Optimal Time-Consistent Monetary Policy in the New Keynesian Model with Repeated Simultaneous Play

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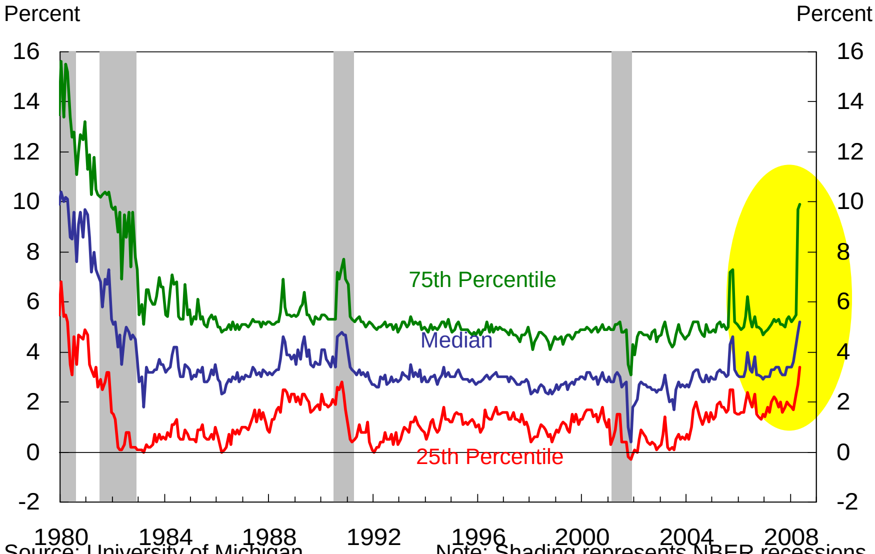
Summary

- There are two definitions of "discretion" in the literature
- These definitions differ in terms of within-period timing of play
- Within-period timing has *major* equilibrium implications
- In the New Keynesian model with repeated Stackelberg play, there are multiple equilibria (King-Wolman, 2004)
- In the New Keynesian model with repeated simultaneous play, there is a unique equilibrium (this paper)
- Empirical relevance: Will the 1970s repeat itself?

Background and Motivation



Source: University of Michigan

Note: Shading represents NBER recessions.

Background and Motivation

Time-consistent (discretionary) policy: Kydland and Prescott (1977)

There are multiple equilibria under discretion:

- Barro and Gordon (1983)
- Chari, Christiano, Eichenbaum (1998)

Critiques of the Barro-Gordon/CEE result:

- enormous number, range of equilibria make theory impossible to test or reject
- equilibria require fantastic sophistication, coordination across continuum of atomistic agents

Background and Motivation

Literature has thus changed focus to *Markov perfect equilibria*:

- Albanesi, Chari, Christiano (2003)
- King and Wolman (2004)

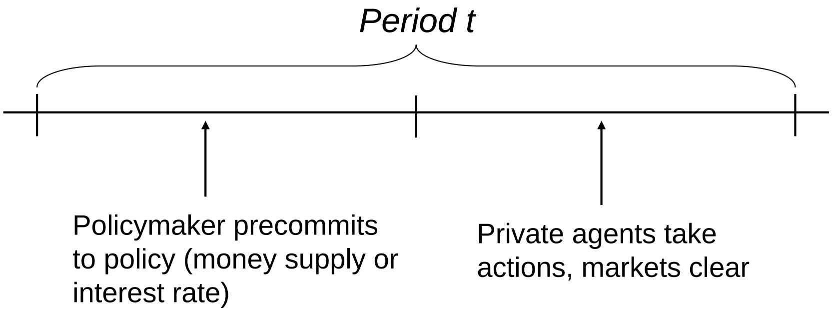
King and Wolman (2004):

- standard New Keynesian model
- assume repeated Stackelberg within-period play
- there are two Markov perfect equilibria

But recall LQ literature:

- Svensson-Woodford (2003, 2004), Woodford (2003)
- Pearlman (1994)
- assume repeated simultaneous within-period play

Repeated Stackelberg Play



- Repeated Stackelberg like “within-period commitment”?
- But policymakers’ actions are much more restricted
- Our results suggest policymaker actually has greater control with repeated simultaneous timing assumption

Comparison: Fiscal Policy

Cohen and Michel (1988), Ortigueira (2005):

- two definitions of discretion in the tax literature
- Brock-Turnovsky (1980), Judd (1998): repeated simultaneous
- Klein, Krusell, Rios-Rull (2004): repeated Stackelberg
- different timing assumption lead to different equilibria, welfare

