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Risk aversion, risk premia, and the labor margin with generalized recursive preferences

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ABSTRACT

A flexible labor margin allows households to absorb shocks to asset values with changes in hours worked as well as changes in consumption. This ability to absorb shocks along both margins alters the household's attitudes toward risk, as shown by Swanson (2012). In the present paper, I extend that analysis to the case of generalized recursive preferences, as in Epstein and Zin (1989) and Weil (1989), including multiplier preferences, as in Hansen and Sargent (2001). Understanding risk aversion for these preferences is important because they are a primary mechanism being used to bring macroeconomic models into closer agreement with asset prices. Traditional, fixed-labor measures of risk aversion show no stable relationship to the equity premium in a standard macroeconomic model, while the closed-form expressions I derive here match the equity premium closely.

1. Introduction

A growing macro-finance literature focuses on bringing standard macroeconomic models into better agreement with basic asset pricing facts, such as the equity premium.[2] In asset pricing models, a crucial parameter is risk aversion, the compensation that households require to hold a risky payoff. At the same time, a key feature of standard macroeconomic models is that households have some ability to vary their labor supply. A fundamental difficulty with this line of research,

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[2] For example, Boldrin et al. (2001), Tallarini (2000), Rudebusch and Swanson (2008, 2012), Uhlig (2007), van Binsbergen et al. (2012), Backus et al. (2008), Gourio (2012, 2013), Palomino (2012), Andreasen (2012a, 2012b), Colacito and Croce (2013), Dew-Becker (2014), Kung (2012), and Swanson (2017) all consider asset pricing in dynamic macroeconomic models with a variable labor margin.

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then, is that much of what is known about risk aversion has been derived under the assumption that household labor is exogenously fixed.[3]

Swanson (2012) addresses this problem when households have standard expected utility preferences. In the present paper, I extend that analysis to generalized recursive preferences, as in Epstein and Zin (1989) and Weil (1989), including multiplier preferences, as in Hansen and Sargent (2001) and Strzalecki (2011). These preferences are a primary mechanism being used to bring macroeconomic models into better agreement with asset prices, so understanding risk aversion in this framework is very important for the macro-finance field.[4] In fact, there is no conventional wisdom as to what the formula for risk aversion should be for these preferences when labor supply can vary, with different authors using different ad hoc generalizations of the traditional, fixed-labor measure. In the present paper, I undertake a systematic and rigorous analysis of this important question.

Intuitively, a flexible labor margin allows households to absorb shocks to asset values with changes in hours worked as well as changes in consumption, which can greatly alter the household's attitudes toward risk. For example, with expected utility and period utility function $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t$, the quantity $-c\,u_{11}/u_1 = \gamma$ is often referred to as the household's coefficient of relative risk aversion, but in fact the household is *risk neutral* with respect to gambles over asset values or wealth (Swanson, 2012). Intuitively, the household is indifferent at the margin between using labor or consumption to absorb a shock to asset values, and the household in this example is clearly risk neutral with respect to gambles over hours.[5] In the present paper, I rigorously derive closed-form expressions for wealth-gamble risk aversion in dynamic equilibrium models with generalized recursive preferences and arbitrary period utility function u, taking into account the effects of the household's flexible labor margin, and I show that those effects can be significant.

I also show, theoretically and numerically, that risk premia on assets in a macroeconomic model are unrelated to traditional, fixed-labor measures of risk aversion unless labor is, in fact, fixed. By contrast, the closed-form expressions for wealth-gamble risk aversion I derive here match risk premia in a standard (flexible-labor) real business cycle model closely. Thus, taking the household's labor margin into account is necessary for there to be a stable relationship between wealth-gamble risk aversion and risk premia in the model.

1.1. Related literature

Arrow (1965) and Pratt (1964) define absolute and relative risk aversion, $-u''(c)/u'(c)$ and $-c\,u''(c)/u'(c)$, in terms of the household's aversion to a fair gamble over a single good in a static, one-period model. When there are multiple goods, the literature has taken two different approaches to generalizing their definition. The first approach, which I follow in this paper, is to consider the household's aversion to a fair gamble over *money* or *wealth*.[6] This approach is taken by Stiglitz (1969) in a static, multiple-good setting, and by Constantinides (1990), Campbell and Cochrane (1999), and others in a dynamic, single-good setting (in which the multiple goods are a single good consumed at different points in time). As shown in Swanson (2012) and in Section 3, below, this approach implies that the household's coefficient of "wealth-gamble risk aversion" is related to the curvature of the household's *indirect* utility function (in the static case) or *value* function (in the dynamic case).

The alternative approach, taken by Kihlstrom and Mirman (1974, 1981) and Nocetti and Smith (2011a, 2011b), defines a household's attitudes toward risk solely as a function of its preferences over consumption and its total consumption bundle. Thus, other features of the economic environment that affect the household's value function but *not* its utility function have no effect on the household's attitudes toward risk in this approach. When there are multiple goods, the household's Kihlstrom–Mirman attitudes toward risk are complicated and multi-dimensional, and cannot be summarized by a single "coefficient of risk aversion" except in very special cases. For example, in Kihlstrom and Mirman (1974), one agent cannot be said to be more or less risk averse than another unless both agents have identical ordinal preferences over all nonstochastic consumption bundles. Similarly, in Kihlstrom and Mirman (1981), increasing, constant, and decreasing relative risk aversion are only defined if the agent has preferences that are homothetic.

For clarity, I will use the phrase "wealth-gamble risk aversion" in this paper to emphasize that I am using the former definition of risk aversion and not the latter. The reason for focusing on wealth gambles is asset pricing: a household's aversion to a wealth gamble is always well-defined under the technical assumptions I make below (e.g., twice-differentiability) and is highly relevant for understanding the household's aversion to holding a risky asset with monetary payoffs, such as

[3] For example, Arrow (1965) and Pratt (1964) define absolute and relative risk aversion, $-u''(c)/u'(c)$ and $-c\,u''(c)/u'(c)$, in a static model with a single consumption good (and no labor). Similarly, Epstein and Zin (1989) and Weil (1989) define risk aversion for generalized recursive preferences in a dynamic model without labor (or, equivalently, in which labor is fixed).

[4] The vast majority of studies cited in footnote 2 take this approach, the exceptions being Boldrin et al. (2001), Rudebusch and Swanson (2008), and Palomino (2012). One of the main advantages of generalized recursive preferences is that they allow risk aversion to be modeled independently from the household's other preference parameters, such as the intertemporal elasticity of substitution.

[5] More generally, when $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$, the household's wealth-gamble risk aversion equals $(\gamma^{-1} + \chi^{-1})^{-1}$, a combination of the parameters on the household's consumption and labor margins, reflecting the fact that the household absorbs shocks along both margins.

[6] Both Arrow (1965) and Pratt (1964) explicitly worked with a "utility function for money" (Pratt, 1964, p. 122) or wealth: "Let $Y =$ wealth, $U(Y) =$ total utility of wealth Y" (Arrow, 1965, pp. 91–92), and defined risk aversion in terms of a fair gamble over money or wealth.

a stock or bond.[7] The Kihlstrom–Mirman (1974, 1981) approach is multi-dimensional in general and provides little insight into how much compensation a household would require to hold a risky asset with monetary payoffs, except in very special cases such as homothetic preferences.[8]

In addition to Swanson (2012), previous studies that consider how the household's labor margin affects wealth-gamble risk aversion include Boldrin et al. (1997) and Uhlig (2007). Boldrin et al. consider some very simple endowment economies for which they can compute closed-form expressions for the value function, and hence wealth-gamble risk aversion. Uhlig (2007) points out that when households have generalized recursive preferences, leisure affects asset prices because the value function V appears in the household's stochastic discount factor, and V depends on leisure. My analysis here differs from theirs by deriving closed-form solutions for wealth-gamble risk aversion in dynamic equilibrium models in general, demonstrating the importance of the labor margin, and showing the tight link between wealth-gamble risk aversion and asset prices in these models.

There is also a large and growing body of empirical evidence that households vary their labor supply in response to financial shocks—i.e., that the wealth effect on labor supply is negative. Imbens et al. (2001) and Cesarini et al. (2017) find that households who win a prize in the lottery reduce their labor supply significantly, even for moderately-sized prizes. Cesarini et al. (2017) find that this adjustment takes place on both the intensive and extensive margins, and for spouses as well as lottery winners themselves. Coile and Levine (2009) document that older workers are less likely to retire after the stock market performs poorly, and Coronado and Perozek (2003) find that households retire earlier when the stock market performs well. Pencavel (1986) and Killingsworth and Heckman (1986) survey estimates of the wealth effect on labor supply and find it to be significantly negative.

The remainder of the paper proceeds as follows. Section 2 defines the dynamic equilibrium framework used in the analysis. Section 3 derives closed-form expressions for wealth-gamble risk aversion within this general framework. Section 4 demonstrates the close connection between wealth-gamble risk aversion and Lucas–Breeden asset prices in the model, both theoretically and with numerical examples. Section 5 verifies the accuracy of the closed-form expressions for wealth-gamble risk aversion using numerical methods. Section 6 extends the results to the case of balanced growth. Section 7 provides the corresponding expressions and additional discussion for the case of multiplier preferences. Section 8 discusses general implications and concludes. An Appendix provides details of proofs and numerical solutions that are outlined in the main text.

2. Dynamic equilibrium framework

2.1. Generalized recursive preferences and value function

Time is discrete and continues forever. At each time t, the household receives a utility flow $u(c_t, l_t)$, where $(c_t, l_t) \in \Omega \subseteq \mathbb{R}^2$ denotes the household's choice of consumption and hours worked in period t. I assume the period utility function u satisfies the following regularity conditions:

Assumption 1. The function $u : \Omega \to \mathbb{R}$ is increasing in its first argument, decreasing in its second, twice-differentiable, and strictly concave.

The household faces a flow budget constraint in each period,

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t, \tag{1}$$

and a no-Ponzi-scheme condition,

$$\lim_{T \to \infty} \prod_{\tau=t}^{T} (1 + r_{\tau+1})^{-1} a_{T+1} \geq 0, \tag{2}$$

where a_t denotes beginning-of-period assets and w_t, r_t, and d_t denote the real wage, real interest rate, and net transfer payments to the household, respectively. There is a finite-dimensional Markov state vector θ_t that is exogenous to the household and governs the processes for w_t, r_t, and d_t. Before choosing (c_t, l_t) in each period t, the household observes θ_t and hence w_t, r_t, and d_t. The state vector and information set of the household's optimization problem at each date t is thus $(a_t; \theta_t)$. Let X denote the domain of $(a_t; \theta_t)$, and $\Gamma : X \to \Omega$ describe the set-valued correspondence of feasible choices for (c_t, l_t) for each given $(a_t; \theta_t)$.

[7] The set of assets with monetary payoffs includes essentially all traded assets in the data, even "real" or "indexed" assets, since those assets' payoffs are transacted in money. Even in the case of a real asset such as an oil well, whose payoffs are in barrels of crude, those payoffs are typically converted into money at a spot rate rather than consumed directly.

[8] Note, however, that when the household has homothetic preferences and faces no constraints other than its budget constraint, the Kihlstrom–Mirman coefficient of risk aversion agrees with the coefficient of wealth-gamble risk aversion that I use in this paper—see Kihlstrom and Mirman (1974, Section 3.1).

Let $(c^t, l^t) \equiv \{(c_\tau, l_\tau)\}_{\tau=t}^{\infty}$ denote a state-contingent plan for household consumption and labor from time t onward, where the explicit state-dependence of the plan is suppressed to reduce notation. Following Epstein and Zin (1989) and Weil (1989), the household has preferences over state-contingent plans ordered by the recursive functional

$$\widetilde{V}(c^t, l^t) = u(c_t, l_t) + \beta \left[E_t \widetilde{V}(c^{t+1}, l^{t+1})^{1-\alpha} \right]^{1/(1-\alpha)}, \tag{3}$$

where $\beta \in (0, 1)$, $\alpha \in \mathbb{R}$, E_t denotes the mathematical expectation conditional on the household's information set at time t, and (c^{t+1}, l^{t+1}) denotes the state-contingent plan (c^t, l^t) from date $t+1$ forward.[9] Note that equation (3) has the same form as expected utility preferences, but with the expectation operator "twisted" and "untwisted" by the coefficient $1-\alpha$. When $\alpha = 0$, (3) reduces to the special case of expected utility. When $\alpha \neq 0$, the intertemporal elasticity of substitution over deterministic consumption paths in (3) is the same as for expected utility, but the household's risk aversion with respect to gambles over future utility flows is amplified (or attenuated) by the additional curvature parameter α.

The household's "generalized value function" $V : X \to \mathbb{R}$ is defined to be the maximized value of (3), subject to the budget constraint (1)–(2). V satisfies the recursive equation

$$V(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} u(c_t, l_t) + \beta \left(E_t V(a_{t+1}; \theta_{t+1})^{1-\alpha} \right)^{1/(1-\alpha)}, \tag{4}$$

where a_{t+1} is given by (1). I discuss technical conditions for the existence and uniqueness of V shortly.

Note that many authors use an alternative notation for the generalized value function,

$$U(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} \left[\widetilde{u}(c_t, l_t)^{\rho} + \beta \left(E_t U(a_{t+1}; \theta_{t+1})^{\widetilde{\alpha}} \right)^{\rho/\widetilde{\alpha}} \right]^{1/\rho}, \tag{5}$$

where $\rho \in \mathbb{R}$, $\rho < 1$. This specification follows Epstein and Zin's (1989) original notation more closely, where those authors take $\widetilde{u}(c_t, l_t) = c_t$ in their framework without labor. However, setting $V = U^{\rho}$, $u = \widetilde{u}^{\rho}$, and $\alpha = 1 - \widetilde{\alpha}/\rho$, this can be seen to correspond exactly to (4).[10] Moreover, (4) has a much clearer relationship than (5) to standard dynamic programming results, regularity conditions, and first-order conditions: for example, (4) requires concavity of u while (5) requires concavity of $\widetilde{u}^{\rho}$, and the Benveniste–Scheinkman equation for (4) is the usual $V_1 = (1+r_t)u_1$ rather than $U_1 = (1+r_t)U^{(1-\rho)/\rho}\widetilde{u}^{\rho-1}\widetilde{u}_1$. That is, the marginal value of wealth in (4) is just the usual marginal utility of consumption rather than something much more complicated.

I require a few technical conditions to ensure that (3)–(4) are well-defined. First, to avoid complex numbers:

Assumption 2. Either $u : \Omega \to [0, \infty)$ or $u : \Omega \to (-\infty, 0]$.

In the latter case, it is natural to take $\widetilde{V} \leq 0$, $V \leq 0$, and interpret (3) as

$$\widetilde{V}(c^t, l^t) = u(c_t, l_t) - \beta \left[E_t(-\widetilde{V}(c^{t+1}, l^{t+1}))^{1-\alpha} \right]^{1/(1-\alpha)}, \tag{3$'$}$$

and similarly for (4). Note that (5) requires this same restriction, for the same reasons.[11]

Technical conditions that ensure the existence and uniqueness of V are tangential to the main points of the present paper, so for simplicity I assume[12]:

Assumption 3. A solution $V : X \to \mathbb{R}$ to the household's generalized Bellman equation (4) exists and is unique, continuous, and concave.

The same technical conditions, plus Assumption 1, guarantee the existence of a unique optimal choice for (c_t, l_t) at each point in time, given $(a_t; \theta_t)$. Let $c_t^* \equiv c^*(a_t; \theta_t)$ and $l_t^* \equiv l^*(a_t; \theta_t)$ denote the household's optimal choices of c_t and l_t as functions of the state $(a_t; \theta_t)$. To avoid boundary solutions for c_t^* and l_t^* that would make derivatives and first-order conditions problematic, I require these solutions to be interior:

9 The case $\alpha = 1$ is understood to correspond to $\widetilde{V}(c^t, l^t) = u(c_t, l_t) + \beta \exp\left[E_t \log \widetilde{V}(c^{t+1}, l^{t+1})\right]$.

10 For the case $\rho < 0$, set $V = -U^{\rho}$ and $u = -\widetilde{u}^{\rho}$. The case $\rho = 0$ corresponds to multiplier preferences, which I consider in Section 7, below.

11 The assumption that either $u \geq 0$ or $u \leq 0$ is not very restrictive in practice. For example, restrictions can be placed on Ω or Γ and a constant added to u such that u never takes on negative (or positive) values. Alternatively, for local analysis around a steady state, the restriction is satisfied so long as $u \neq 0$ in steady state, since then $u \geq 0$ or $u \leq 0$ holds locally. Note that Assumption 2 is not required for multiplier preferences; see Section 7.

12 Stokey and Lucas (1989), Alvarez and Stokey (1998), and Rincón-Zapatero and Rodríguez-Palmero (2003) provide different sets of sufficient conditions that ensure Assumption 3 is satisfied for the case $\alpha = 0$. Sufficient conditions for general α and general period utility functions $u(c_t, l_t)$ have not yet been derived in the literature, but there are proofs for important special cases. For example, Epstein and Zin (1989) prove the existence and uniqueness of V for general α when there is a single consumption good and no labor, which also applies here if consumption and leisure form an aggregate good, such as when $u(c_t, l_t)$ is Cobb–Douglas or CES. (In this case, we must include the household's "full labor income" in its budget constraint, but Epstein and Zin (1991) show how this can be handled within their framework.) Similarly, the results of Marinacci and Montrucchio (2010) for the case of a single consumption good also apply here when consumption and leisure form an aggregate good.

Assumption 4. For any $(a_t;\theta_t) \in X$, the household's optimal choice (c_t^*, l_t^*) exists, is unique, and lies in the interior of $\Gamma(a_t;\theta_t)$.

Intuitively, Assumption 4 requires the partial derivatives of u to grow sufficiently large toward the boundary that only interior solutions for c_t^* and l_t^* are optimal for any $(a_t;\theta_t) \in X$.[13]

It follows that V can be written as

$$V(a_t;\theta_t) = u(c_t^*, l_t^*) + \beta \left(E_t V(a_{t+1}^*;\theta_{t+1})^{1-\alpha}\right)^{1/(1-\alpha)}, \tag{6}$$

where $a_{t+1}^* \equiv (1+r_t)a_t + w_t l_t^* + d_t - c_t^*$.

Assumptions 1–4 guarantee that V is continuously differentiable with respect to a and satisfies the Benveniste–Scheinkman equation, but I will require slightly more than this below:

Assumption 5. For any $(a_t;\theta_t)$ in the interior of X, the second derivative of V with respect to its first argument, $V_{11}(a_t;\theta_t)$, exists.

