varpi}{v-\varpi}>0$ and $\frac{dR_1^*}{d\varpi}<0$. Further, $(\alpha+\Delta)h^{*1}+(\alpha-\Delta)h^{*2}-2\Delta h^{*1}h^{*2} = p_1^* - p_2^*$. Hence when Player 1 is the favorite in equilibrium ($p_1^* > p_2^*$), an increase in ϖ tends to decrease aggregate welfare via the first component in (III.8). However the second component works towards reversing this effect while the last component cannot be signed. When $p_1^* < p_2^*$, the first two components in (III.8) tend to increase aggregate welfare but the sign of the last term cannot be determined a priori. Hence the impact of an increase in ϖ on aggregate welfare cannot be determined unequivocally in either of these scenarios and therefore is generally ambiguous.

IV. Corner equilibria cannot exist when $h(.)$ is given by (8) and p_i is given by either (1) or (17)

Recall that for p_i given by either (1) or (17), $R_i = 0$ is strictly preferred to $R_i \geq v_i$ for any R_j $i, j = 1, 2$, $i \neq j$. Hence the only feasible corner solution in the strategy space $R_i \in [0, \max\{v_1, v_2\}]$ is one where $R_i = 0$.

Next we demonstrate that when $h(.)$ is given by (8) and p_i is given by (1) or (17), $R_i^* = 0$ cannot be the solution to maximization of $U^i(R_1, R_2) = p_i(R_1, R_2)v_i - R_i$ with respect to R_i for $i = 1, 2$.

Suppose that p_i is given by (1). In this case, the expected payoffs to the players are given by,

$$U^i(R_i, R_j) = \left\{\frac{1}{2} + \frac{\alpha}{2}\left[\sqrt{\frac{R_i}{K}} - \sqrt{\frac{R_j}{K}}\right]\right\}v_i - R_i \text{ for } i, j = 1, 2 \text{ and } i \neq j \quad \text{(IV.1)}$$

Using (IV.1), we obtain the net marginal benefit from R_i for each player $i = 1, 2$, as shown below.

$$\frac{\partial U^i}{\partial R_i} = \frac{v_i}{4\sqrt{R_i K}} - 1 \quad \text{(IV.2)}$$

Notice that the above net marginal benefit approaches ∞ as R_i tends to 0. Hence each player is always induced to increase her expenditure beyond 0. From this it follows that $R_i^* \neq 0$ when persuasion function is given by (1).

Suppose now that p_i is given by the persuasion function (17) and assumptions 2 and 6 also hold. In this case, using (6), the expected payoffs to the players are given by:

$$U^1(R_1, R_2) = \left\{\tfrac{1}{2} + \tfrac{1}{2}\left[(\alpha+\Delta)\sqrt{\frac{R_1}{K}} - (\alpha-\Delta)\sqrt{\frac{R_2}{K}} - \frac{2\Delta}{K}\sqrt{R_1 R_2}\right]\right\}(v+\omega) - R_1 \quad \text{(IV.3)}$$

$$U^2(R_1,R_2)=\left\{\tfrac{1}{2}+\tfrac{1}{2}\left[(\alpha-\Delta)\sqrt{\frac{R_2}{K}}-(\alpha+\Delta)\sqrt{\frac{R_1}{K}}+\frac{2\Delta}{K}\sqrt{R_1R_2}\right]\right\}(v-\omega)-R_2 \tag{IV.4}$$

Using (IV.3) and (IV.4), we examine the net marginal benefit from R_i $i=1,2$ to each player, beginning with the Player 1 as given by (IV.5).

$$\frac{\partial U^1}{\partial R_1}=\frac{1}{4\sqrt{R_1K}}\left\{(\alpha+\Delta)-2\Delta\sqrt{\frac{R_2}{K}}\right\}(v+\omega)-1 \tag{IV.5}$$

For any given R_1, the lowest value (IV.5) can take is,

$$\frac{\partial U^1}{\partial R_1}=\frac{1}{4\sqrt{R_1K}}(\alpha-\Delta)(v+\omega)-1 \tag{IV.6}$$

Notice that the above net marginal benefit approaches ∞ as R_1 tends to 0. Hence for any given R_2, Player 1 is induced to increase her expenditure beyond 0. Hence $R_1^*=0$ cannot be an equilibrium.

The net marginal benefit of Player 2 from R_2 is given by:

$$\frac{\partial U^2}{\partial R_2}=\frac{1}{4\sqrt{R_2K}}\left\{(\alpha-\Delta)+2\Delta\sqrt{\frac{R_1}{K}}\right\}(v-\omega)-1 \tag{IV.7}$$

The lowest value (IV.7) assumes for any given R_2 is:

$$\frac{\partial U^2}{\partial R_2}=\frac{1}{4\sqrt{R_2K}}(\alpha-\Delta)(v-\omega)-1 \tag{IV.8}$$

Notice that the net marginal benefit as given by (IV.8) approaches ∞ as R_2 tends to 0. Hence for any given R_1, Player 2 is always induced to increase her expenditure beyond 0. Hence $R_2^*=0$ cannot be an equilibrium.

Hence it is clear from this section that when $h(.)$ is given by (8), corner equilibria can be ruled out when p_i is given by either (1) or (17).