In this paper:

- defining repeated simultaneous play is more subtle: Walras
- timing assumption changes not just payoffs, welfare, but *multiplicity* of equilibria

The Game Γ_0

Discretion is a game between private sector and central bank

For clarity, begin definition of game without central bank:

- assume interest rate process $\{r_t\}$ is i.i.d.
- call this game Γ_0

Game Γ_0:

- time is discrete, continues forever
- Γ_0 begins at t_0, but inherits history h^{t_0}
- define:
  - players
  - payoffs
  - information sets
  - action spaces

Game Γ_0: Players and Payoffs

1. Firms indexed by $i \in [0, 1]$:

produce differentiated products; face Dixit-Stiglitz demand curves; have production function $y_t(i) = l_t(i)$; hire labor at wage rate w_t; payoff each period is profit:

$$\Pi_t(i) = p_t(i) y_t(i) - w_t l_t(i)$$

2. Households indexed by $j \in [0, 1]$:

supply labor $L_t(j)$; consume final good $C_t(j)$; borrow or lend a one-period nominal bond $B_t(j)$; payoff each period is utility flow:

$$u(C_s(j), L_s(j)) = \frac{C_s(j)^{1-\varphi} - 1}{1 - \varphi} - \chi_0 \frac{L_s(j)^{1+\chi}}{1 + \chi}$$

Note: there is a final good aggregator that is *not* a player of Γ_0

Game Γ_0: Information Sets

Individual households and firms are anonymous:

- only aggregate variables and aggregate outcomes are publicly observed

Information set of each firm i at time t is thus:

- history of aggregate outcomes: $\{C_s, L_s, P_s, r_s, w_s, \Pi_s\}$, $s < t$
- history of firm i's own actions

Information set of each household j at time t is thus:

- history of aggregate outcomes: $\{C_s, L_s, P_s, r_s, w_s, \Pi_s\}$, $s < t$
- history of household j's own actions

Aggregate Resource Constraints

In games of industry competition:

- Bertrand
- Cournot
- Stackelberg

Action spaces are just real numbers: e.g., price, quantity

In a macroeconomic game, there are aggregate resource constraints that must be respected, e.g.:

- total labor supplied by households must equal total labor demanded by firms
- total output supplied by firms must equal total consumption demanded by households
- money supplied by central bank must equal total money demanded by households (in game Γ_1)

Walrasian Auctioneer

To ensure that aggregate resource constraints are respected, we introduce a Walrasian auctioneer

- Instead of playing a price p_t, firms now play a price *schedule* $p_t(X_t)$, where X_t denotes aggregate variables realized at t
- this is just the usual NK assumption that firms take wages, interest rate, aggregates at time t as given

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Walrasian auctioneer then determines the equilibrium X_t that satisfies aggregate resource constraints

Game Γ_0: Action Spaces

1. Firms
   - set prices for two periods in Taylor contracts; must supply whatever output is demanded at posted price
   - firms in $[0, 1/2)$:
     for t odd, action space is set of measurable functions $p_t(X_t)$
     for t even, action space is trivial
   - firms in $[1/2, 1)$:
     for t even, action space is set of measurable functions $p_t(X_t)$
     for t odd, action space is trivial
2. Households
   - in each period, action space is set of measurable functions $(C_t(X_t), L_t(X_t))$

Game Γ_0: Action Spaces

Note:

- all firms *i* and households *j* play simultaneously in each period *t*
- Walrasian auctioneer clears markets, aggregate resource constraints

Also, do not confuse *action spaces* here with *strategies*:

- a *strategy* is a mapping from history h^t to the action space
- here, action spaces are functions of aggregate variables realized at *t*
- but strategies are unrestricted, may depend on arbitrary history of aggregate variables (until we impose Markovian restriction)

Game Γ_0: Firm Optimality Conditions

Each firm that resets price faces a standard NK optimal pricing condition:

$$p_t^*(i) = (1+\theta)\,\frac{E_{it}P_t^{(1+\theta)/\theta}Y_t w_t + E_{it}Q_{t,t+1}P_{t+1}^{(1+\theta)/\theta}Y_{t+1}w_{t+1}}{E_{it}P_t^{(1+\theta)/\theta}Y_t + E_{it}Q_{t,t+1}P_{t+1}^{(1+\theta)/\theta}Y_{t+1}},$$

$$= (1+\theta)\,\frac{P_t^{(1+\theta)/\theta}Y_t w_t + E_t Q_{t,t+1}P_{t+1}^{(1+\theta)/\theta}Y_{t+1}w_{t+1}}{P_t^{(1+\theta)/\theta}Y_t + E_t Q_{t,t+1}P_{t+1}^{(1+\theta)/\theta}Y_{t+1}}.$$

$E_{it} \rightarrow E_t$ because firm can play functions of variables dated t

Game Γ_0: Household Optimality Conditions

Each household j faces a standard dynamic programming problem with initial bond holdings $B_{t-1}(j)$.