Assumption 5 also implies differentiability of the optimal policy functions, c^* and l^*, with respect to a_t. Santos (1991) provides relatively mild sufficient conditions for this assumption to be satisfied when $\alpha = 0$; intuitively, u must be strongly concave.

2.2. Representative household and steady state assumptions

Up to this point, the analysis has focused on a single household in isolation, leaving the other households of the model and the production side of the economy unspecified. Implicitly, the other households and production sector jointly determine the process for θ_t (and hence w_t, r_t, and d_t) via general equilibrium, but the assumptions above and much of the analysis below does not depend on those details and can simply treat θ_t as an exogenous Markov process from the point of view of a single household. However, in order to move from more complicated, general expressions for wealth-gamble risk aversion to more concrete, closed-form expressions, I adopt three standard assumptions from the macroeconomics literature:[14]

Assumption 6. The household is *infinitesimal*.

Assumption 7. The household is *representative*.

Assumption 8. The model has a *nonstochastic steady state*, at which $x_t = x_{t+k}$ for all $k = 1, 2, \ldots$, and $x \in \{c, l, a, w, r, d, \theta\}$.

Assumption 6 implies that an individual household's choices for c_t and l_t have no effect on the aggregate quantities w_t, r_t, d_t, and θ_t, determined in general equilibrium. Thus, Assumption 6 explicitly states an assumption that was already implicit in the framework above, in the statement that θ_t (and w_t, r_t and d_t) is exogenous to the household.

Assumption 7 is a restriction on the general equilibrium of the economy: that aggregate consumption demand and labor supply are determined by a household sector that has the same preferences as the given household. Assumption 7 implies that, when the economy is at the nonstochastic steady state, the given household will find it optimal to choose the steady-state values of c and l, given a and θ. (Throughout the text, a variable without a time subscript denotes its steady-state value).[15]

Finally, Assumption 8 is another restriction on the general equilibrium of the economy, in particular on how that equilibrium behaves when there are no aggregate shocks and no aggregate uncertainty.

It's important to note that Assumptions 7–8 do not prohibit offering an individual household a hypothetical gamble of the type described below. The steady state of the model serves only as a reference point around which the *aggregate* variables w, r, d, and θ and the *other households'* choices of c, l, and a can be predicted with certainty. This reference point is important because it is there that closed-form expressions for wealth-gamble risk aversion can be computed.

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[13] Interiority is a standard requirement for differentiability of the household's optimal policy functions—see, e.g., Santos (1991). There are two main ways this assumption can be imposed: through Inada-type conditions on the period utility function u, or through restrictions on the domain X of the state variables a_t and θ_t. For example, if the household's period utility function is Cobb–Douglas over consumption and leisure, $u(c_t, l_t) = c_t^{\chi}(1-l_t)^{1-\chi}$, then the household's optimal choices of c_t and $1 - l_t$ will be strictly positive by the Inada-type conditions. To ensure $l_t > 0$ as well, the domain X must exclude cases where the household is so wealthy or wages are so low that the household would want to choose negative labor supply.

[14] Alternative assumptions about the nature of the other households in the model or the production sector may also allow for closed-form expressions for wealth-gamble risk aversion. However, the assumptions used here are the most standard.

[15] Let the exogenous state θ_t contain the variances of any shocks to the model, so that $(a;\theta)$ denotes the nonstochastic steady state, with the variances of any shocks (other than the hypothetical gamble described in the next section) set equal to zero; $c(a;\theta)$ corresponds to the household's optimal consumption choice at the nonstochastic steady state, etc.

Finally, many dynamic models do not have a steady state *per se*, but rather a balanced growth path, as in King et al. (1988). The results below carry through essentially unchanged to the case of balanced growth. For ease of exposition, Sections 3–5 restrict attention to the case of a steady state, while Section 6 shows the adjustments required under the more general:

Assumption 8′. *The model has a* balanced growth path *that can be renormalized to a nonstochastic steady state after a suitable change of variables.*

3. Wealth-gamble risk aversion

3.1. The coefficient of absolute wealth-gamble risk aversion

The household's attitudes toward a risky wealth gamble at time t generally depend on the household's state vector at time t, $(a_t;\theta_t)$. Given this state, I consider the household's aversion to a hypothetical one-shot gamble in period t of the form

$$a_{t+1} = (1+r_t)a_t + w_t l_t + d_t - c_t + \sigma\varepsilon_{t+1}, \tag{7}$$

where ε_{t+1} is a random variable representing the gamble, with bounded support $[\underline{\varepsilon}, \bar{\varepsilon}]$, mean zero, unit variance, independent of θ_τ for all times τ, and independent of a_τ, c_τ, and l_τ for all $\tau \le t$. A few words about (7) are in order: First, the gamble is dated $t+1$ to clarify that its outcome is not in the household's information set at time t. Second, c_t cannot be made the subject of the gamble without substantial modifications to the household's optimization problem, because c_t is a choice variable under control of the household at time t. However, (7) is clearly equivalent to a one-shot gamble over net transfers d_t or asset returns r_t, both of which are exogenous to the household.[16] Indeed, thinking of the gamble as being over r_t helps to illuminate the connection between (7) and the price of risky assets, which I will discuss further in Section 4. As shown there, the household's aversion to the gamble in (7) is directly linked to the premium households require to hold risky assets.

Following Arrow (1965) and Pratt (1964), I ask what one-time fee μ the household would be willing to pay in period t to avoid the gamble in (7):

$$a_{t+1} = (1+r_t)a_t + w_t l_t + d_t - c_t - \mu. \tag{8}$$

Again following Arrow and Pratt, I define the quantity μ that makes the household just indifferent between (7) and (8), for infinitesimal σ and μ, to be the household's coefficient of absolute wealth-gamble risk aversion. Formally, this corresponds to the following:

Definition 1. Let $(a_t;\theta_t)$ be an interior point of X. Let $\hat{V}(a_t;\theta_t;\sigma)$ denote the value function for the household's optimization problem inclusive of the one-shot gamble (7), and let $\mu(a_t;\theta_t;\sigma)$ denote the value of μ that satisfies $V(a_t - \frac{\mu}{1+r_t};\theta_t) = \hat{V}(a_t;\theta_t;\sigma)$. The household's *coefficient of absolute wealth-gamble risk aversion at* $(a_t;\theta_t)$, denoted $R^a(a_t;\theta_t)$, is given by $R^a(a_t;\theta_t) = \lim_{\sigma\to 0}\mu(a_t;\theta_t;\sigma)/(\sigma^2/2)$.

In Definition 1, $\mu(a_t;\theta_t;\sigma)$ denotes the household's "willingness to pay" to avoid a one-shot gamble of size σ in (7). As in Arrow (1965) and Pratt (1964), R^a denotes the limit of the household's willingness to pay per unit of variance as this variance becomes small. Note that $R^a(a_t;\theta_t)$ depends on the economic state because $\mu(a_t;\theta_t;\sigma)$ depends on that state. Proposition 1 shows that $\hat{V}(a_t;\theta_t;\sigma)$, $\mu(a_t;\theta_t;\sigma)$, and $R^a(a_t;\theta_t)$ in Definition 1 are well-defined and derives the expression for $R^a(a_t;\theta_t)$:

Proposition 1. *Let $(a_t;\theta_t)$ be an interior point of X. Given Assumptions 1–6, $\hat{V}(a_t;\theta_t;\sigma)$, $\mu(a_t;\theta_t;\sigma)$, and $R^a(a_t;\theta_t)$ exist and*

$$R^a(a_t;\theta_t) = \frac{-E_t\left[V(a^*_{t+1};\theta_{t+1})^{-\alpha}V_{11}(a^*_{t+1};\theta_{t+1}) - \alpha\, V(a^*_{t+1};\theta_{t+1})^{-\alpha-1}V_1(a^*_{t+1};\theta_{t+1})^2\right]}{E_t\, V(a^*_{t+1};\theta_{t+1})^{-\alpha}V_1(a^*_{t+1};\theta_{t+1})}, \tag{9}$$

where V_1 and V_{11} denote the first and second partial derivatives of V with respect to its first argument. Given Assumptions 7–8, equation (9) can be evaluated at the nonstochastic steady state to yield:

$$R^a(a;\theta) = \frac{-V_{11}(a;\theta)}{V_1(a;\theta)} + \alpha\,\frac{V_1(a;\theta)}{V(a;\theta)}. \tag{10}$$

[16] In this case, the realized transfer $d_t + \sigma\varepsilon_{t+1}$, or asset return $r_t + \sigma\varepsilon_{t+1}$, would not be in the household's time-t information set, $(a_t;\theta_t)$.

Proof. See Appendix.[17] □

Equation (9) is valid quite generally and requires only the technical Assumptions 1–5 on preferences and that θ_t is exogenous to the household. However, (9) is also quite complicated and can only be solved numerically in general; equation (10) is much simpler (and can be simplified further, below), which shows the benefits of assuming a representative agent nonstochastic steady state to use as a reference point.

Equation (10) can be decomposed into the sum of two components: the first term on the right-hand side is essentially intratemporal and does not depend on the parameter α, while the second term captures the household's *additional* aversion to risky utility flows in the future and is closely related to α. For $u, V \geq 0$, larger values of α imply higher levels of wealth-gamble risk aversion.[18] For $u, V \leq 0$, the opposite is true: *lower* values of α (especially negative values) imply higher levels of wealth-gamble risk aversion.

Definition 1 and Proposition 1 imply that wealth-gamble risk aversion is related to the curvature of the value function V with respect to wealth rather than the curvature of period utility u with respect to consumption. Because the value function depends on the economic state θ_t (and, implicitly, on the stochastic process governing θ_t), wealth-gamble risk aversion also depends on θ_t (and its stochastic process). This state-dependence is standard in the literature on wealth-gamble risk aversion: for example, Stiglitz (1969) defines risk aversion in terms of a fair gamble over money or wealth, with absolute and relative wealth-gamble risk aversion of $-U_{yy}/U_y$ and $-yU_{yy}/U_y$, where the indirect utility function U "is a function of both [income] y and [prices] p: $U = U(y, p)$. Arrow and Pratt assumed in effect that p is constant" (Stiglitz, 1969, footnote 11). In Stiglitz (1969), risk aversion clearly depends on the economic state—the price vector p—as well as the agent's preferences and total consumption bundle. Constantinides (1990) and Campbell and Cochrane (1999) work in a dynamic framework, where the household's aversion to money or wealth gambles implies that it is the *value* function that is relevant, as in the present paper: "I define the RRA coefficient in terms of an atemporal gamble that changes the current level of capital by the outcome of the gamble... $RRA = -WV_{ww}/V_w$ ... The RRA coefficient is a function of wealth and of the state variables $x(t)$," (Constantinides, 1990, p. 527); and "Risk aversion measures attitudes toward pure wealth bets and is therefore conventionally captured by the second partial derivative of the value function with respect to individual wealth..." (Campbell and Cochrane, 1999, p. 243), and risk aversion "depends on individual wealth W and on aggregate variables that describe asset prices or investment opportunities..." (Campbell and Cochrane, 1999, p. 244).[19] Note that this state-dependence is not a feature of the Kihlstrom–Mirman (1974, 1981) approach to risk aversion mentioned in the Introduction; that approach defines the household's attitudes toward risk in a multi-dimensional way that is independent of the economic state (and its stochastic process) and depends only on the household's preferences and total consumption bundle.

Definition 1, above, is stated more formally than in Stiglitz (1969), Constantinides (1990), and Campbell and Cochrane (1999), but the intuition is the same. The formalization in Definition 1 and Proposition 1 is important because there is no conventional wisdom for what the correct wealth-gamble risk aversion formula is in the case of generalized recursive preferences with multiple goods.

A practical difficulty with Definition 1 and Proposition 1 is that closed-form solutions for V do not exist in general, even for the simplest dynamic models with labor. I solve this problem by observing that V_1 and V_{11} often can be computed even when closed-form solutions for V cannot be. For example, the Benveniste–Scheinkman equation,

$$V_1(a_t; \theta_t) = (1 + r_t)\, u_1(c_t^*, l_t^*), \tag{11}$$

states that the marginal value of a dollar of assets equals the marginal utility of consumption times $1 + r_t$ (the interest rate appears here because beginning-of-period assets in the model generate income in period t).[20] In (11), u_1 is a known function. Although closed-form solutions for the functions c^* and l^* are not known in general, the points c_t^* and l_t^* often

[17] When generalized recursive preferences are written in the form (5), the corresponding expressions are

$$R^a(a_t; \theta_t) = \frac{-E_t\left[U(a_{t+1}^*; \theta_{t+1})^{\widetilde{\alpha}-1} U_{11}(a_{t+1}^*; \theta_{t+1}) + (\widetilde{\alpha} - 1) U(a_{t+1}^*; \theta_{t+1})^{\widetilde{\alpha}-2} U_1(a_{t+1}^*; \theta_{t+1})^2\right]}{E_t\, U(a_{t+1}^*; \theta_{t+1})^{\widetilde{\alpha}-1} U_1(a_{t+1}^*; \theta_{t+1})}$$

and

$$R^a(a; \theta) = \frac{-U_{11}(a; \theta)}{U_1(a; \theta)} + (1 - \widetilde{\alpha}) \frac{U_1(a; \theta)}{U(a; \theta)}\,.$$

[18] Sufficiently low or negative values of α imply risk-loving behavior, $R^a(a; \theta) < 0$. The parameter α also determines the household's preference for early vs. late resolution of uncertainty, as discussed in Kreps and Porteus (1978) and Epstein and Zin (1989). For $u, V \geq 0$, the household prefers early resolution of uncertainty if $\alpha > 0$, and late resolution if $\alpha < 0$; for $u, V \leq 0$, the household prefers early resolution if $\alpha < 0$, and late resolution if $\alpha > 0$. These conditions correspond to the criterion $\widetilde{\alpha} < \rho$ in (5), emphasized by Epstein and Zin (1989).

[19] See also Farmer (1990) and Flavin and Nakagawa (2008), who use essentially the same definition and come to the same conclusions in their dynamic frameworks.

[20] The Benveniste–Scheinkman equation (11) holds for generalized recursive preferences as well as expected utility. See the proof of Proposition 1 in the Appendix.

are known—for example, when they are evaluated at the nonstochastic steady state reference point, c and l. Thus, we can compute V_1 at the nonstochastic steady state by evaluating (11) at that point. Similarly, we know $V(a;\theta) = u(c,l)/(1-\beta)$ at the nonstochastic steady state.

The second derivative V_{11} can be computed by noting that (11) holds for general a_t; hence we can differentiate it to yield:

$$V_{11}(a_t;\theta_t) = (1+r_t)\left[u_{11}(c_t^*,l_t^*)\,\frac{\partial c_t^*}{\partial a_t} + u_{12}(c_t^*,l_t^*)\,\frac{\partial l_t^*}{\partial a_t}\right]. \tag{12}$$

Again, we can compute V_{11} at the nonstochastic steady state reference point by evaluating the right-hand side of (12) at that point. All that remains is to find the derivatives $\partial c_t^*/\partial a_t$ and $\partial l_t^*/\partial a_t$.

I solve for $\partial l_t^*/\partial a_t$ by differentiating the household's intratemporal optimality condition,

$$-u_2(c_t^*,l_t^*) = w_t\,u_1(c_t^*,l_t^*), \tag{13}$$

with respect to a_t, and rearranging terms to yield:

$$\frac{\partial l_t^*}{\partial a_t} = -\lambda_t\,\frac{\partial c_t^*}{\partial a_t}\,, \tag{14}$$

where

$$\lambda_t \equiv \frac{w_t u_{11}(c_t^*,l_t^*) + u_{12}(c_t^*,l_t^*)}{u_{22}(c_t^*,l_t^*) + w_t u_{12}(c_t^*,l_t^*)} = \frac{u_1(c_t^*,l_t^*)u_{12}(c_t^*,l_t^*) - u_2(c_t^*,l_t^*)u_{11}(c_t^*,l_t^*)}{u_1(c_t^*,l_t^*)u_{22}(c_t^*,l_t^*) - u_2(c_t^*,l_t^*)u_{12}(c_t^*,l_t^*)}\,. \tag{15}$$

Note that, if consumption and leisure in period t are normal goods, then $\lambda_t > 0$, although I do not require this restriction below. It now only remains to solve for the derivative $\partial c_t^*/\partial a_t$.

Intuitively, $\partial c_t^*/\partial a_t$ should not be too difficult to compute: it is just the household's marginal propensity to consume today out of a change in assets, which can be deduced from the household's Euler equation and budget constraint:

Lemma 2. *Given Assumptions 1–8, the household's marginal propensity to consume out of wealth, evaluated at the nonstochastic steady state, satisfies*

$$\frac{\partial c_t^*}{\partial a_t} = E_t\frac{\partial c_{t+1}^*}{\partial a_t} = E_t\frac{\partial c_{t+k}^*}{\partial a_t}, \qquad k = 1,2,3,\ldots, \tag{16}$$

and

$$\frac{\partial c_t^*}{\partial a_t} = \frac{r}{1+w\lambda}\,. \tag{17}$$

Proof. See Appendix.[21] □

In other words, starting at the nonstochastic steady state, the household's optimal change in consumption today in response to an increase in assets must be the same as the expected change in consumption tomorrow, and the expected change in consumption at each future date $t+k$. Note that this equality does not follow from the steady-state assumption—for example, in a model with internal habits, the individual household's optimal consumption response to a change in assets increases gradually over time, even starting from steady state.

According to Lemma 2, the household's optimal response to a unit increase in assets is to raise consumption in every period by the extra asset income, r (essentially the permanent income hypothesis), adjusted downward by the amount $1+w\lambda$, which takes into account the household's decrease in hours worked and labor income. Thus, Lemma 2 represents a modified permanent income hypothesis that takes into account the household's labor margin.

We can now compute the household's coefficient of absolute wealth-gamble risk aversion. Substituting (11), (12), (14), and (17) into (10) proves the following:

Proposition 3. *Given Assumptions 1–8, the household's coefficient of absolute wealth-gamble risk aversion, $R^a(a_t;\theta_t)$, evaluated at the nonstochastic steady state, satisfies*

$$R^a(a;\theta) = \frac{-u_{11}+\lambda u_{12}}{u_1}\,\frac{r}{1+w\lambda} + \alpha\,\frac{r\,u_1}{u}\,, \tag{18}$$

where u_1, u_{11}, and u_{12} denote the corresponding partial derivatives of u evaluated at the steady state (c,l), and λ is given by (15) evaluated at steady state.

[21] The notation $\frac{\partial c_{t+1}^*}{\partial a_t}$ is taken to mean $\frac{\partial c_{t+1}^*}{\partial a_{t+1}}\frac{da_{t+1}^*}{da_t} = \frac{\partial c_{t+1}^*}{\partial a_{t+1}}\left[1+r_{t+1}+w_t\frac{\partial l_t^*}{\partial a_t} - \frac{\partial c_t^*}{\partial a_t}\right]$, and analogously for $\frac{\partial c_{t+2}^*}{\partial a_t}$, $\frac{\partial c_{t+3}^*}{\partial a_t}$, etc.

There are several features of Proposition 3 worth noting. First, when $\alpha = 0$, equation (18) reduces to the expressions derived in Swanson (2012) for the case of expected utility. When $\alpha = 0$ and labor is fixed ($\lambda = 0$), wealth-gamble risk aversion in (18) reduces further to $-ru_{11}/u_1$, which is just the usual Arrow–Pratt formula multiplied by r, a scaling factor that translates between units of wealth and current-period consumption.[22] When $u \geq 0$ everywhere, R^a is increasing in α, and when $u \leq 0$, R^a is decreasing in α, as observed after Proposition 1. Multiplying u by a constant has no effect on R^a, but an additive translation of u *does* affect R^a if $\alpha \neq 0$, because it changes the "twisted" expectation in equation (4).[23]

When $\lambda \neq 0$, households partially offset shocks to income through changes in hours worked. Even when consumption and labor are additively separable in u (so $u_{12} = 0$), c_t^* and labor supply are indirectly connected through the household's budget constraint. When $u_{12} \neq 0$, wealth-gamble risk aversion is further attenuated or amplified by the direct interaction between consumption and labor in utility, u_{12}. Note, however, that regardless of the signs of λ and u_{12}, R^a is always reduced when households can vary their labor supply:

Corollary 4.

$$R^a(a;\theta) \leq \frac{-ru_{11}}{u_1} + \alpha \frac{ru_1}{u}. \tag{19}$$

Note that the right-hand side of (19) is the formula for wealth-gamble risk aversion when labor is exogenously fixed ($\lambda = 0$).

Proof. Substituting in for λ and w, the first term in (18) can be written as:

$$\frac{-ru_{11}}{u_1} \frac{u_{11}u_{22} - u_{12}^2}{u_{11}u_{22} - 2\frac{u_2}{u_1}u_{11}u_{12} + \left(\frac{u_2}{u_1}\right)^2 u_{11}^2} = \frac{-ru_{11}}{u_1} \frac{1}{1 + \dfrac{\left(\frac{u_2}{u_1}u_{11} - u_{12}\right)^2}{u_{11}u_{22} - u_{12}^2}}. \tag{20}$$

Strict concavity of u implies $u_{11}u_{22} - u_{12}^2 > 0$, hence the right-hand side of (20) is less than or equal to $-ru_{11}/u_1$. □

The reduction in (20) can be substantial if the discriminant, $u_{11}u_{22} - u_{12}^2$, is small. The second term in (18)–(19), $\alpha ru_1/u$, is not directly affected by a change from a fixed-labor to flexible-labor assumption, however.

I provide examples of wealth-gamble risk aversion calculations in Section 3.3, below, after first defining relative wealth-gamble risk aversion.

3.2. The coefficient of relative wealth-gamble risk aversion

The distinction between absolute and relative wealth-gamble risk aversion lies in the size of the hypothetical gamble faced by the household. If the household faces a one-shot gamble of size A_t in period t,

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t + A_t \sigma \varepsilon_{t+1}, \tag{21}$$

or the household can pay a one-time fee $A_t \mu$ in period t to avoid this gamble, then it follows from Proposition 1 that $\lim_{\sigma \to 0} 2\mu(\sigma)/\sigma^2$ for this gamble is given by

$$A_t R^a(a_t; \theta_t). \tag{22}$$

The natural definition of A_t, considered by Arrow (1965) and Pratt (1964), is the household's wealth at time t. The gamble in (21) is then over a fraction of the household's wealth and (22) is referred to as the household's coefficient of relative wealth-gamble risk aversion.

In models with labor, however, total household wealth can be more difficult to define because of the presence of human capital. There are two natural definitions of human capital in these models, leading to two measures of household wealth A_t and hence two coefficients of relative wealth-gamble risk aversion (22). Note that the household's financial assets a_t is *not* a good measure of wealth A_t, because a_t for an individual household may be zero or negative at some points in time.

[22] A gamble over a lump sum of \$$x$ is equivalent here to a gamble over an annuity of \$$rx$. Thus, even though V_{11}/V_1 differs from u_{11}/u_1 by a factor of r, this difference is exactly the same as a change from lump-sum to annuity units. Thus, the difference in scale is just one of units.

[23] When generalized recursive preferences are written in the form (5), $w = -\tilde{u}_2/\tilde{u}_1$, $\lambda = \dfrac{w\tilde{u}_{11} + \tilde{u}_{12}}{\tilde{u}_{22} + w\tilde{u}_{12}}$, and

$$R^a(a;\theta) = \left[\frac{-\tilde{u}_{11} + \lambda\tilde{u}_{12}}{\tilde{u}_1} + (\rho - 1)\frac{-\tilde{u}_1 + \lambda\tilde{u}_2}{\tilde{u}}\right] \frac{r}{1 + w\lambda} + (\rho - \tilde{\alpha})\frac{r\tilde{u}_1}{\tilde{u}}.$$

This expression is somewhat more complicated than (18), owing to the more complicated derivatives of (5). When $\lambda = 0$ and $\tilde{u} = c$, this reduces to $(1 - \tilde{\alpha})/c$, the traditional fixed-labor measure of absolute wealth-gamble risk aversion in Epstein and Zin (1989), Weil (1989), and Example 1, below.

When the household's time endowment is not well-defined, such as when $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}$ and no upper bound $\bar{l}$ on l_t is specified, or $\bar{l}$ is specified but is arbitrary, it is most natural to define human capital as the present discounted value of labor income, $w_t l_t^*$. Equivalently, total household wealth A_t equals the present discounted value of consumption, which follows from the budget constraint (1)–(2). This leads to the following definition:

Definition 2. Let $(a_t; \theta_t)$ be an interior point of X. The household's *consumption-wealth coefficient of relative wealth-gamble risk aversion*, denoted $R^c(a_t; \theta_t)$, is given by (22) with wealth $A_t = A_t^c \equiv (1+r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau} c_\tau^*$, the present discounted value of household consumption, where $m_{t,\tau}$ denotes the stochastic discount factor $\prod_{s=t}^{\tau-1} m_{s+1}$, and m_{s+1} is given by (37).

The factor $(1+r_t)^{-1}$ in the definition expresses wealth A_t^c in beginning- rather than end-of-period-t units. In steady state, $A^c = c/r$ and $R^c(a;\theta)$ is given by

$$R^c(a;\theta) = \frac{-A^c\, V_{11}(a;\theta)}{V_1(a;\theta)} + \alpha\, \frac{A^c\, V_1(a;\theta)}{V(a;\theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1+w\lambda} + \alpha \frac{c u_1}{u}\,. \tag{23}$$

Alternatively, when the household's time endowment $\bar{l}$ is well specified, it's natural to define human capital as the present discounted value of the household's time endowment, $w_t \bar{l}$. Equivalently, total household wealth A_t equals the present discounted value of consumption c_t^* plus leisure $w_t(\bar{l} - l_t^*)$, from (1)–(2). This corresponds to the following definition:

Definition 3. Let $(a_t; \theta_t)$ be an interior point of X. The household's *consumption-and-leisure-wealth coefficient of relative wealth-gamble risk aversion*, denoted $R^{cl}(a_t; \theta_t)$, is given by (22) with wealth $A_t = A_t^{cl} \equiv (1+r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau} \big(c_\tau^* + w_\tau(\bar{l} - l_\tau^*)\big)$.

In steady state, $A^{cl} = \big(c + w(\bar{l} - l)\big)/r$, and

$$R^{cl}(a;\theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c + w(\bar{l} - l)}{1+w\lambda} + \alpha \frac{\big(c + w(\bar{l} - l)\big) u_1}{u}\,. \tag{24}$$

The closed-form expressions (23)–(24) are closely related, differing only by the ratio of the two gambles, $(c + w(\bar{l} - l))/c$.[24] Assuming consumption and leisure are positive, $|R^{cl}(a_t;\theta_t)| > |R^c(a_t;\theta_t)|$, because the size of the gamble is larger.

For expositional purposes, define

$$R^{fl}(a;\theta) \equiv \frac{-c\, u_{11}}{u_1} + \alpha \frac{c\, u_1}{u}\,, \tag{25}$$

the coefficient of relative wealth-gamble risk aversion when labor is exogenously fixed (see Example 1, below). R^{fl} thus ignores the household's ability to offset portfolio value shocks by varying labor supply. By Corollary 4, $R^c(a;\theta) \le R^{fl}(a;\theta)$. However, $R^{cl}(a;\theta)$ may be greater or less than $R^{fl}(a;\theta)$, depending on the importance of leisure in the household's total consumption bundle.

3.3. Examples

Example 1. Following Epstein and Zin (1989) and Weil (1989), consider the case where utility depends only on consumption,

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma}\,, \tag{26}$$

with $\gamma > 0$, $c_t \ge 0$, and l_t fixed exogenously at some $l \in \mathbb{R}$ for all t.[25] Leisure is arbitrary in this example—any $\bar{l} > l$ is observationally equivalent—so R^{cl} from Definition 3 is not well-defined, but from Definition 2 we have

$$R^c(a;\theta) = \frac{-c\, u_{11}}{u_1} + \alpha \frac{c\, u_1}{u} = \gamma + \alpha(1-\gamma)\,, \tag{27}$$

which motivates the definition of R^{fl} above. Note that if the household's generalized value function is written using (5) rather than (4), with $\rho \equiv 1-\gamma$, then $1 - \widetilde{\alpha} = \gamma + \alpha(1-\gamma)$ and $R^c(a;\theta) = 1 - \widetilde{\alpha}$. This is the usual definition of risk aversion for generalized recursive preferences in an endowment economy.

[24] Both Definitions 2 and 3 represent a proper generalization of the traditional definition of wealth-gamble risk aversion in an endowment economy. First, both definitions reduce to R^{fl}, defined below, when there is no labor in the model. Second, in steady state the household consumes exactly the flow of income from its wealth, rA, consistent with standard permanent income theory (where one must include the value of leisure $w(\bar{l} - l)$ as part of consumption when the value of leisure is included in wealth).

[25] In this example, Assumptions 1–8 need to be modified in a straightforward way to the one-dimensional case.

Example 2. Following van Binsbergen et al. (2012), among others,[26] a natural way to incorporate leisure into the preferences in (26) is to let

$$u(c_t, l_t) = \frac{\left(c_t^{\chi}(1-l_t)^{1-\chi}\right)^{1-\gamma}}{1-\gamma}, \tag{28}$$

where $\gamma > 0$, $\gamma \neq 1$, $\chi \in (0,1)$, $c_t > 0$, and $l_t \in (0,1)$.[27] In this example, the household can be regarded as consuming a single, composite good in each period formed from the Cobb–Douglas aggregate of consumption and leisure. A natural definition of wealth-gamble risk aversion is thus $\gamma + \alpha(1-\gamma) = 1 - \widetilde{\alpha}$, the coefficient of relative wealth-gamble risk aversion from Example 1 applied to the single, composite good. Indeed, this is the definition used by van Binsbergen et al. (2012). It is also the value implied by Definition 3 of the present paper, which includes the value of leisure in household wealth:

$$R^{cl}(a;\theta) = \frac{-u_{11} + \lambda u_{12}}{u_1}\frac{c + w(1-l)}{1 + w\lambda} + \alpha\frac{\big(c + w(1-l)\big)u_1}{u} \;=\; \gamma + \alpha(1-\gamma). \tag{29}$$

The consumption-wealth coefficient R^c from Definition 2 excludes leisure from household wealth and thus is smaller than (29), because the gamble is smaller:

$$R^{c}(a;\theta) \;=\; \frac{-u_{11} + \lambda u_{12}}{u_1}\frac{c}{1 + w\lambda} + \alpha\frac{c\,u_1}{u} \;=\; \gamma\chi + \alpha(1-\gamma)\chi. \tag{30}$$

In this example, the Cobb–Douglas functional form implies $R^c(a;\theta) = \chi R^{cl}(a;\theta)$.[28] In the next section, I compare these two measures to the risk premium on an equity security in a standard macroeconomic model, solved numerically.

Note that neither (29) nor (30) corresponds to the fixed-labor measure of wealth-gamble risk aversion, $R^{fl}(a;\theta) = \frac{-c\,u_{11}}{u_1} + \alpha\frac{c\,u_1}{u} = \big(1 - \chi(1-\gamma)\big) + \alpha(1-\gamma)$, a point emphasized by Swanson (2012) for the case of expected utility ($\alpha = 0$). The fixed-labor measure R^{fl} ignores the household's ability to offset portfolio value shocks by varying hours of work; as a result, R^{fl} does not generally correspond to the household's willingness to hold a risky asset and is not closely related to the equilibrium prices of such assets, as I will show in the next section.

Finally, a number of other authors consider an alternative parameterization of (28),[29]

$$u(c_t, l_t) = \frac{\left(c_t(1-l_t)^{\nu}\right)^{1-\gamma}}{1-\gamma}, \tag{31}$$

where $\gamma > 0$, $\nu > 0$, $c_t > 0$, $l_t \in (0,1)$, and $\gamma > \nu/(1+\nu)$ for concavity. For this parameterization, $R^{fl}(a;\theta) = \gamma + \alpha(1-\gamma) = 1 - \widetilde{\alpha}$, while

$$R^{cl}(a;\theta) \;=\; \big(1 - (1-\gamma)(1+\nu)\big) + \alpha(1-\gamma)(1+\nu) \tag{32}$$

and

$$R^{c}(a;\theta) \;=\; \frac{1 - (1-\gamma)(1+\nu)}{1+\nu} + \alpha(1-\gamma). \tag{33}$$

These expressions follow from Definitions 2 and 3 directly, or from the fact that (31) can be rewritten as $u(c_t, l_t) = \big(c_t^{1/(1+\nu)}(1-l_t)^{\nu/(1+\nu)}\big)^{(1-\gamma)(1+\nu)}$.

Example 3. Following Rudebusch and Swanson (2009) and many authors in the New Keynesian DSGE literature,[30] consider the additively separable period utility function

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta\frac{l_t^{1+\chi}}{1+\chi}, \tag{34}$$

[26] See also Andreasen (2012a, 2012b), Gourio (2013), Colacito and Croce (2013), and Dew-Becker (2014).

[27] When $\gamma \in (0,1)$, then $u > 0$, wealth-gamble risk aversion is increasing in α, and $\alpha > 0$ corresponds to preferences that are more risk averse than expected utility. When $\gamma > 1$, then $u < 0$, wealth-gamble risk aversion is decreasing in α, and $\alpha < 0$ corresponds to preferences that are more risk averse than expected utility.

[28] That is, $c/\big(c + w(1-l)\big) = \chi$. One might be surprised that $R^c(a;\theta) \to 0$ as $\chi \to 0$. However, as $\chi \to 0$, $w/c \to \infty$, so consumption becomes trivial to insure with tiny variations in labor supply.

[29] See Gourio (2012), Uhlig (2007), Backus et al. (2008), and Kung (2012).

[30] See, e.g., Erceg et al. (2000), Woodford (2003), Christiano et al. (2005), and Galí (2008). These New Keynesian DSGE studies do not use generalized recursive preferences, however.

where $\chi > 0$, $\eta > 0$, $c_t > 0$, $l_t > 0$, and $\gamma > 1$.[31] Leisure is essentially arbitrary in this example, since different assumptions regarding $\bar{l}$ have essentially no effect on the equilibrium. Thus, $R^{cl}(a;\theta)$ is not well-defined, but

$$R^c(a;\theta) \;=\; \frac{\gamma}{1+\frac{\gamma}{\chi}\frac{wl}{c}} + \frac{\alpha(1-\gamma)}{1+\frac{\gamma-1}{1+\chi}\frac{wl}{c}}\,. \tag{35}$$

As in Swanson (2012), we can simplify (35) a bit further by assuming $c \approx wl$, an assumption made in *this paragraph only* and nowhere else in the paper.[32] In this case,

$$R^c(a;\theta) \;\approx\; \frac{\gamma}{1+\frac{\gamma}{\chi}} + \frac{\alpha(1-\gamma)}{1+\frac{\gamma-1}{1+\chi}}\,. \tag{36}$$

Equation (36) is less than $R^{fl}(a;\theta) = \gamma + \alpha(1-\gamma)$, by an amount that can be dramatic if either of the denominators in (36) is large. On the other hand, as $\chi \to \infty$, the household's labor margin becomes inflexible and $R^c \to R^{fl}$.

4. Wealth-gamble risk aversion and asset pricing

The analysis above shows that a household's aversion to gambles over asset values or wealth depends on its ability to offset the outcome of those gambles with changes in hours worked. In this section, I show the relationship between wealth-gamble risk aversion and risk premia in the Lucas–Breeden stochastic discounting framework. Risk premia in this framework are closely related to the definition of wealth-gamble risk aversion in the previous section, and are generally unrelated to traditional measures of risk aversion that hold household labor fixed.

4.1. The SDF, risk premia, and wealth-gamble risk aversion

For generalized recursive preferences (4) with labor, Rudebusch and Swanson (2012) show that the household's stochastic discount factor is given by

$$m_{t+1} \equiv \beta\,\frac{u_1(c^*_{t+1}, l^*_{t+1})}{u_1(c^*_t, l^*_t)}\,\frac{V(a^*_{t+1};\theta_{t+1})^{-\alpha}}{\left(E_t V(a^*_{t+1};\theta_{t+1})^{1-\alpha}\right)^{-\alpha/(1-\alpha)}}. \tag{37}$$

Let p^i_t denote the ex-dividend time-t price of an asset i that pays stochastic dividend d^i_t each period. In equilibrium, p^i_t satisfies

$$p^i_t \;=\; E_t m_{t+1}(d^i_{t+1} + p^i_{t+1}). \tag{38}$$

Let $1 + r^i_{t+1}$ denote the realized gross return on the asset,

$$1 + r^i_{t+1} \;\equiv\; \frac{d^i_{t+1} + p^i_{t+1}}{p^i_t}\,, \tag{39}$$

and define the risk premium on the asset, ψ^i_t, to be its expected excess return,

$$\psi^i_t \;\equiv\; E_t r^i_{t+1} - r^f_{t+1}\,, \tag{40}$$

where $1 + r^f_{t+1} \equiv 1/E_t m_{t+1}$ denotes the risk-free rate. Then

$$\begin{aligned}\psi^i_t &= \frac{E_t m_{t+1} E_t(d^i_{t+1} + p^i_{t+1}) - E_t m_{t+1}(d^i_{t+1} + p^i_{t+1})}{p^i_t E_t m_{t+1}} \\ &= \frac{-\mathrm{Cov}_t(m_{t+1}, r^i_{t+1})}{E_t m_{t+1}}\,,\end{aligned} \tag{41}$$

where Cov_t denotes the covariance conditional on information at time t.