Optimality conditions are standard:

$$\begin{aligned}
C_t^*(j)^{-\varphi} &= E_{jt}\beta(1+r_t)\frac{P_t}{P_{t+1}}\,C_{t+1}^*(j)^{-\varphi}, \\
\chi_0 L_t^*(j)^{\chi} &= E_{jt}\frac{w_t}{P_t}C_t^*(j)^{-\varphi}, \\
E_{jt}\sum_{T=t}^{\infty} R_{t,T}P_T C_T^*(j) &= B_{t-1}(j) + E_{jt}\sum_{T=t}^{\infty} R_{t,T}\left[w_T L_T^*(j) + \Pi_T\right],
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\end{aligned}
$$

Note: $E_{jt} \rightarrow E_t$ once we establish symmetry across households

Private Sector Equilibrium and Markov Equilibrium

3. Private Sector Equilibrium and Markov Perfect Equilibrium
   - Private Sector Equilibrium
   - State Variables of the Game Γ_0
   - Markov Perfect Equilibrium in the Game Γ_0
   - Markov Perfect Equilibrium Conditions

Game Γ_0: Private Sector Equilibrium

Definition 1: Given the i.i.d. stochastic process for $\{r_t\}$ and initial conditions $p_{t_0-1}(i)$ and $B_{t_0-1}(j)$ for all firms i and households j, we define a Private Sector Equilibrium (PSE) to be a subgame perfect equilibrium of the game Γ_0.

In particular, a PSE implies a collection of stochastic processes for $\{L_t, r_t, P_t, w_t, Y_t, \Pi_t, Q_{t,t+1}, p_t(i), y_t(i), l_t(i), C_t(j), L_t(j), B_t(j)\}$ for $t \geq t_0$ and for all i, j that satisfy: (i) the price optimality condition (14) of the firm's maximization problem; (ii) the consumption and labor optimality conditions (15)–(17) of the household's maximization problem; (iii) the Dixit-Stiglitz aggregation and demand conditions (7)–(9) of the competitive goods aggregator; and (iv) the aggregate resource constraints (10)–(12) imposed by the Walrasian auctioneer.

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Game Γ_0: State Variables

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- distribution of household bond holdings, $B_{t-1}(j)$, $j \in [0, 1]$
- two measures of the distribution of inherited prices:

$$\int p_{t-1}(i)^{-1/\theta}\, di$$

and

$$\int p_{t-1}(i)^{-(1+\theta)/\theta}\, di$$

Game Γ_0: State Variables

Proposition 1: Suppose that $B_{t-1}(j)$ is the same for all households $j \in [0, 1]$ except possibly a set S of measure zero. Then the optimal action $(C_t^*(j), L_t^*(j)) \in L(\Omega, \mathbb{R}_+^2)$ is the same for every household $j \notin S$. We denote this optimal action by (C_t^*, L_t^*).

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Proof: The household optimality conditions:

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\begin{aligned}
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\end{aligned}
$$

for households j_1 and j_2 are identical if $B_{t-1}(j_1) = B_{t-1}(j_2)$.

Game Γ_0: State Variables

Proposition 2: The optimal choice of price schedule $p_t^*(i) \in L(\Omega, \mathbb{R}_+)$ is the same for all firms i that reset price in period t. We denote this optimal price schedule, given by (14), by p_t^*.

Game Γ_0: State Variables

Proposition 2: The optimal choice of price schedule $p_t^*(i) \in L(\Omega, \mathbb{R}_+)$ is the same for all firms i that reset price in period t. We denote this optimal price schedule, given by (14), by p_t^*.

Proof: The right-hand side of firm optimality condition:

$$p_t^*(i) = (1+\theta)\frac{P_t^{(1+\theta)/\theta} Y_t w_t + E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1} w_{t+1}}{P_t^{(1+\theta)/\theta} Y_t + E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1}},$$

is identical for all firms i.