[31] The last restriction ensures consistency with Assumption 2. Alternatively, one could impose restrictions on Ω such that $u(\cdot,\cdot) < 0$ for all $(c_t, l_t) \in \Omega$. Under either of these assumptions, $u < 0$, wealth-gamble risk aversion is decreasing in α, and $\alpha < 0$ corresponds to preferences that are more risk averse than expected utility.

[32] In steady state, $c = ra + wl + d$, so $c = wl$ holds exactly if there is neither capital nor transfers in the model. In any case, $ra + d$ is typically small, since $r \approx .01$.

We can relate the stochastic discount factor to wealth-gamble risk aversion as follows.[33] First, differentiate (37) at the nonstochastic steady state, conditional on information at time t, to yield

$$dm_{t+1} = \frac{\beta}{u_1}\left[u_{11}dc^*_{t+1} + u_{12}dl^*_{t+1}\right] - \frac{\alpha\beta}{V}\, dV_{t+1} \tag{42}$$

to first order, where $dx_t \equiv x_t - x$, the time-t deviation of variable x from steady state. From the household's intratemporal optimality condition (13),

$$dl^*_{t+1} = -\lambda dc^*_{t+1} - \frac{u_1}{u_{22} + wu_{12}}\, dw_{t+1} \tag{43}$$

to first order. In contrast to (14), in equation (43) I keep track of changes in w_{t+1} (and r_{t+1} and d_{t+1}, below) because the asset return r^i_{t+1} may be correlated with these macroeconomic variables. Any such correlation will affect the risk premium ψ^i_t through the covariance in (41).[34]

The corresponding expression for dc^*_{t+1} is more complicated, so I state it as a lemma:

Lemma 5. *To first order in a neighborhood of the nonstochastic steady state,*

$$\begin{aligned} dc^*_{t+1} = {} & \frac{r}{1+w\lambda}\left[da_{t+1} + E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1+r)^k}\,(l\,dw_{t+k} + dd_{t+k} + a\,dr_{t+k})\right] \\ & + \frac{u_1u_{12}}{u_{11}u_{22} - u_{12}^2}\, dw_{t+1} + \frac{-u_1}{u_{11} - \lambda u_{12}}\, E_{t+1}\sum_{k=1}^{\infty}\frac{1}{(1+r)^k}\left(\frac{r\lambda}{1+w\lambda}\, dw_{t+k} - \beta dr_{t+k+1}\right). \end{aligned} \tag{44}$$

Proof. The expression follows from the household's Euler equation, budget constraint, and equation (43). See the Appendix for details. □

Like (43), equation (44) keeps track of changes in w_{t+k}, r_{t+k}, and d_{t+k}, because the asset return r^i_{t+1} may be correlated with these macroeconomic variables, which would affect ψ^i_t through the covariance in (41). In the special case where r^i_{t+1} is uncorrelated with all macroeconomic variables, then these terms can be ignored and (43)–(44) reduce to (14) and (17).

The term in square brackets in (44) describes the change in household wealth—including nonfinancial wealth—and thus the first line of (44) describes the wealth effect on consumption. The last line of (44) describes the substitution effect: changes in consumption due to changes in current and future wages and interest rates.[35]

For notational simplicity, let $d\hat{A}_{t+1} \equiv da_{t+1} + E_{t+1}\sum_{k=1}^{\infty}(1+r)^{-k}(l\,dw_{t+k} + dd_{t+k} + a\,dr_{t+k})$, the change in household wealth in (44). Then it's straightforward to show:

[33] Intuitively, one can start to see the close relationship between the risk premium and wealth-gamble risk aversion as follows: since $u_1(c^*_t, l^*_t) = V_1(a_t;\theta_t)/(1+r_t)$,

$$m_{t+1} = \beta\,\frac{V_1(a^*_{t+1};\theta_{t+1})}{V_1(a_t;\theta_t)}\,\frac{V(a^*_{t+1};\theta_{t+1})^{-\alpha}}{\left(E_tV(a^*_{t+1};\theta_{t+1})^{1-\alpha}\right)^{-\alpha/(1-\alpha)}}\,\frac{1+r_t}{1+r_{t+1}}.$$

Then, to first order around the nonstochastic steady state, conditional on information at time t,

$$\begin{aligned} dm_{t+1} &= \beta\,\frac{V_{11}da^*_{t+1} + V_{12}d\theta_{t+1}}{V_1} - \alpha\beta\,\frac{V_1da^*_{t+1} + V_2d\theta_{t+1}}{V} - \beta\,\frac{dr_{t+1}}{1+r} \\ &= -\beta R^a(a;\theta)da^*_{t+1} + \left(\frac{\beta V_{12}}{V_1} - \frac{\alpha\beta V_2}{V}\right)d\theta_{t+1} - \beta\,\frac{dr_{t+1}}{1+r}, \end{aligned}$$

assuming V is differentiable with respect to θ at the steady state. It follows that

$$\psi^i_t \approx R^a(a;\theta)\mathrm{Cov}_t(dr^i_{t+1}, da^*_{t+1}) + \left(\frac{-V_{12}}{V_1} + \alpha\,\frac{V_2}{V}\right)\mathrm{Cov}_t(dr^i_{t+1}, d\theta_{t+1}) + \beta\,\mathrm{Cov}_t(dr^i_{t+1}, dr_{t+1})$$

near the steady state. Here, ψ^i_t increases linearly with R^a, by an amount that depends on the covariance of the asset return with the household's financial wealth.

However, this decomposition is problematic for several reasons. First, the covariance involving da^*_{t+1} ignores the household's nonfinancial wealth, such as the present value of future transfers and labor income. Instead, the asset's covariance with nonfinancial wealth is relegated to the *second* term above, since θ determines the household's current and future wages w and transfers d. But this covariance is expressed in terms of the "black box" state variable θ rather than nonfinancial wealth itself, and the coefficient $(-V_{12}/V_1 + \alpha V_2/V)$ on this covariance is neither clearly related nor unrelated to wealth-gamble risk aversion. Thus, the decomposition in the main text is more useful, albeit somewhat more complicated.

[34] In contrast, the idiosyncratic gamble faced by the household in (7) was independent of all macroeconomic variables by assumption.

[35] The household's intertemporal elasticity of substitution is given by $-u_1/\big(c(u_{11} - \lambda u_{12})\big)$, so the last term in (44) describes intertemporal substitution effects on consumption of changes in future wages and interest rates.

Lemma 6. *To first order in a neighborhood of the nonstochastic steady state,*

$$dV_{t+1} = u_1(1+r)\, d\hat{A}_{t+1}. \tag{45}$$

Proof. The expression follows from (6), (43), and (44). See the Appendix for details. □

Lemma 6 states that the change in household welfare equals the marginal utility of consumption times the change in household wealth. The factor $1+r$ appears in (45) because a change in beginning-of-period-t assets produces $1+r$ units of extra consumption in period t.

Equations (42)–(45) then imply the following decomposition:

Proposition 7. *To first order in a neighborhood of the nonstochastic steady state,*

$$dm_{t+1} = -R^a(a;\theta)\,\beta\, d\hat{A}_{t+1} + \beta\, d\Phi_{t+1}\,, \tag{46}$$

where $d\Phi_{t+1} \equiv E_{t+1}\sum_{k=1}^{\infty}(1+r)^{-k}\big(\beta dr_{t+k+1} - \frac{r\lambda}{1+w\lambda}\, dw_{t+k}\big)$*, the intertemporal substitution term from (44). To second order in a neighborhood of the nonstochastic steady state,*

$$\psi_t^i = R^a(a;\theta)\,\mathrm{Cov}_t(dr_{t+1}^i, d\hat{A}_{t+1}) - \mathrm{Cov}_t(dr_{t+1}^i, d\Phi_{t+1})\,. \tag{47}$$

Proof. Substituting (43)–(45) into (42) yields (46). Substituting (46) into (41) yields (47). (Recall that $V = u/(1-\beta)$ and $\beta = E_t m_{t+1}$ at steady state.) Finally, $\mathrm{Cov}(dx, dy)$ is accurate to second order when dx and dy are accurate to first order. □

The first term in (47) shows that ψ_t^i increases locally linearly with R^a, by an amount that depends on the covariance between the asset return and household wealth, including nonfinancial wealth. This link between risk premia and wealth-gamble risk aversion should not be too surprising: Propositions 1 and 3 described the risk premium for extremely simple, idiosyncratic gambles over household wealth, while Proposition 7 shows that the same coefficient also appears in the household's aversion to more general financial market gambles that may be correlated with macroeconomic variables such as interest rates, wages, and transfers.

The second term in (47) corresponds to Merton's (1973) "changes in investment opportunities" in the ICAPM framework. Even if $R^a = 0$—that is, even if the household is risk-neutral for money or wealth gambles—ψ_t^i can be nonzero. This is because even a risk-neutral household can benefit from an asset that pays off well when the price of the household's total consumption bundle is low. An asset that pays off well when current and future wages are low (and hence leisure is cheap) or current and future interest rates are high (and hence future consumption is cheap) is preferable to an asset that pays off poorly in those situations. Even a risk-neutral household is willing to pay a premium for such an asset—implying a lower ψ_t^i—and this effect is captured by the second term in (47). The fact that the household faces a consumption-leisure tradeoff as well as a current-vs.-future consumption tradeoff implies that the second term in (47) is more general than just changes in the household's investment opportunities. Indeed, the second term in (47) is better described as changes in *purchasing* opportunities—opportunities to purchase leisure and future consumption more cheaply. An increase in Merton's investment opportunities is just an opportunity to purchase future consumption more cheaply.

Finally, the decomposition (47) can be written in terms of relative rather than absolute wealth-gamble risk aversion using Definitions 2–3:[36]

Corollary 8. *In terms of relative wealth-gamble risk aversion, the risk premium in (47) can be written as:*

$$\psi_t^i = R^c(a;\theta)\,\mathrm{Cov}_t\left(dr_{t+1}^i, \frac{d\hat{A}_{t+1}}{A^c}\right) - \mathrm{Cov}_t(dr_{t+1}^i, d\Phi_{t+1}) \tag{48}$$

or

$$\psi_t^i = R^{cl}(a;\theta)\,\mathrm{Cov}_t\left(dr_{t+1}^i, \frac{d\hat{A}_{t+1}}{A^{cl}}\right) - \mathrm{Cov}_t(dr_{t+1}^i, d\Phi_{t+1})\,, \tag{49}$$

where A^c *and* A^{cl} *are as in Definitions 2–3.*

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[36] Note that $d\hat{A}_{t+1}$ differs slightly from dA_{t+1}^c and dA_{t+1}^{cl}, which is why (48) and (49) are not written in terms of $d\log A_{t+1}^c$ or $d\log A_{t+1}^{cl}$.

4.2. Numerical examples

Two numerical examples help to illustrate the relationship between wealth-gamble risk aversion and risk premia derived above. Both examples use a standard real business cycle (RBC) model to generate the behavior of macroeconomic variables and the dividend on an equity security, which I take to be a consumption claim. This framework is both simple and very standard in the macro-finance literature cited in the Introduction.

The economy consists of a unit continuum of representative households and a unit continuum of perfectly competitive representative firms. Each household has optimization problem (1)–(4) and period utility function to be specified shortly. Each firm has production function

$$y_t = Z_t k_t^{1-\zeta} l_t^{\zeta} , \tag{50}$$

where y_t, k_t, and l_t denote firm output, beginning-of-period capital, and labor input, respectively. The productivity parameter Z_t is common across firms and follows the exogenous process

$$\log Z_t = \rho_z \log Z_{t-1} + \varepsilon_t, \tag{51}$$

where ε_t is i.i.d. with mean zero and variance σ_ε^2. Labor and capital are supplied by households at the competitive wage and rental rates w_t and r_t^k. Capital is the only asset in the economy that is in nonzero net supply. Households accumulate capital according to

$$k_{t+1} = (1 + r_t)k_t + w_t l_t - c_t, \tag{52}$$

where $r_t \equiv r_t^k - \delta$, δ is the capital depreciation rate, and c_t denotes household consumption.

For simplicity, I define an equity security to be a claim on the aggregate consumption stream, where aggregate consumption $C_t = c_t$ in equilibrium. The ex-dividend price of the equity claim, p_t, then satisfies

$$p_t = E_t m_{t+1}(C_{t+1} + p_{t+1}) \tag{53}$$

in equilibrium, where m_{t+1} is given by (37). I define the equity premium, ψ_t, to be the expected excess return

$$\psi_t \equiv \frac{E_t(C_{t+1} + p_{t+1})}{p_t} - (1 + r_t^f) . \tag{54}$$

I follow standard calibrations in the literature and take a period in the model to be one quarter in the data, set β to .99, δ to .025, ζ to .7, and σ_ε to .01. I consider the cases $\rho_z < 1$ and $\rho_z = 1$ in the examples below. Once the period utility function is specified, I solve the model using perturbation methods, as in Rudebusch and Swanson (2012) and Swanson (2012). This involves computing a nonstochastic steady state for the model (or transformed version of the model) and an nth-order Taylor series approximation to the true nonlinear solution for the model's endogenous variables around the steady state.[37] Additional details of the solution algorithm and computer code are provided in the Appendix and in Swanson et al. (2006). Aruoba et al. (2006) solve a standard RBC model using a variety of numerical methods and find that a fifth-order perturbation solution is among the most accurate methods globally as well as being faster to compute than other standard methods.

Example 4. Consider first the additively separable period utility function from Rudebusch and Swanson (2009) and Example 3,

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi} . \tag{55}$$

Set $\rho_z = 0.9$, $\gamma = 5$, $\chi = 1.5$, and $\alpha = -10$ as baseline values, and consider how the equity premium and wealth-gamble risk aversion vary as each of γ, χ, and α are varied in turn.[38] For each set of parameter values, we can solve the model as described above.

Fig. 1 plots the equity premium and wealth-gamble risk aversion as functions of χ, γ, and α. The solid black line in each panel plots the equity premium, ψ, against the right axis. The equity premium in this model is very small, less than 25 basis points per year in each of the panels; this is a manifestation of Rouwenhorst's (1995) and Lettau and Uhlig's (2000) finding that the equity premium is an even larger puzzle in RBC models with endogenous labor than in an endowment economy, because households can endogenously smooth their consumption in response to shocks. The dashed blue line in each panel plots the coefficient of relative wealth-gamble risk aversion, $R^c(a;\theta)$ from equation (35), against the left axis.

[37] Results in the figures below are for $n = 5$, a fifth-order approximation, but results are very similar for $n = 4$ and $n = 6$, suggesting that the Taylor series has essentially converged by $n = 5$ for the range of state variables considered in the figures.

[38] To allow for balanced growth or $\rho_z = 1$, the preference specification (55) would have to be modified, as in Rudebusch and Swanson (2012) and Swanson (2017). For simplicity, I do not consider those modifications here.

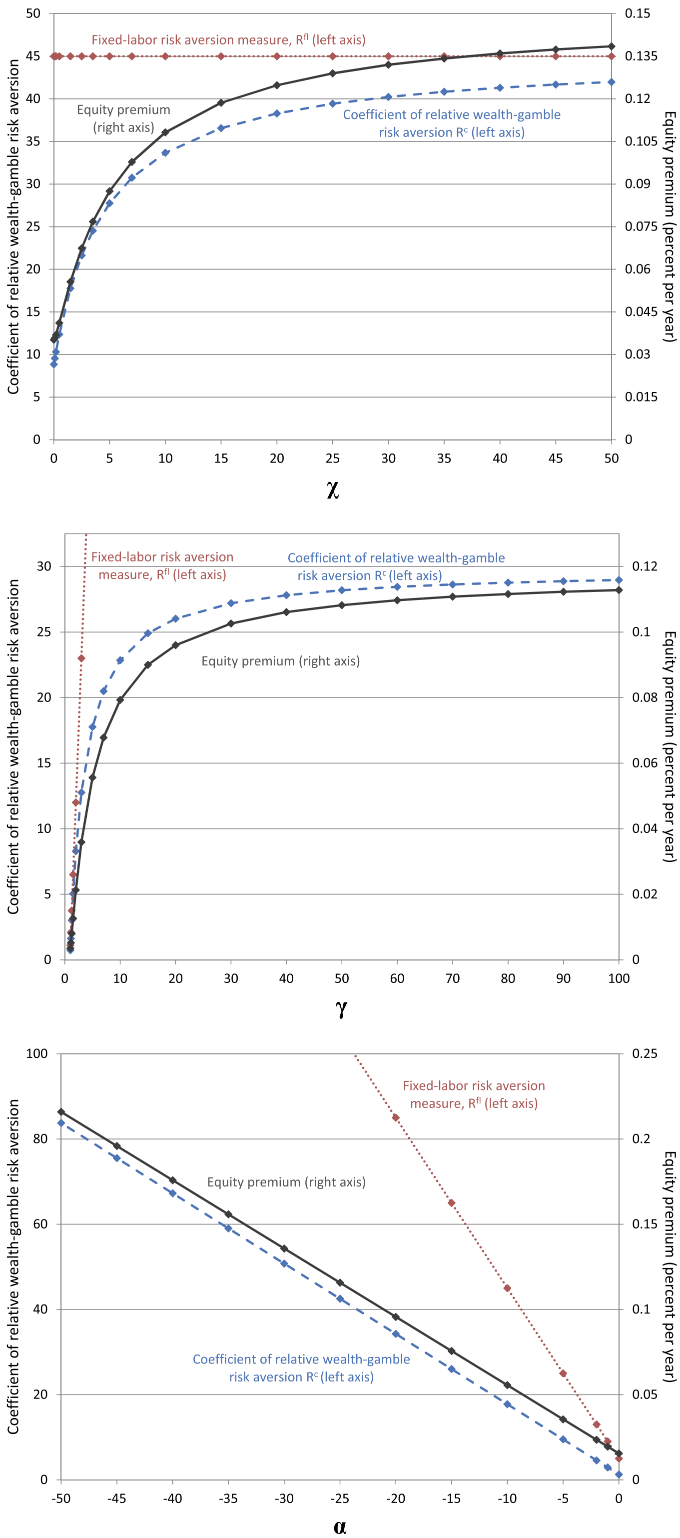

Fig. 1. The equity premium and wealth-gamble risk aversion in a real business cycle model with generalized recursive preferences and period utility $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$. Solid black lines depict the equity premium, dashed blue lines the coefficient of relative wealth-gamble risk aversion R^c, and dotted red lines the traditional, fixed-labor measure of risk aversion, $R^{fl} = \gamma + \alpha(1-\gamma) = 1 - \widetilde{\alpha}$. In the top panel, χ ranges from .01 to 50 while γ is fixed at 5 and α at -10; in the middle panel, γ ranges from 1.01 to 100 while χ is fixed at 1.5 and α at -10; in the bottom panel, α ranges from -50 to 0 while χ is fixed at 1.5 and γ at 5. In each panel, the equity premium is closely related to R^c and is essentially unrelated to R^{fl}. See text for details. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

For comparison, the dotted red line in each panel plots the fixed-labor measure of risk aversion, $R^{fl}(a;\theta) = \gamma + \alpha(1-\gamma)$, also against the left axis.

In each of the three panels in Fig. 1, the equity premium tracks R^c closely, and is essentially unrelated to R^{fl}. In the top panel, R^{fl} is independent of χ and thus is constant at 45, yet the equity premium varies by a factor of four, along with R^c. In the middle panel, R^{fl} increases linearly with γ, ranging from about 1 to 1090 (values above 32 are off the chart and not depicted), while the equity premium is a concave function of γ that corresponds closely to R^c. In the bottom panel, the equity premium varies about linearly with α and R^c, but grows more slowly than R^{fl}.[39] Note that, in the bottom panel, more negative values of α imply greater wealth-gamble risk aversion because $u \le 0$; also, the equity premium does not converge to zero as $R^c \to 0$ due to the additional ICAPM term in (47) reflecting changes in purchasing opportunities discussed earlier.

It's important to note that varying χ, γ, and α in Fig. 1 has general equilibrium effects on the economy as well as effects on the household's coefficients R^c and R^{fl}. For example, as the parameter χ or γ varies, the response of aggregate consumption to a productivity shock will typically change as well, which can change the covariance terms in (48) as well as the coefficient R^c. The equity premium in Fig. 1 will vary for both of these reasons—changes in R^c and changes in covariance due to general equilibrium effects—so we should not expect the equity premium to vary exactly in line with wealth-gamble risk aversion R^c. Nevertheless, Fig. 1 suggests that R^c is in fact closely related to the equity premium in this general equilibrium model, much more so than the traditional, fixed-labor coefficient R^{fl}.

Intuitively, lower values of χ imply a more flexible labor margin, which gives the household more ability to insure itself from consumption fluctuations. This can be seen clearly in Fig. 2, which plots first-order impulse response functions for consumption, labor, and the capital stock to a one percent positive shock to productivity Z_t. In each panel, the solid black line depicts the impulse response for the baseline parameterization of the model and the dashed and dotted lines plot impulse response functions for the cases $\chi = 5$ and $\chi = 0.1$, respectively. For all three parameterizations, consumption rises in response to the productivity shock, labor rises on impact and then falls, and household savings increases (as evidenced by the rise in the capital stock). When χ is lower, the household's labor margin is more flexible and the household reduces labor supply by more, on net, in response to the shock, thereby smoothing consumption. Note how this intuition holds despite the fact that labor initially *rises* on impact as a result of the substitution effect on labor supply. Thus, the fact that the short-run correlation between labor and consumption is positive in the model does not prevent the household from using labor supply to smooth its consumption in response to shocks.

Example 5. Next, consider the Cobb–Douglas preference specification from van Binsbergen et al. (2012) and Example 2,

$$u(c_t, l_t) = \frac{\left(c_t^{\chi}(1-l_t)^{1-\chi}\right)^{1-\gamma}}{1-\gamma}. \tag{56}$$

Following Gourio (2013), set $\rho_z = 1$, $\gamma = 0.5$, $\chi = 0.3$, and $\alpha = 19$,[40] and consider how the equity premium and wealth-gamble risk aversion vary as χ, γ, and α are varied in turn. For each set of parameter values, I solve the model as described above.

Fig. 3 plots the equity premium and wealth-gamble risk aversion as functions of χ, γ, and α. As in Fig. 1, the solid black line in each panel depicts the equity premium, ψ, the dashed blue line plots the consumption-wealth coefficient of relative wealth-gamble risk aversion, $R^c(a;\theta)$, and the dotted red line graphs the traditional, fixed-labor risk aversion measure, $R^{fl}(a;\theta)$. As in Fig. 1, the equity premium in Fig. 3 tracks R^c closely, and is essentially unrelated to R^{fl}. In the top panel, R^{fl} is nearly constant at a value of about 10, yet the equity premium varies by a factor of about ten, along with R^c. (As in Fig. 1, the equity premium does not quite converge to zero along with R^c due to the changes in purchasing opportunities ICAPM term in (47).) In the middle panel, R^{fl} increases linearly as γ falls, ranging from about 1 to 19.5 (values above 12 are not depicted), but the equity premium increases at a more moderate pace corresponding to R^c. For example, a value of $\psi = 10$ bp is associated with $R^c \approx 5$ in the top panel of Fig. 3, while a value of $\psi = 10$ bp in the middle panel requires $R^c \approx 5$ vs. $R^{fl} \approx 16$, at $\gamma \approx .2$. In the bottom panel, the equity premium increases about linearly with α and R^c, while R^{fl} again grows too quickly. Thus, as in Fig. 1, the traditional, fixed-labor measure of risk aversion R^{fl} is unrelated to the equity premium in Fig. 3, while the coefficient of relative wealth-gamble risk aversion R^c tracks the equity premium closely.

Household leisure is well-defined in this example, so we can also plot the consumption-and-leisure-wealth coefficient of relative wealth-gamble risk aversion, $R^{cl}(a;\theta) = \gamma + \alpha(1-\gamma) = 1 - \widetilde{\alpha}$, as the dash-dotted green line in each panel of Fig. 3. Perhaps surprisingly, R^{cl} is not closely related to the equity premium ψ. In the top panel of Fig. 3, R^{cl} is independent of χ and thus constant at 10, while ψ varies by a factor of about ten. In the middle and bottom panels, R^{cl} grows linearly along with R^{fl} at a rate much greater than ψ. The reason for this discrepancy in this example is that changes in χ, γ, and α have general equilibrium effects on macroeconomic variables like w_t, r_t, and aggregate consumption C_t. These effects change the covariance terms in (47)–(49) as well as the household's wealth-gamble risk aversion coefficients, which can obscure

[39] The equity premium ψ, R^c, and R^{fl} all vary about linearly with α, but the magnitude of R^{fl} does not agree with ψ. For example, in the top panel of Fig. 1, an equity premium of about 14 bp corresponds to risk aversion around 45 by either measure R^c or R^{fl}. In the bottom panel of Fig. 1, ψ of about 14 bp also corresponds to R^c of about 50 (at $\alpha \approx -30$), but would require $R^{fl} \approx 125$.

[40] Gourio sets $1 - \widetilde{\alpha} = \gamma + \alpha(1-\gamma) = 10$.

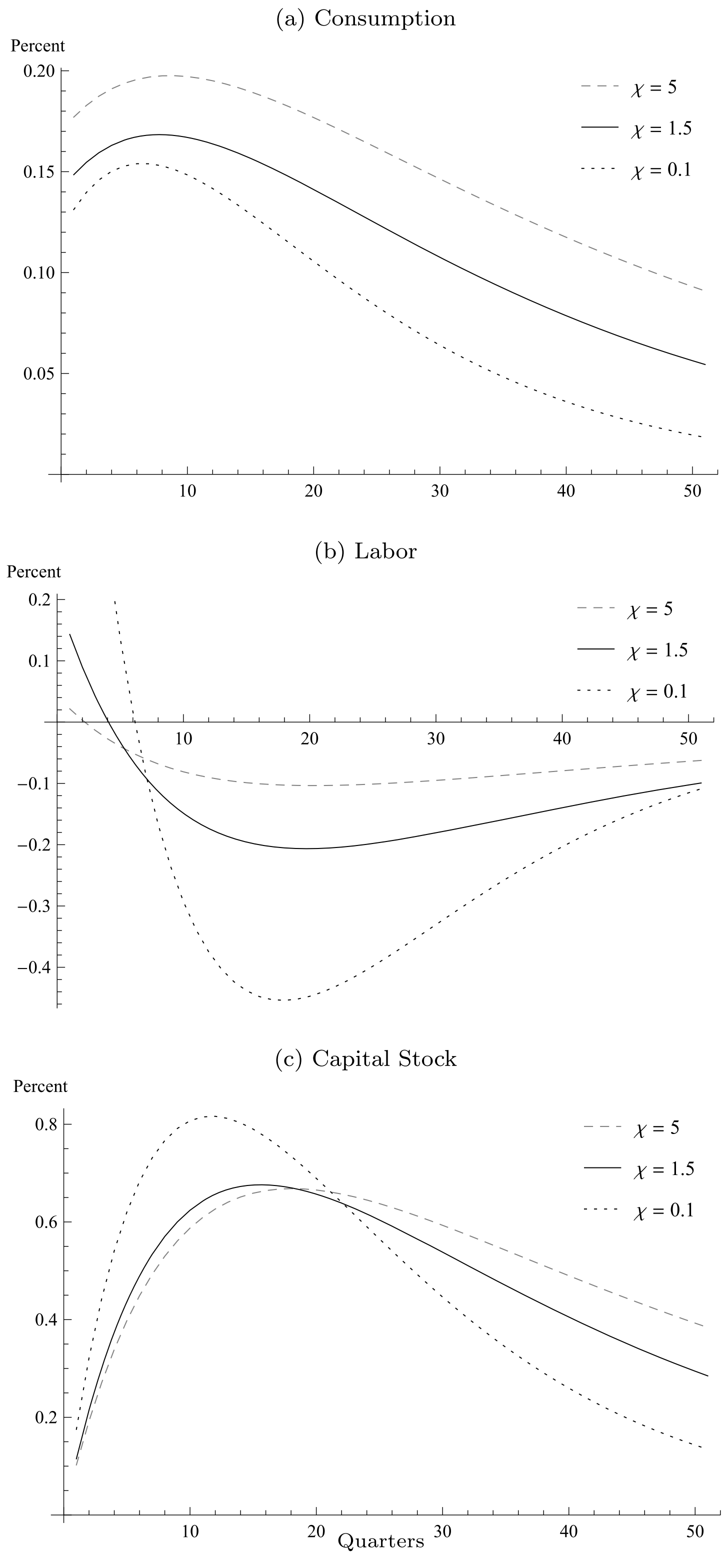

Fig. 2. Impulse response functions for (a) consumption, (b) labor, and (c) the capital stock to a 1% technology shock in the real business cycle model from Example 4 and Fig. 1, with generalized recursive preferences and period utility $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$. In each panel, $\gamma = 5$, $\alpha = -10$, and $\chi \in \{0.1, 1.5, 5\}$. When χ is lower, the household varies labor supply by more to smooth consumption, even though labor and consumption comove positively in the short run. See text for details.

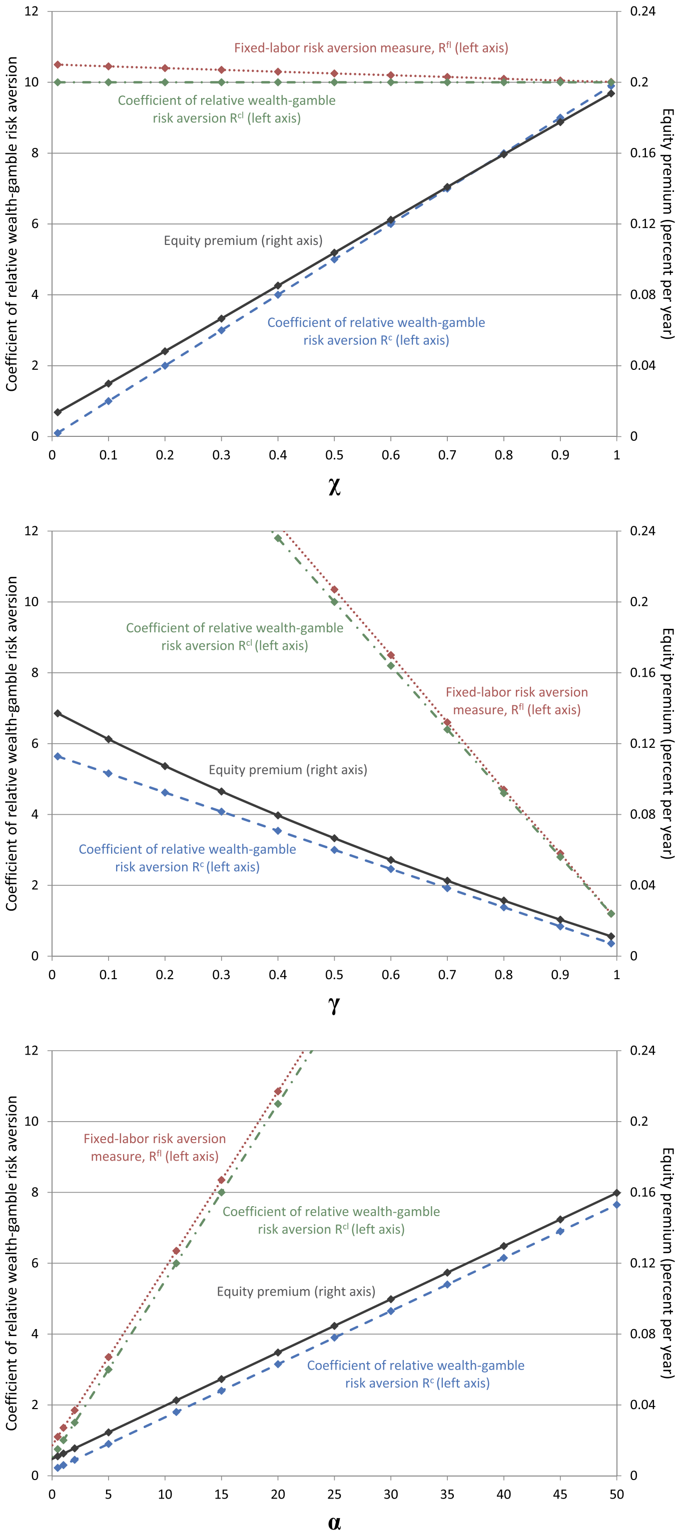

Fig. 3. The equity premium and wealth-gamble risk aversion in an RBC model with generalized recursive preferences and period utility $u(c_t, l_t) = (c_t^{\chi}(1 - l_t)^{1-\chi})^{1-\gamma}/(1-\gamma)$. Solid black lines depict the equity premium, dashed blue lines the coefficient of relative wealth-gamble risk aversion R^c, dotted red lines the fixed-labor measure of risk aversion R^{fl}, and dash-dot green lines the coefficient of relative wealth-gamble risk aversion R^{cl}. In the top panel, χ ranges from .01 to .99 while γ is fixed at 0.5 and α at 19; in the middle panel, γ ranges from .01 to .99 while χ is fixed at 0.3 and α at 19; in the bottom panel, α ranges from 0 to 50 while χ is fixed at 0.3 and γ at 0.5. In each panel, the equity premium is closely related to R^c and is essentially unrelated to R^{fl} and R^{cl}. See text for details. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the relationship between wealth-gamble risk aversion and the equity premium in the model. I now discuss those general equilibrium effects in more detail.

4.3. R^c vs. R^{cl} and the equity premium in general equilibrium

When studying an individual household, changes in the household's preference parameters, such as χ, γ, and α, have no effect on aggregate variables, given the usual assumption that the household is infinitesimal. However, Examples 4–5 compute the general equilibrium of a representative agent economy; thus, changes in the representative household's preference parameters χ, γ, and α do affect macroeconomic variables in those examples.

Looking at Example 5 and Fig. 3 in particular, it may seem surprising that R^{cl} is not more closely related to the equity premium, given that consumption and leisure form a composite good in those preferences. Instead, the consumption-wealth coefficient R^c provides a better measure. Looking at the decomposition of the equity premium in Corollary 8, Fig. 3 says that the covariance $\mathrm{Cov}_t(dr^i_{t+1}, d\hat{A}_{t+1}/A^c)$ is much closer to being invariant with respect to changes in χ, γ, and α than is the covariance $\mathrm{Cov}_t(dr^i_{t+1}, d\hat{A}_{t+1}/A^{cl})$.[41]

Note first that—unlike the traditional, fixed-labor measure R^{fl}—both R^c and R^{cl} recognize that households will vary their labor supply to insure themselves from portfolio fluctuations. The issue here is simply whether the value of leisure should be included in household wealth when measuring relative wealth-gamble risk aversion, with R^{cl} including the value of leisure and R^c excluding it. In a model with two consumption goods (and no labor) and period utility $u(c_{1t}, c_{2t}) = \left(c_{1t}^{\chi} c_{2t}^{1-\chi}\right)^{1-\gamma}/(1-\gamma)$, it would seem bizarre to equate household wealth to the present value of consumption of one of the goods, excluding the value of the other. Yet that is essentially what the results in Fig. 3 and Example 5 suggest.

The key difference in Example 5 is that consumption and leisure appear separately elsewhere in the model (e.g., in the production function), which is inconsistent with the composite good interpretation. In a model with two consumption goods, varying the parameter χ between 0 and 1 might change the relative sizes of the two consumption good sectors in steady state, but would not affect aggregate consumption or have any other aggregate general equilibrium implications. In contrast, varying the parameter χ in Example 5 has important general equilibrium effects on steady-state capital, labor, wealth, and other aggregate variables.[42]

To see the effects of χ on the steady state and the covariance term $\mathrm{Cov}_t(dr^i_{t+1}, d\hat{A}_{t+1})$ in Example 5, first note that the steady-state interest rate $r = (1-\beta)/\beta$ and marginal product of capital $r^k = (1-\zeta)y/k$, so the output-capital ratio satisfies

$$\frac{y}{k} = \frac{1}{1-\zeta}\left(\frac{1-\beta}{\beta} + \delta\right), \tag{57}$$

which is independent of χ. From the production function, $(l/k) = (y/k)^{1/\zeta}$, and the aggregate resource constraint implies $(c/k) = (y/k) + \delta$; thus, the ratios y/k, l/k, and c/k are all invariant with respect to χ, as is the steady-state wage $w = \zeta\,\dfrac{(y/k)}{(l/k)}$. Finally, the household's period utility function implies $\chi w(1-l) = (1-\chi)c$, and thus

$$k = \frac{w}{w\,(l/k) + \dfrac{1-\chi}{\chi}(c/k)}. \tag{58}$$

The wage w and ratios l/k and c/k are invariant with respect to χ, so the aggregate equilibrium level of k is increasing in χ, ranging from 0 to $(y/k)^{-1/\zeta}$ as χ ranges from 0 to 1.