Game Γ_0: State Variables

Starting from symmetric initial conditions in period t_0:

- Propositions 1 and 2 show that the distributions $B_{t-1}(\cdot)$ and $p_{t-1}(\cdot)$ are degenerate for all times $t \geq t_0$ along the equilibrium path in any subgame perfect equilibrium of Γ_0
- We henceforth restrict definition of game Γ_0 to case of symmetric initial conditions in period t_0

Note: we will not write out how play evolves off of the equilibrium path (if a positive measure of firms or households were to deviate), but simply assert that agents will continue to play according to their optimality conditions (Phelan-Stachetti, 2001)

Game Γ_0: Markov Perfect Equilibrium

Definition 2: A Markov Perfect Equilibrium (MPE) of the game Γ_0 is a set of strategies for households and firms that, at each date t, depend only on the state variables of Γ_0 at time t, and yield a Nash equilibrium in every proper subgame of Γ_0.

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Note:

- state variables of general game correspond to coarsest partition of original game tree into equivalence classes that preserve payoffs and action spaces (Fudenberg-Tirole, 1993)
- for Γ_0, can define action spaces, payoffs in real terms
- normalize Γ_0 by p_{t-1}

Game Γ_0: Markov Perfect Equilibrium Conditions

Now consolidate and simplify necessary conditions for an MPE.

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First:

$$P_t = \left[\int_0^1 p_t(i)^{-1/\theta} di\right]^{-\theta} \iff \frac{p_t}{P_t} = 2^{-\theta}\left(1 + x_t^{1/\theta}\right)^{\theta}.$$

Game Γ_0: Markov Perfect Equilibrium Conditions

Then, consolidating necessary conditions yields:

$$\int_0^1 l_t(i)di = L_t \iff \frac{L_t}{Y_t} = 2^{\theta} \frac{1 + x_t^{(1+\theta)/\theta}}{\left(1 + x_t^{1/\theta}\right)^{1+\theta}},$$

$$\text{firm optimality} \iff 2^{-\theta}\left(1 + x_t^{1/\theta}\right)^{\theta} = (1+\theta)\frac{\chi_0\left[Y_t L_t^{\chi} + \beta\left(1 + x_t^{1/\theta}\right)^{1+\theta} h_{1t}\right]}{Y_t^{1-\varphi} + \beta(1 + x_t^{1/\theta})h_{2t}},$$

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$$h_{1t} \equiv E_t \frac{Y_{t+1} L_{t+1}^{\chi}}{\left(1 + x_{t+1}^{-1/\theta}\right)^{1+\theta}},$$

$$h_{2t} \equiv E_t \frac{Y_{t+1}^{1-\varphi}}{1 + x_{t+1}^{-1/\theta}},$$

$$h_{3t} \equiv E_t Y_{t+1}^{-\varphi}(1 + x_{t+1}^{-1/\theta}),$$

Game Γ_0: Markov Perfect Equilibrium Conditions

Proposition 4: Given the i.i.d. stochastic process for $\{r_t\}$ *and symmetric initial conditions* $p_{t_0-1}(i) = p_{t_0-1} \in \mathbb{R}_+$ *and* $B_{t_0-1}(j) = 0$ *for all firms i and households j, necessary conditions for an equilibrium path of a Markov Perfect Equilibrium (MPE) of the game* Γ_0 *are that, for all* $t \geq t_0$*: (i)* (L_t, x_t, Y_t) *satisfy households' and firms' optimality conditions (19)-(21), taking* r_t *and* (h_{1t}, h_{2t}, h_{3t}) *as given; (ii)* (h_{1t}, h_{2t}, h_{3t}) *satisfy conditions (22)-(24) for rational expectations; and (iii) households' and firms' strategies along the equilibrium path are independent of history and independent of time.*

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Game Γ_0: Markov Perfect Equilibrium Conditions

Proposition 5: Along the equilibrium path of a Markov Perfect Equilibrium of the game Γ_0, there exist positive real numbers h_1, h_2, and h_3 such that $(h_{1t}, h_{2t}, h_{3t}) = (h_1, h_2, h_3)$ for all times t.

Game Γ_0: Markov Perfect Equilibrium Conditions

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Proof:

- h_{1t}, h_{2t}, h_{3t} are conditional expectations of variables in $t+1$
- variables in $t+1$ depend only on variables dated $t+1$ or later
- r_t is i.i.d. over time
- no sunspots or time-dependence (Markov)
- $\implies h_{1t}$, h_{2t}, h_{3t} are the same in every period t

Game Γ_0: Markov Perfect Equilibrium Conditions

Proposition 5: Along the equilibrium path of a Markov Perfect Equilibrium of the game Γ_0, there exist positive real numbers h_1, h_2, and h_3 such that $(h_{1t}, h_{2t}, h_{3t}) = (h_1, h_2, h_3)$ for all times t.