Thus, varying the parameter χ in Example 5 changes not just the composition of the consumption-leisure aggregate good, but also the equilibrium level of k and household wealth A^c and A^{cl}, among other variables. This, in turn, affects the crucially important covariance $\mathrm{Cov}_t(dr^i_{t+1}, d\hat{A}_{t+1})$ in Proposition 7. In particular, $\mathrm{Cov}_t(dr^i_{t+1}, d\hat{A}_{t+1})$ is roughly proportional to steady-state k, because $d\hat{A}_{t+1} = da_{t+1} + E_{t+1}\sum_{k=1}^{\infty}(1+r)^{-k}(l\,dw_{t+k} + a\,dr_{t+k})$ scales about linearly with k.[43]

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[41] As discussed below, the second covariance term in Corollary 8, $\mathrm{Cov}_t(dr^i_{t+1}, d\Phi_{t+1})$, does not vary much with changes in the household's preference parameters in Fig. 3.

[42] In partial equilibrium, the interpretation of consumption and leisure as a composite good for the household in Example 5 is valid. The issue is that the composite good interpretation is invalid in the general equilibrium model, because labor appears separately in the production function. Fig. 3 plots the general equilibrium relationship between the equity premium (and wealth-gamble risk aversion) and the parameters χ, γ, and α.

[43] Household assets $a = k$ and the ratio l/k is constant, so a and l scale linearly with k. (Labor scales linearly from $l = 0$ to 1, which are attained when $\chi = 0$ and 1, respectively.) In contrast, dr_{t+1} and dw_{t+1} hardly change with k because the marginal products of capital and labor, $(1-\zeta)y_t/k_t$ and $\zeta y_t/l_t$, are invariant to changes in steady-state k. The term da_{t+1} grows about linearly with k because technology shocks in the model are multiplicative, so the effects of technology shocks scale. Thus, $d\hat{A}_{t+1}$ scales about linearly with k. The return r^i_{t+1} on the consumption claim hardly changes with k because both sides of the household's Euler equation scale linearly with k. Thus, $\mathrm{Cov}_t(dr^i_{t+1}, d\hat{A}_{t+1})$ varies roughly linearly with k.

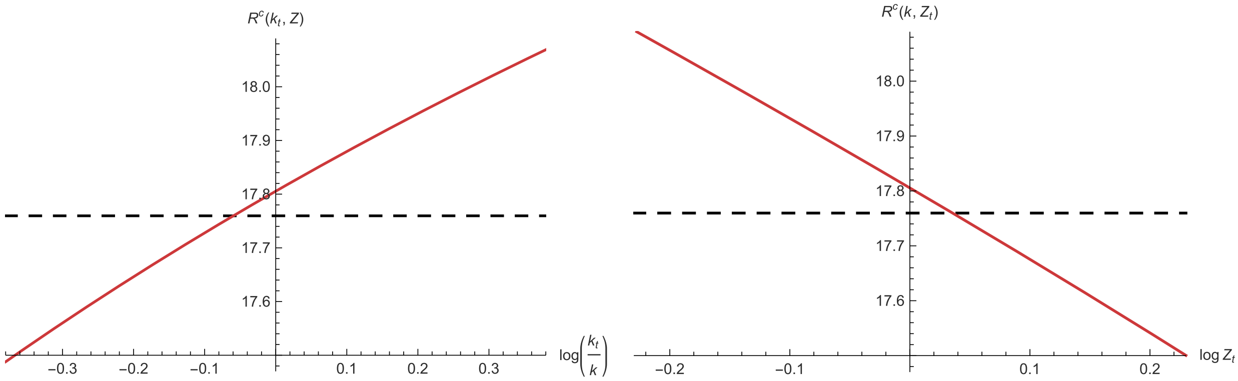

Fig. 4. Coefficient of relative wealth-gamble risk aversion R^c as a function of the state $(k_t; Z_t)$ in a real business cycle model with generalized recursive preferences and period utility $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$. Dashed black lines depict the closed-form, steady-state value $R^c(k; Z)$, solid red lines the numerical solution for $R^c(k_t; Z_t)$. In the left panel, $\log(k_t/k)$ ranges from -0.38 to 0.38 while $\log Z_t$ is fixed at 0; in the right panel, Z_t ranges from -0.23 to 0.23 while k_t is fixed at k. In both panels, $R^c(k_t; Z_t)$ is close to $R^c(k; Z)$ and never near the traditional, fixed-labor value of $R^{fl} = 45$. See text for details. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Finally, household wealth A^c is proportional to k.[44] As a result, $\mathrm{Cov}_t(dr^i_{t+1}, d\hat{A}_{t+1}/A^c)$ in Corollary 8 is roughly invariant with respect to χ, implying a tight, linear relationship between $R^c(a;\theta)$ and the equity premium ψ.[45] This close relationship is clearly visible in Fig. 3.

By contrast, A^{cl}, the leisure-inclusive measure of household wealth, is *not* proportional to k. The value of leisure, $w(1-l)$, *decreases* with k (because w is invariant and $1-l$ decreases). As a result, A^{cl} has no simple relationship to k and $\mathrm{Cov}_t(dr^i_{t+1}, d\hat{A}_{t+1}/A^{cl})$ varies substantially with changes in χ. Thus, there is no stable relationship between R^{cl} and the equity premium in Corollary 8 and Example 5, as is evident in Fig. 3.

For any given macroeconomic model, the equity premium depends not just on R^c but also on the two covariance terms in Corollary 8—the covariance of the equity return with household wealth and with changes in purchasing opportunities. To the extent that these covariances change as parameters of the model are varied, the relationship between the equity premium and R^c will be weaker. Nevertheless, for standard macroeconomic models like those considered in this section, the wealth-gamble risk aversion measure R^c seems to provide a good benchmark, much more so than the traditional, fixed-labor risk aversion measure R^{fl}.

5. Wealth-gamble risk aversion away from the steady state

The closed-form expressions for wealth-gamble risk aversion derived in Section 3 hold exactly only at the model's nonstochastic steady state. For values of $(a_t; \theta_t)$ away from steady state, these expressions are only approximations. In this section, I evaluate the accuracy of those approximations by computing wealth-gamble risk aversion numerically away from the steady state for the standard real business cycle model described above.

The setup and parameterization of the model are the same as described previously. I assume that households have the same additively separable preferences as in Examples 3–4, with parameter values $\gamma = 5$, $\chi = 1.5$, and $\alpha = -10$. The state variables of the model are k_t and Z_t.[46] The household's consumption-wealth coefficient of relative wealth-gamble risk aversion at the steady state, $R^c(k; Z)$, is given by equation (35); for the parameter values above, this implies $R^c(k; Z) = 17.76$, a little more than one-third the traditional, fixed-labor measure of $1 - \widetilde{\alpha} = \gamma + \alpha(1-\gamma) = 45$.

For values of $(k_t; Z_t)$ away from the steady state, equations (9) and (11)–(15) remain valid, and can be used to compute $R^c(k_t; Z_t)$ numerically. I append equations for R^c, V_1, V_{11}, λ_t, and $\partial c^*_t/\partial a_t$ to the standard set of RBC equilibrium conditions and solve them using the same fifth-order perturbation method as in the previous section. (See the Appendix for a complete list of equations and additional details regarding the numerical solution algorithm.)

Fig. 4 graphs the result as a function of $\log(k_t/k)$ and $\log Z_t$ over a wide range of values for these variables, about ± 10 standard deviations (equal to about ± 38 percent and ± 23 percent for $\log k_t$ and $\log Z_t$, respectively).[47] The horizontal dashed black lines in Fig. 4 report the constant, closed-form value for wealth-gamble risk aversion at the nonstochastic

[44] Because consumption and hence the present discounted value of consumption scale linearly with k.

[45] The second covariance term in Corollary 8, $\mathrm{Cov}_t(dr^i_{t+1}, d\Phi_{t+1})$, is not strictly invariant to changes in χ, but this term is much smaller than the first and thus does not have a substantial effect on ψ in Fig. 3.

[46] The household's endogenous state variable is its own holdings of capital, k_t. The exogenous state variables of the model are Z_t and the aggregate capital stock, K_t. Thus, the state vector of the household's optimization problem could be written more precisely as $(k_t; Z_t, K_t)$, or even $(k_t; Z_t, K_t, \sigma^2_\varepsilon)$, since the nonstochastic steady state requires setting $\sigma^2_\varepsilon = 0$. However, in equilibrium, $k_t = K_t$, so for simplicity I write the state vector in this example as $(k_t; Z_t)$.

[47] The unconditional standard deviations of $\log Z_t$ and $\log(k_t/k)$ are about 2.3 and 3.8 percent, respectively. The ergodic mean of $\log Z_t$ is zero and that of $\log(k_t/k)$ is about .006, or 0.6 percent.

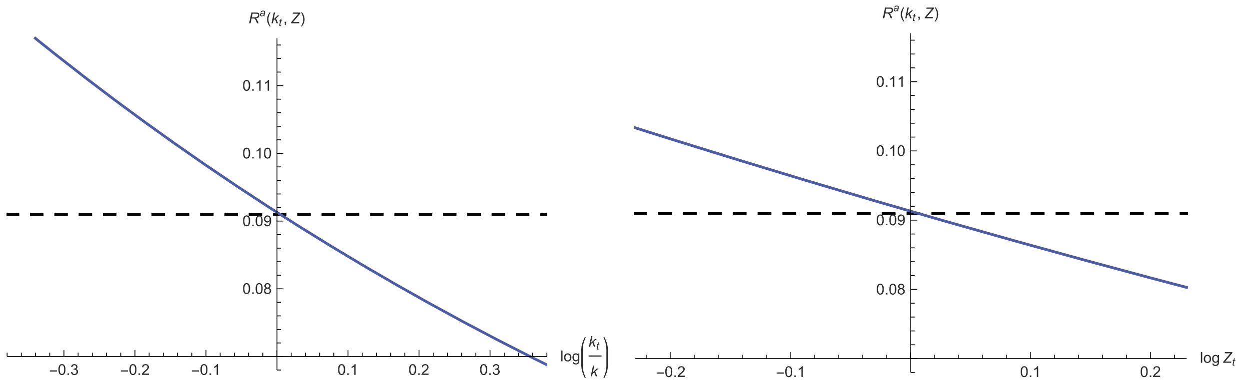

Fig. 5. Coefficient of absolute wealth-gamble risk aversion R^a as a function of the state $(k_t; Z_t)$ in a real business cycle model with generalized recursive preferences and period utility $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$. Dashed black lines depict the closed-form, steady-state value $R^a(k; Z)$, solid blue lines the numerical solution for $R^a(k_t; Z_t)$. Absolute risk aversion is decreasing with both k_t and Z_t. See notes to Fig. 4 and text for details. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

steady state, $R^c(k; Z)$, equal to 17.76. The solid red lines in the figure plot the numerical solution for $R^c(k_t; Z_t)$ for general values of k_t and Z_t.[48] The key point of Fig. 4 is that, even over the very wide range of values of the state variables considered, the household's coefficient of relative wealth-gamble risk aversion ranges between about 17.5 and 18.1, very close to $R^c(k; Z)$, and never near the traditional, fixed-labor value of $R^{fl} = 45$. Thus, the closed-form expressions in Section 3 seem to provide a good approximation to the household's wealth-gamble risk aversion in a standard model even far away from steady state.

It's also interesting that the household's wealth-gamble risk aversion is countercyclical with respect to the state variables k_t and Z_t. This can be seen most clearly in Fig. 5, which depicts the household's coefficient of absolute wealth-gamble risk aversion, $R^a(k_t; Z_t)$, over the same range of values for k_t and Z_t as in Fig. 4. The absolute wealth-gamble risk aversion coefficient of .09 implies that the household is willing to pay about 9 cents to avoid a fair gamble with a standard deviation of one dollar. This willingness to pay varies from about 7 to 12 cents over the range of values for the state variables in Fig. 5, with higher values of the states corresponding to higher household wealth and lower risk aversion.

Looking back at Fig. 4, relative wealth-gamble risk aversion is not countercyclical in that figure with respect to k_t because household wealth—and thus the size of the hypothetical gamble faced by the household—is increasing in k_t and Z_t. Indeed, for higher k_t, the increase in wealth is sufficiently large that the household's relative wealth-gamble risk aversion increases with k_t, even though absolute wealth-gamble risk aversion decreases.

6. Balanced growth

In previous sections, I abstracted from growth for simplicity, but the results carry through essentially unchanged to the case of balanced growth. I briefly collect the corresponding expressions in this section and provide proofs in the Appendix.

King et al. (1988, 2002) provide a detailed discussion of balanced growth. Along a balanced growth path, $x \in \{l, r\}$ satisfies $x_{t+k} = x_t$ for $k = 1, 2, \ldots$, and I drop the time subscript to denote the constant value. For $x \in \{a, c, w, d\}$, $x_{t+k} = G^k x_t$ for $k = 1, 2, \ldots$, for some $G \in (0, 1+r)$, and I use x_t^{bg} to denote the balanced growth path value. For notational simplicity, I denote the balanced growth path value of θ_t by θ_t^{bg}, although the elements of θ may grow at different constant rates over time (or remain constant).

Lemma 9. *Given Assumptions 1–7 and 8′, for all $k = 1, 2, \ldots$ along the balanced growth path: i) $\lambda_{t+k}^{bg} = G^{-k}\lambda_t^{bg}$, where λ_t^{bg} denotes the balanced growth path value of λ_t, ii) $\partial c_{t+k}^*/\partial a_t = G^k \partial c_t^*/\partial a_t$, iii) $\partial l_{t+k}^*/\partial a_t = \partial l_t^*/\partial a_t$, and iv) $\partial c_t^*/\partial a_t = (1+r-G)/(1+w_t^{bg}\lambda_t^{bg})$.*

Proof. See Appendix. □

Note that $w_t^{bg}\lambda_t^{bg}$ in Lemma 9 is constant over time because w and λ grow at reciprocal rates. The larger is G, the smaller is $\partial c_t^*/\partial a_t$, since the household chooses to absorb a greater fraction of asset shocks in future periods.

Proposition 10. *Given Assumptions 1–7 and 8′, absolute wealth-gamble risk aversion, evaluated along the balanced growth path, satisfies*

[48] The red lines do not intersect the black lines at the vertical axis because c_t^* and l_t^* evaluated at $k_t = k$ and $Z_t = Z$ do not equal the nonstochastic steady state values c and l due to the presence of uncertainty (e.g., precautionary savings).

$$R^a(a_t^{bg};\theta_t^{bg}) = \frac{-V_{11}(a_{t+1}^{bg};\theta_{t+1}^{bg})}{V_1(a_{t+1}^{bg};\theta_{t+1}^{bg})} + \alpha\frac{V_1(a_{t+1}^{bg};\theta_{t+1}^{bg})}{V(a_{t+1}^{bg};\theta_{t+1}^{bg})} \tag{59}$$

and

$$R^a(a_t^{bg};\theta_t^{bg}) = \frac{-u_{11}+\lambda_t^{bg}u_{12}}{u_1}\,\frac{\frac{1+r}{G}-1}{1+w_t^{bg}\lambda_t^{bg}} + \alpha\left(\frac{1+r}{G}-1\right)\frac{u_1}{u}, \tag{60}$$

where u_i and u_{ij} denote the corresponding partial derivatives of u evaluated at (c_t^{bg}, l). If $u(c_t,l_t) = \log c_t + v(\bar{l}-l_t)$ for some function v, then u in (60) must be interpreted to mean $\log c_t + v(\bar{l}-l_t) + \frac{\log G}{1-\frac{G}{1+r}}$.

Proof. See Appendix. □

Note that (60) agrees with Proposition 3 when $G = 1$. The larger is G, the smaller is R^a, since larger G implies greater household wealth and ability to absorb shocks to asset values.

Corollary 11. *Given Assumptions 1–7 and 8′, relative wealth-gamble risk aversion, evaluated along the balanced growth path, satisfies*

$$R^c(a_t^{bg};\theta_t^{bg}) = \frac{-u_{11}+\lambda_t^{bg}u_{12}}{u_1}\,\frac{c_t^{bg}}{1+w_t^{bg}\lambda_t^{bg}} + \alpha\frac{c_t^{bg}u_1}{u} \tag{61}$$

and

$$R^{cl}(a_t^{bg};\theta_t^{bg}) = \frac{-u_{11}+\lambda_t^{bg}u_{12}}{u_1}\,\frac{c_t^{bg}+w_t^{bg}(\bar{l}-l)}{1+w_t^{bg}\lambda_t^{bg}} + \alpha\frac{(c_t^{bg}+w_t^{bg}(\bar{l}-l))u_1}{u}. \tag{62}$$

If $u(c_t,l_t) = \log c_t + v(\bar{l}-l_t)$ for some function v, then u in (61)–(62) must be interpreted to mean $\log c_t + v(\bar{l}-l_t) + \frac{\log G}{1-\frac{G}{1+r}}$.

Proof. See Appendix. □

Thus, the expressions for relative wealth-gamble risk aversion are essentially unchanged by balanced growth.

7. Multiplier preferences

Multiplier preferences are a version of generalized recursive preferences defined by Hansen and Sargent (2001) and Strzalecki (2011). I briefly review those preferences here and derive the corresponding expressions for wealth-gamble risk aversion with labor.

Households with multiplier preferences order state-contingent consumption and labor plans according to the recursive functional

$$\widetilde{W}(c^t,l^t) = (1-\beta)\,u(c_t,l_t) - \beta\phi^{-1}\log E_t \exp\big(-\phi\widetilde{W}(c^{t+1},l^{t+1})\big), \tag{63}$$

rather than (3), where β is the household's discount factor and $\phi \in \mathbb{R}$. The preferences (63) can be regarded as a special case of (5), corresponding to $\rho = 0$. Let $W(a_t;\theta_t)$ denote the maximized value of (63), subject to (1)–(2):

$$W(a_t;\theta_t) = \max_{(c_t,l_t)\in\Gamma(a_t;\theta_t)} (1-\beta)\,u(c_t,l_t) - \beta\phi^{-1}\log E_t \exp\big(-\phi W(a_{t+1};\theta_{t+1})\big). \tag{64}$$

Hansen and Sargent (2001) show how (63)–(64) can be derived from microfoundations based on household optimization in the presence of concerns regarding model misspecification.[49] Maximizing (64) instead of expected utility ensures that the household achieves a reasonable discounted sum of utility flows for a range of empirically plausible processes for θ_t.

As ϕ approaches 0, (64) converges to expected utility. For $\phi \neq 0$, the intertemporal elasticity of substitution is the same as for expected utility, but the household's wealth-gamble risk aversion is amplified (or attenuated) by the additional curvature parameter ϕ.

From a practical perspective, an advantage of multiplier preferences is that they are well-defined even when u takes on both positive and negative values, so Assumption 2 can be dropped. Modifying the other assumptions and definitions to correspond to W rather than V is straightforward and gives the following:

[49] These microfoundations can be used to derive values of $\phi \geq 0$. The case $\phi < 0$, corresponding to risk-loving behavior, cannot be microfounded this way.

Proposition 12. *Let $(a_t;\theta_t)$ be an interior point of X. Given Assumptions 1 and 3–6, $\hat{W}(a_t;\theta_t;\sigma)$, $\mu(a_t;\theta_t;\sigma)$, and $R^a(a_t;\theta_t)$ exist, and*

$$R^a(a_t;\theta_t) = \frac{-E_t \exp\big(-\phi W(a^*_{t+1};\theta_{t+1})\big)\big[W_{11}(a^*_{t+1};\theta_{t+1}) - \phi W_1(a^*_{t+1};\theta_{t+1})^2\big]}{E_t \exp\big(-\phi W(a^*_{t+1};\theta_{t+1})\big)W_1(a^*_{t+1};\theta_{t+1})}. \tag{65}$$

Given Assumptions 7–8, (65) can be evaluated at the steady state to yield:

$$R^a(a;\theta) = \frac{-W_{11}(a;\theta)}{W_1(a;\theta)} + \phi W_1(a;\theta). \tag{66}$$

Proof. The proof follows along exactly the same lines as Proposition 1. □

Even though the preferences (64) can be derived from a concern for robustness rather than risk, the household acts in a way that is observationally equivalent to having higher wealth-gamble risk aversion. That is, if one confronts a Hansen–Sargent household with the hypothetical gamble in (7), the household's concerns about the stochastic process $\{\theta_t\}$ manifest themselves as an increased aversion to the gamble; as a result, the household behaves exactly as if it were certain about the economic environment but had a higher level of risk aversion governed by ϕ. Higher values of ϕ correspond to higher levels of wealth-gamble risk aversion, with sufficiently negative values of ϕ corresponding to risk-loving behavior.

Proposition 13. *Given Assumptions 1 and 3–8, the household's coefficient of absolute wealth-gamble risk aversion for multiplier preferences, evaluated at steady state, satisfies*

$$R^a(a;\theta) = \frac{-u_{11}+\lambda u_{12}}{u_1}\frac{r}{1+w\lambda} + \phi r u_1. \tag{67}$$

Proof. The proof follows along exactly the same lines as Proposition 3. □

Corollary 14. *Given Assumptions 1 and 3–8, the household's coefficients of relative wealth-gamble risk aversion for multiplier preferences, evaluated at steady state, satisfy*

$$R^c(a;\theta) = \frac{-u_{11}+\lambda u_{12}}{u_1}\frac{c}{1+w\lambda} + \phi c u_1 \tag{68}$$

and

$$R^{cl}(a;\theta) = \frac{-u_{11}+\lambda u_{12}}{u_1}\frac{c+w(\bar{l}-l)}{1+w\lambda} + \phi\big(c+w(\bar{l}-l)\big)u_1. \tag{69}$$

A feature of multiplier preferences worth emphasizing is that wealth-gamble risk aversion depends on ϕu_1 rather than $\phi u_1/u$. As a result, additive shifts of the period utility function u have no effect on wealth-gamble risk aversion, while multiplicative scalings of u *do* affect wealth-gamble risk aversion, in a way that is isomorphic to varying ϕ.[50]

Importantly, the expressions (67)–(69) *only* hold when the period utility function $u(c_t,l_t)$ is premultiplied by $(1-\beta)$, as in (63) and (64). Without that scaling factor, the second terms in (67)–(69) each need to be multiplied by $(1-\beta)^{-1}$. If $\beta = .99$, this is observationally equivalent to increasing ϕ by a factor of 100, an *enormous* increase in wealth-gamble risk aversion for what might seem like a simple renormalization. Note that some authors (e.g., Tallarini, 2000; Barillas et al., 2009) pre-multiply their period utility functions by a factor of $(1-\beta)$, while others (e.g., Boyarchenko, 2012; Bidder and Smith, 2013) do not, so there is no convention in the literature, and the discussion in these papers typically gives the reader no indication that wealth-gamble risk aversion differs by a factor of about 100 across the two alternatives. Thus, it's important that researchers bear in mind the dramatic effect different scalings of u have on wealth-gamble risk aversion and risk premia in these models.

Example 6. Tallarini (2000) considers the multiplier specification (64) with period utility

$$u(c_t,l_t) = \frac{1}{1+\xi}\log c_t + \frac{\xi}{1+\xi}\log(\bar{l}-l_t), \tag{70}$$

where $\xi \geq 0$.[51] The household's coefficients of relative wealth-gamble risk aversion are given by

[50] For standard Epstein–Zin–Weil preferences, it is the other way around: multiplicative transformations of u have no effect on wealth-gamble risk aversion, while additive shifts do have effects.

[51] Tallarini writes equation (64) in terms of $U_t \equiv W(a_t;\theta_t)(1-\beta)/(1+\xi)$. Rewriting this equation in the form (64) gives period utility function (70).

$$R^c(a;\theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1+w\lambda} + \phi c u_1 \; = \; \frac{1+\phi}{1+\xi} \tag{71}$$

and

$$R^{cl}(a;\theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c + w(\bar{l} - l)}{1+w\lambda} + \phi\big(c + w(\bar{l}-l)\big) u_1 \; = \; 1+\phi\,. \tag{72}$$

Neither of these equals the traditional, fixed-labor measure of risk aversion used by Tallarini,

$$R^{fl}(a;\theta) = \frac{-c u_{11}}{u_1} + \phi c u_1 \; = \; 1 + \frac{\phi}{1+\xi}\,. \tag{73}$$

This last measure ignores the fact that households will vary their labor endogenously in response to shocks. Note that $R^c \le R^{fl}$, as always, although in this particular example the difference is not very large quantitatively—Tallarini calibrates $\xi \approx 3$, implying $R^{fl} - R^c \approx 3/4$.

8. Discussion and conclusions

There are four main points to take away from this analysis. First, I've derived closed-form expressions for household wealth-gamble risk aversion when the household has generalized recursive preferences and can vary its labor supply. Traditional studies of risk aversion, such as Arrow (1965), Pratt (1964), Epstein and Zin (1989), and Weil (1989), assume household labor is fixed; as a result, the risk aversion coefficients in those studies are not representative of the household's aversion to holding risky assets when labor supply can vary. For reasonable parameterizations, the traditional, fixed-labor measure of risk aversion can overstate the household's actual aversion to money or wealth gambles by a factor of as much as ten, as in Fig. 3. Fixed-labor measures of risk aversion are also unrelated to the equity premium in a standard RBC model, as shown in Section 4, while the flexible-labor measure R^c derived in this paper is closely related.

Second, for Hansen and Sargent (2001)-type multiplier preferences, including vs. excluding a factor of $1-\beta$ in period utility leads to huge differences in wealth-gamble risk aversion, by a factor of about 100. Researchers must be very careful to account for any scale factor in utility when computing risk aversion in models with multiplier preferences.

Third, I've shown how the ICAPM generalizes to the case of endogenous labor. Expected excess returns on assets depend not only on their risk, but also on their correlation with the prices of future consumption and leisure.

Fourth and finally, the closed-form expressions for wealth-gamble risk aversion I derive above, and the methods of the paper more generally, are potentially useful for asset pricing in any dynamic model with multiple goods in the utility function. Models with home production, money in the utility function, or tradeable and nontradeable goods can imply very different household attitudes toward risk than traditional measures of risk aversion might suggest.

Appendix A. Proofs of propositions and numerical solution details

Proof of Proposition 1. Since $(a_t;\theta_t)$ is an interior point of X, $V(a_t + \frac{\sigma\underline{\varepsilon}}{1+r_t};\theta_t)$ and $V(a_t + \frac{\sigma\bar{\varepsilon}}{1+r_t};\theta_t)$ exist for sufficiently small σ, and $V(a_t + \frac{\sigma\underline{\varepsilon}}{1+r_t};\theta_t) \le \hat{V}(a_t;\theta_t;\sigma) \le V(a_t + \frac{\sigma\bar{\varepsilon}}{1+r_t};\theta_t)$, hence $\hat{V}(a_t;\theta_t;\sigma)$ exists. Moreover, since $V(\cdot;\cdot)$ is continuous and increasing in its first argument, the intermediate value theorem implies there exists a unique $-\mu(\sigma) \in [\sigma\underline{\varepsilon}, \sigma\bar{\varepsilon}]$ with $V(a_t - \frac{\mu(\sigma)}{1+r_t};\theta_t) = \hat{V}(a_t;\theta_t;\sigma)$.

For generalized recursive preferences, the household's first-order optimality conditions for c_t^* and l_t^*,

$$u_1(c_t^*, l_t^*) = \beta\big(E_t V(a_{t+1}^*;\theta_{t+1})^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V(a_{t+1}^*;\theta_{t+1})^{-\alpha} V_1(a_{t+1}^*;\theta_{t+1}), \tag{A.1}$$

$$u_2(c_t^*, l_t^*) = -\beta w_t\big(E_t V(a_{t+1}^*;\theta_{t+1})^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V(a_{t+1}^*;\theta_{t+1})^{-\alpha} V_1(a_{t+1}^*;\theta_{t+1}), \tag{A.2}$$

are slightly more complicated than the case of expected utility considered in Swanson (2012). Note that (A.1) and (A.2) are related by the usual $u_2(c_t^*, l_t^*) = -w_t u_1(c_t^*; l_t^*)$, and when $\alpha = 0$, (A.1) and (A.2) reduce to the standard optimality conditions for expected utility.

For an infinitesimal fee $d\mu$ in (8), the first-order change in household welfare (4) is given by

$$-V_1(a_t;\theta_t)\,\frac{d\mu}{1+r_t}. \tag{A.3}$$

Differentiating (6) with respect to a_t yields

$$\begin{aligned} V_1(a_t;\theta_t) = {} & u_1(c_t^*, l_t^*)\frac{\partial c_t^*}{\partial a_t} + u_2(c_t^*, l_t^*)\frac{\partial l_t^*}{\partial a_t} \\ & + \beta\big(E_t V(a_{t+1}^*;\theta_{t+1})^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V(a_{t+1}^*;\theta_{t+1})^{-\alpha} V_1(a_{t+1}^*;\theta_{t+1}) \left[(1+r_t) - \frac{\partial c_t^*}{\partial a_t} + w_t\frac{\partial l_t^*}{\partial a_t}\right]. \end{aligned} \tag{A.4}$$

Applying (A.1)–(A.2) to (A.4) gives the envelope theorem,

$$V_1(a_t;\theta_t) = \beta(1+r_t)\big(E_t V(a^*_{t+1};\theta_{t+1})^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V(a^*_{t+1};\theta_{t+1})^{-\alpha} V_1(a^*_{t+1};\theta_{t+1}) \tag{A.5}$$

and the Benveniste–Scheinkman equation (11),

$$V_1(a_t;\theta_t) = (1+r_t)u_1(c^*_t, l^*_t)\,. \tag{A.6}$$

From (A.5), (A.3) equals

$$-\beta\big(E_t V(a^*_{t+1};\theta_{t+1})^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V(a^*_{t+1};\theta_{t+1})^{-\alpha} V_1(a^*_{t+1};\theta_{t+1})\, d\mu\,. \tag{A.7}$$

Turning now to the gamble in (7), the household's optimal choices for consumption and labor in period t, c^*_t and l^*_t, will generally depend on the size of the gamble σ—for example, the household may undertake precautionary saving when faced with this gamble. Thus, in this section we write $c^*_t \equiv c^*(a_t;\theta_t;\sigma)$ and $l^*_t \equiv l^*(a_t;\theta_t;\sigma)$ to emphasize this dependence on σ. The household's value function, inclusive of the one-shot gamble in (7), satisfies

$$\hat{V}(a_t;\theta_t;\sigma) = u(c^*_t, l^*_t) + \beta E_t\big(V(a^*_{t+1};\theta_{t+1})^{1-\alpha}\big)^{1/(1-\alpha)}, \tag{A.8}$$

where $a^*_{t+1} \equiv (1+r_t)a_t + w_t l^*_t + d_t - c^*_t + \sigma\varepsilon_{t+1}$. Because (7) describes a one-shot gamble in period t, it affects assets a^*_{t+1} in period $t+1$ but does not appear in future periods' budget constraints and otherwise does not affect the household's optimization problem from period $t+1$ onward; as a result, the household's value-to-go at time $t+1$ is just $V(a^*_{t+1};\theta_{t+1})$, which does not depend on σ except through a^*_{t+1}.

Differentiating (A.8) with respect to σ, the first-order effect of the gamble on household welfare is:

$$\left[u_1\frac{\partial c^*}{\partial\sigma} + u_2\frac{\partial l^*}{\partial\sigma} + \beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_1\cdot\left(w_t\frac{\partial l^*}{\partial\sigma} - \frac{\partial c^*}{\partial\sigma} + \varepsilon_{t+1}\right)\right] d\sigma\,, \tag{A.9}$$

where the arguments of u_1, u_2, V, and V_1 are suppressed to simplify notation. Optimality of c^*_t and l^*_t implies that the terms involving $\partial c^*/\partial\sigma$ and $\partial l^*/\partial\sigma$ cancel, as in the usual envelope theorem (these derivatives vanish at $\sigma=0$ anyway, for the reasons discussed below). Moreover, $E_t V^{-\alpha} V_1 \varepsilon_{t+1} = 0$ because ε_{t+1} is independent of θ_{t+1} and a^*_{t+1}, evaluating the latter at $\sigma = 0$. Thus, the first-order cost of the gamble is zero, as in Arrow (1965) and Pratt (1964).

To second order, the effect of the gamble on household welfare is

$$\begin{aligned}
\Bigg\{ & u_{11}\left(\frac{\partial c^*}{\partial\sigma}\right)^2 + 2u_{12}\frac{\partial c^*}{\partial\sigma}\frac{\partial l^*}{\partial\sigma} + u_{22}\left(\frac{\partial l^*}{\partial\sigma}\right)^2 + u_1\frac{\partial^2 c^*}{\partial\sigma^2} + u_2\frac{\partial^2 l^*}{\partial\sigma^2} \\
& + \alpha\beta\big(E_t V^{1-\alpha}\big)^{(2\alpha-1)/(1-\alpha)}\left[E_t V^{-\alpha} V_1\cdot\left(w_t\frac{\partial l^*}{\partial\sigma} - \frac{\partial c^*}{\partial\sigma} + \varepsilon_{t+1}\right)\right]^2 \\
& - \alpha\beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V^{-\alpha-1}\left[V_1\cdot\left(w_t\frac{\partial l^*}{\partial\sigma} - \frac{\partial c^*}{\partial\sigma} + \varepsilon_{t+1}\right)\right]^2 \\
& + \beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_{11}\cdot\left(w_t\frac{\partial l^*}{\partial\sigma} - \frac{\partial c^*}{\partial\sigma} + \varepsilon_{t+1}\right)^2 \\
& + \beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_1\cdot\left(w_t\frac{\partial^2 l^*}{\partial\sigma^2} - \frac{\partial^2 c^*}{\partial\sigma^2}\right)\Bigg\}\frac{d\sigma^2}{2}\,.
\end{aligned} \tag{A.10}$$

The terms involving $\partial^2 c^*/\partial\sigma^2$ and $\partial^2 l^*/\partial\sigma^2$ cancel due to the optimality of c^*_t and l^*_t. The derivatives $\partial c^*/\partial\sigma$ and $\partial l^*/\partial\sigma$ vanish at $\sigma=0$ (there are two ways to see this: first, the linearized version of the model is certainty equivalent; alternatively, if the distribution of ε is symmetric about zero, the gamble in (7) is isomorphic for positive and negative σ, hence c^* and l^* must be symmetric about $\sigma=0$, implying the derivatives vanish). Finally, ε_{t+1} is independent of θ_{t+1} and a^*_{t+1}, evaluating the latter at $\sigma=0$. Since ε_{t+1} has unit variance, (A.10) reduces to

$$\beta\big(E_t V^{1-\alpha}\big)^{\alpha/(1-\alpha)}\left(E_t V^{-\alpha} V_{11} - \alpha E_t V^{-\alpha-1} V_1^2\right)\frac{d\sigma^2}{2}\,. \tag{A.11}$$

Equating (A.7) to (A.11) allows us to solve for $d\mu$ as a function of $d\sigma^2$. Thus, $\lim_{\sigma\to 0} 2\mu(\sigma)/\sigma^2$ exists and is given by

$$\frac{-E_t V^{-\alpha} V_{11} + \alpha E_t V^{-\alpha-1} V_1^2}{E_t V^{-\alpha} V_1}\,. \tag{A.12}$$

Since (A.12) is already evaluated at $\sigma=0$, to evaluate it at the nonstochastic steady state, set $a_{t+1} = a$ and $\theta_{t+1} = \theta$ to get

$$\frac{-V_{11}(a;\theta)}{V_1(a;\theta)} + \alpha \frac{V_1(a;\theta)}{V(a;\theta)}. \qquad \square \tag{A.13}$$

Proof of Lemma 2. Equations (A.1), (A.4), and the envelope theorem imply the household's intertemporal optimality (Euler) condition,

$$u_1(c_t^*, l_t^*) = \beta\big(E_t V(a_{t+1}^*;\theta_{t+1})^{1-\alpha}\big)^{\alpha/(1-\alpha)} E_t V(a_{t+1}^*;\theta_{t+1})^{-\alpha}(1+r_{t+1})\, u_1(c_{t+1}^*, l_{t+1}^*). \tag{A.14}$$

Differentiating (A.14) with respect to a_t at the nonstochastic steady state implies

$$u_{11}\left(\frac{\partial c_t^*}{\partial a_t} - E_t \frac{\partial c_{t+1}^*}{\partial a_t}\right) = -u_{12}\left(\frac{\partial l_t^*}{\partial a_t} - E_t \frac{\partial l_{t+1}^*}{\partial a_t}\right) \tag{A.15}$$

in a neighborhood of the steady state, where the arguments of the u_{ij} are suppressed to reduce notation. Using (14), this implies

$$(u_{11} - \lambda u_{12})\left(\frac{\partial c_t^*}{\partial a_t} - E_t \frac{\partial c_{t+1}^*}{\partial a_t}\right) = 0 \tag{A.16}$$

and thus

$$E_t \frac{\partial c_{t+1}^*}{\partial a_t} = \frac{\partial c_t^*}{\partial a_t}, \tag{A.17}$$

since $u_{11} - \lambda u_{12} < 0$ by concavity. Equations (A.14)–(A.17) can be iterated forward to yield

$$E_t \frac{\partial c_{t+k}^*}{\partial a_t} = \frac{\partial c_t^*}{\partial a_t}, \qquad k = 1, 2, \ldots, \tag{A.18}$$

whatever the initial response $\partial c_t^*/\partial a_t$. From (14) and (A.18), it also follows that

$$E_t \frac{\partial l_{t+k}^*}{\partial a_t} = \frac{\partial l_t^*}{\partial a_t}, \qquad k = 1, 2, \ldots \tag{A.19}$$

It remains to solve for $\partial c_t^*/\partial a_t$. The household's intertemporal budget constraint (1)–(2), evaluated at steady state, implies

$$\frac{1+r}{r}\frac{\partial c_t^*}{\partial a_t} = (1+r) + w\frac{1+r}{r}\frac{\partial l_t^*}{\partial a_t}. \tag{A.20}$$

Substituting (14) into (A.20) and solving for $\partial c_t^*/\partial a_t$ yields

$$\frac{\partial c_t^*}{\partial a_t} = \frac{r}{1+w\lambda}. \qquad \square \tag{A.21}$$

Proof of Lemma 5. Differentiating the household's Euler equation (A.14) at the nonstochastic steady state implies

$$u_{11}(dc_t^* - E_t dc_{t+1}^*) + u_{12}(dl_t^* - E_t dl_{t+1}^*) = \beta u_1 E_t dr_{t+1}, \tag{A.22}$$

which, applying (43), becomes

$$(u_{11} - \lambda u_{12})(dc_t^* - E_t dc_{t+1}^*) - \frac{u_1 u_{12}}{u_{22} + w u_{12}}(dw_t - E_t dw_{t+1}) = \beta u_1 E_t dr_{t+1}. \tag{A.23}$$

Note that (A.23) implies, for each $k = 1, 2, \ldots$,

$$E_t dc_{t+k}^* = dc_t^* - \frac{u_1 u_{12}}{u_{11}u_{22} - u_{12}^2}(dw_t - E_t dw_{t+k}) - \frac{\beta u_1}{u_{11} - \lambda u_{12}} E_t \sum_{i=1}^{k} dr_{t+i}. \tag{A.24}$$

Combining (1)–(2), differentiating, and evaluating at the nonstochastic steady state yields

$$E_t \sum_{k=0}^{\infty} \frac{1}{(1+r)^k}(dc_{t+k}^* - w\, dl_{t+k}^* - l\, dw_{t+k} - dd_{t+k} - a\, dr_{t+k}) = (1+r)\, da_t. \tag{A.25}$$

Substituting (43) and (A.24) into (A.25), and solving for dc_t^*, yields

$$dc_t^* = \frac{r}{1+r}\frac{1}{1+w\lambda}\left[(1+r)da_t + E_t\sum_{k=0}^{\infty}\frac{1}{(1+r)^k}(l\,dw_{t+k} + dd_{t+k} + adr_{t+k})\right]$$
$$+\frac{u_1u_{12}}{u_{11}u_{22}-u_{12}^2}dw_t + \frac{1}{1+r}\frac{-u_1}{u_{11}-\lambda u_{12}}E_t\sum_{k=0}^{\infty}\frac{1}{(1+r)^k}\left[\frac{r\lambda}{1+w\lambda}dw_{t+k} - \beta dr_{t+k+1}\right]. \quad \square \tag{A.26}$$

Proof of Lemma 6. Differentiating equation (6) and evaluating at the nonstochastic steady state implies

$$dV_t = u_1 dc_t^* + u_2 dl_t^* + \beta E_t dV_{t+1}. \tag{A.27}$$

Solving (A.27) forward and applying (43) yields

$$dV_t = \sum_{k=0}^{\infty}\beta^k u_1(1+w\lambda)E_t dc_{t+k}^* - \sum_{k=0}^{\infty}\beta^k\frac{u_1u_2}{u_{22}+wu_{12}}E_t dw_{t+k}\,. \tag{A.28}$$

Substituting (A.24) into (A.28) and simplifying yields

$$dV_t = \frac{1+r}{r}u_1(1+w\lambda)dc_t^* - \frac{1+r}{r}\frac{u_1^2u_{12}(1+w\lambda)}{u_{11}u_{22}-u_{12}^2}dw_t$$
$$+\sum_{k=0}^{\infty}\beta^k\frac{u_1(u_1u_{12}-u_2u_{11})}{u_{11}u_{22}-u_{12}^2}E_t dw_{t+k} - \frac{u_1^2(1+w\lambda)}{u_{11}-\lambda u_{12}}\frac{1}{1-\beta}\sum_{k=1}^{\infty}\beta^{k+1}E_t dr_{t+k}\,. \tag{A.29}$$

Substituting (A.26) into (A.29) and simplifying gives

$$dV_t = u_1(1+r)da_t + u_1E_t\sum_{k=0}^{\infty}\frac{1}{(1+r)^k}(ldw_{t+k} + dd_{t+k} + adr_{t+k}). \quad \square \tag{A.30}$$

Proof of Lemma 9. i) The household's Euler equation (A.14), evaluated along the (nonstochastic) balanced growth path, implies

$$u_1(c_t^{bg}, l) = \beta(1+r)u_1(c_{t+1}^{bg}, l) = \beta(1+r)u_1(Gc_t^{bg}, l). \tag{A.31}$$

Similarly, for labor,

$$u_2(c_t^{bg}, l) = \beta(1+r)\frac{w_t^{bg}}{w_{t+1}^{bg}}u_2(c_{t+1}^{bg}, l) = \beta(1+r)G^{-1}u_2(Gc_t^{bg}, l). \tag{A.32}$$

As in King et al. (2002), assume that preferences u are consistent with balanced growth for all initial asset stocks and wages in a neighborhood of a_t^{bg} and w_t^{bg}, and hence for all initial values of (c_t, l_t) in a neighborhood of (c_t^{bg}, l). Thus, (A.31) and (A.32) can be differentiated to yield:

$$u_{11}(c_t^{bg}, l) = \beta(1+r)G\,u_{11}(Gc_t^{bg}, l), \tag{A.33}$$
$$u_{12}(c_t^{bg}, l) = \beta(1+r)\,u_{12}(Gc_t^{bg}, l), \tag{A.34}$$
$$u_{22}(c_t^{bg}, l) = \beta(1+r)G^{-1}\,u_{22}(Gc_t^{bg}, l). \tag{A.35}$$

Substituting (A.33)–(A.35) into (15) gives

$$\lambda_{t+1}^{bg} = \frac{w_{t+1}^{bg}u_{11}(c_{t+1}^{bg}, l) + u_{12}(c_{t+1}^{bg}, l)}{u_{22}(c_{t+1}^{bg}, l) + w_{t+1}^{bg}u_{12}(c_{t+1}^{bg}, l)} = G^{-1}\lambda_t^{bg}\,, \tag{A.36}$$

ii) Assumptions 1–6 imply (11)–(15) in the text and the Euler equation (A.14). Hence

$$\left(u_{11}(c_t^{bg}, l) - \lambda_t^{bg}u_{12}(c_t^{bg}, l)\right)\frac{\partial c_t^*}{\partial a_t} = \beta(1+r)\left(u_{11}(c_{t+1}^{bg}, l) - \lambda_{t+1}^{bg}u_{12}(c_{t+1}^{bg}, l)\right)\frac{\partial c_{t+1}^*}{\partial a_t}\,. \tag{A.37}$$

Solving for $\partial c_{t+1}^*/\partial a_t$ and using (A.33)–(A.36) yields $\partial c_{t+1}^*/\partial a_t = G\,\partial c_t^*/\partial a_t$.

iii) Follows from (14), (A.33)–(A.36), and ii).

iv) Use the household's budget constraint (1)–(2) and ii) to solve for $\partial c_t^*/\partial a_t$. $\square$

Proof of Proposition 10. Proposition 1 implies (59). Assumptions 1–6 imply (11)–(15). Substituting (11)–(14) and Lemma 9(iv) into (59) gives

$$R^a(a_t^{bg};\theta_t^{bg}) = \frac{-u_{11}(c_{t+1}^{bg},l)+\lambda_{t+1}^{bg}u_{12}(c_{t+1}^{bg},l)}{u_1(c_{t+1}^{bg},l)}\,\frac{1+r-G}{1+w_{t+1}^{bg}\lambda_{t+1}^{bg}}+\alpha\,\frac{(1+r)u_1(c_{t+1}^{bg},l)}{V(a_{t+1}^{bg};\theta_{t+1}^{bg})}\,. \tag{A.38}$$

Expressing $V(a_{t+1}^{bg};\theta_{t+1}^{bg})$ in terms of period utility u is made slightly more complicated by the presence of balanced growth, since now the arguments of u are not constant but rather grow over time.

King et al. (1988, 2002) show that, to be consistent with balanced growth, $u(c_t,l_t)$ must have the functional form

$$u(c_t,l_t) = \frac{c_t^{1-\gamma}}{1-\gamma}\,v(\bar{l}-l_t) \tag{A.39}$$

or, as $\gamma\to 1$,

$$u(c_t,l_t) = \log c_t + v(\bar{l}-l_t), \tag{A.40}$$

where $v(\cdot)$ in (A.39) or (A.40) is differentiable, increasing, and concave, but otherwise unrestricted. Since the balanced growth path is nonstochastic, the allowable functional forms for $u(c_t,l_t)$ are the same for the case of generalized recursive preferences as they are for expected utility.

If u has the form (A.39), then

$$V(a_t^{bg};\theta_t^{bg}) = \frac{1}{1-\beta G^{1-\gamma}}\,u(c_t^{bg},l) \tag{A.41}$$

and

$$\beta V(a_{t+1}^{bg};\theta_{t+1}^{bg}) = V(a_t^{bg};\theta_t^{bg}) - u(c_t^{bg},l) = \frac{\beta G^{1-\gamma}}{1-\beta G^{1-\gamma}}\,u(c_t^{bg},l). \tag{A.42}$$

Moreover, $\beta(1+r)=G^\gamma$. Substituting (A.31), (A.33)–(A.36), and (A.42) into (A.38) then completes the proof.

If u has the form (A.40), then

$$V(a_t^{bg};\theta_t^{bg}) = \frac{1}{1-\beta}\,u(c_t^{bg},l) + \frac{\beta}{(1-\beta)^2}\log G, \tag{A.43}$$

$$\beta V(a_{t+1}^{bg};\theta_{t+1}^{bg}) = \frac{\beta}{1-\beta}\,u(c_t^{bg},l) + \frac{\beta}{(1-\beta)^2}\log G, \tag{A.44}$$

and $\beta(1+r)=G$. Substituting (A.31), (A.33)–(A.36), and (A.44) into (A.38) yields

$$R^a(a_t^{bg};\theta_t^{bg}) = \frac{-u_{11}+\lambda_t^{bg}u_{12}}{u_1}\,\frac{\frac{1+r}{G}-1}{1+w_t^{bg}\lambda_t^{bg}}+\alpha\left(\frac{1+r}{G}-1\right)\frac{u_1}{u+\frac{1+r}{1+r-G}\log G}\,. \tag{A.45}$$

This differs from (60) by the addition of the constant term $\frac{\log G}{1-\frac{G}{1+r}}$ to u. Thus, in the case of log preferences, u in (60) must be interpreted to include the additive constant $\frac{\log G}{1-\frac{G}{1+r}}$. □

Proof of Corollary 11. As in Definitions 2–3, define wealth A_t^{bg} in beginning- rather than end-of-period-t units; this requires multiplying by $(1+r)^{-1}G$ rather than just $(1+r)^{-1}$. Then the present discounted value of consumption along the balanced growth path is given by $A_t^{bg}=c_t^{bg}/\big(\frac{1+r}{G}-1\big)$, and the present discounted value of leisure by $w_t^{bg}(\bar{l}-l)/\big(\frac{1+r}{G}-1\big)$. Substituting these into Proposition 10 completes the proof. □

Numerical solution procedure for Sections 4–5

The equations of the RBC model itself are standard:

$$Y_t = Z_t K_{t-1}^{1-\theta} L_t^{\theta}, \tag{A.46}$$

$$K_t = (1-\delta)K_{t-1} + Y_t - C_t, \tag{A.47}$$

$$\eta L_t^{\chi}/C_t^{-\gamma} = w_t, \tag{A.48}$$

$$r_t = (1-\theta)Y_t/K_{t-1} - \delta, \tag{A.49}$$

$$w_t = \theta Y_t/L_t, \tag{A.50}$$

$$\log Z_t = \rho \log Z_{t-1} + \varepsilon_t, \tag{A.51}$$

where, for concreteness, I use the additively separable preference specification from Examples 3 and 5 in (A.48) and throughout this section. In equations (A.46)–(A.51), note that K_{t-1} denotes the capital stock at the beginning of period t (or the end of period $t-1$), so the notation differs slightly from the main text for compatibility with the numerical algorithm below.

Because of the generalized recursive structure of household preferences, the household's Euler equation (A.14) involves the value function. Following Rudebusch and Swanson (2012), I use two equations to compute the value function in the model, as follows:

$$V_t = \frac{C_t^{1-\gamma}}{1-\gamma} - \eta \frac{L_t^{1+\chi}}{1+\chi} + \beta \, \mathrm{VTWIST}_t^{1/(1-\alpha)}, \tag{A.52}$$

$$\mathrm{VTWIST}_t = E_t V_{t+1}^{1-\alpha}. \tag{A.53}$$

The household's Euler equation (A.14) then can be written as

$$C_t^{-\gamma} = \beta E_t (1+r_{t+1}) \left(V_{t+1} / \mathrm{VTWIST}_t^{1/(1-\alpha)} \right)^{-\alpha} C_{t+1}^{-\gamma}. \tag{A.54}$$

To compute wealth-gamble risk aversion in the model, we can append the following auxiliary variables and equations to the system (A.46)–(A.54)[52]:

$$\lambda_t = (\gamma/\chi) L_t / C_t, \tag{A.55}$$

$$\mathrm{CARA}_t = \frac{E_t V_{t+1}^{-\alpha} \left[(1+r_{t+1})(\gamma C_{t+1}^{-\gamma-1} \mathrm{DCDA}_{t+1}) + \alpha (1+r_{t+1})^2 C_{t+1}^{-2\gamma} / V_{t+1} \right]}{\mathrm{V1EXP}_t}, \tag{A.56}$$

$$\mathrm{V1EXP}_t = E_t V_{t+1}^{-\alpha} (1+r_{t+1}) C_{t+1}^{-\gamma}. \tag{A.57}$$

$$\begin{aligned} -\gamma C_t^{-\gamma-1} \mathrm{DCDA}_t = \beta E_t \big[& -\gamma \mathrm{VTWIST}_t^{\alpha/(1-\alpha)} V_{t+1}^{-\alpha} (1+r_{t+1}) C_{t+1}^{-\gamma-1} \mathrm{DCDA}_{t+1} \\ & - \alpha \mathrm{VTWIST}_t^{\alpha/(1-\alpha)} V_{t+1}^{-\alpha-1} (1+r_{t+1})^2 C_{t+1}^{-2\gamma} \\ & + \alpha \mathrm{VTWIST}_t^{(\alpha/(1-\alpha))-1} \mathrm{V1EXP}_t^2 \big] \big[(1+r_t) - (1+w_t\lambda_t) \mathrm{DCDA}_t \big], \end{aligned} \tag{A.58}$$

$$\mathrm{PDVC}_t = C_t + \beta E_t C_{t+1}^{-\gamma} / C_t^{-\gamma} \left(V_{t+1} / \mathrm{VTWIST}_t^{1/(1-\alpha)} \right)^{-\alpha} \mathrm{PDVC}_{t+1}. \tag{A.59}$$

$$\mathrm{CRRA}_t = \mathrm{CARA}_t \mathrm{PDVC}_t / (1+r_t). \tag{A.60}$$

Equation (A55) corresponds to (14), (A.56)–(A.57) to Proposition 1, and (A.59)–(A.60) to Definition 2. Equations (11)–(12) are used to rewrite V_1 and V_{11} in terms of derivatives of u. The variable DCDA_t corresponds to $\partial c_t^* / \partial a_t$; equation (A.58) is the derivative of (A.14) with respect to a_t, which determines how $\partial c_t^* / \partial a_t$ evolves over time. Note that

$$\frac{\partial c_{t+1}^*}{\partial a_t} = \frac{\partial c_{t+1}^*}{\partial a_{t+1}^*} \left[(1+r_t) - w_t \lambda_t \frac{\partial c_t^*}{\partial a_t} - \frac{\partial c_t^*}{\partial a_t} \right], \tag{A.61}$$

which is used in (A.58).

We can then solve the system of equations (A.46)–(A.60) numerically using the Perturbation AIM algorithm of Swanson et al. (2006) to compute a fifth-order Taylor series approximate solution around the nonstochastic steady state. These nth-order Taylor series approximations are guaranteed to be arbitrarily accurate in a neighborhood of the nonstochastic steady state, but importantly also converge globally within the domain of convergence of the Taylor series as the order of the approximation n becomes large. In practice, the solution seemed to converge globally over the range of values considered for the state variables in Figs. 1–5 by about the third or fourth order, so solutions higher than the fifth order are not reported. Aruoba et al. (2006) solve a standard real business cycle model like (A.46)–(A.60) using a variety of numerical methods, including second- and fifth-order perturbation methods, and find that the perturbation solutions are among the most accurate methods globally, as well as being the fastest to compute.

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[52] These are somewhat more complicated versions of the equations in Swanson (2012), due to the presence of generalized recursive preferences in the present paper.

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