Proof:

- h_{1t}, h_{2t}, h_{3t} are conditional expectations of variables in $t+1$
- variables in $t+1$ depend only on variables dated $t+1$ or later
- r_t is i.i.d. over time
- no sunspots or time-dependence (Markov)
- $\implies$ h_{1t}, h_{2t}, h_{3t} are the same in every period t

Note that this does not rule out the possibility of multiple MPE:

- there may be multiple sets of (h_1, h_2, h_3) each of which can support an MPE
- any given (h_1, h_2, h_3) may be able to support multiple MPE

Game Γ_0: Markov Perfect Equilibrium Conditions

Proposition 6: Let $(L_{t_1}, x_{t_1}, Y_{t_1}, h_{1t_1}, h_{2t_1}, h_{3t_1}, r_{t_1})$ *and* $(L_{t_2}, x_{t_2}, Y_{t_2}, h_{1t_2}, h_{2t_2}, h_{3t_2}, r_{t_2})$ *lie on the equilibrium path of an MPE of* Γ_0. *Then* $(L_{t_1}, x_{t_1}, Y_{t_1}, h_{1t_1}, h_{2t_1}, h_{3t_1}, r_{t_1}) = (L_{t_2}, x_{t_2}, Y_{t_2}, h_{1t_2}, h_{2t_2}, h_{3t_2}, r_{t_2})$.

That is, along the equilibrium path, any MPE of Γ_0 *must be constant over time.*

Game Γ_0: Markov Perfect Equilibrium Conditions

Proposition 6: Let $(L_{t_1}, x_{t_1}, Y_{t_1}, h_{1t_1}, h_{2t_1}, h_{3t_1}, r_{t_1})$ *and* $(L_{t_2}, x_{t_2}, Y_{t_2}, h_{1t_2}, h_{2t_2}, h_{3t_2}, r_{t_2})$ *lie on the equilibrium path of an MPE of* Γ_0*. Then* $(L_{t_1}, x_{t_1}, Y_{t_1}, h_{1t_1}, h_{2t_1}, h_{3t_1}, r_{t_1}) = (L_{t_2}, x_{t_2}, Y_{t_2}, h_{1t_2}, h_{2t_2}, h_{3t_2}, r_{t_2})$*.*

That is, along the equilibrium path, any MPE of Γ_0 *must be constant over time.*

Proof:

- household, firm strategies are independent of history, time
- h_1, h_2, and h_3 are independent of time (Prop. 5)
- $\implies$ any MPE is independent of time.

The Game Γ_1

Now, extend the game Γ_0 to include an optimizing central bank:

- interest rate r_t is set by central bank each period
- call this game Γ_1

First two sets of players (firms and households) are defined exactly as in Γ_0

Game Γ_1: Central Bank

3. Central bank:

sets one-period nominal interest rate r_t; payoff each period is given by average household welfare:

$$\int \frac{C_s(j)^{1-\varphi} - 1}{1-\varphi} - \chi_0 \frac{L_s(j)^{1+\chi}}{1+\chi}\, dj$$

Central bank's information set is the history of aggregate outcomes: $\{C_s, L_s, P_s, r_s, w_s, \Pi_s\}$, $s < t$

Note:

- central bank has no ability to commit to future actions (discretion)
- central bank is *monolithic,* while private sector is *atomistic*

Within-Period Timing of Play

Repeated Stackelberg play:

- each period divided into two halves
- first, central bank precommits to a value for r_t (or m_t)
- second, firms and households play simultaneously
- Walrasian auctioneer determines equilibrium

Repeated simultaneous play:

- firms, households, and central bank all play simultaneously
- Walrasian auctioneer determines equilibrium

Simultaneous Play: Example

Linearized New Keynesian model:

$$\begin{aligned} y_t &= E_t y_{t+1} - \alpha r_t \\ \pi_t &= \beta E_t \pi_{t+1} + \gamma y_t \end{aligned}$$

Under repeated simultaneous play, a Taylor rule is valid:

$$r_t = a\pi_t + by_t$$

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Under repeated Stackelberg play, corresponding rule would be:

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although note that this rule is not Markov (model has no state variables).

Why Assume Simultaneous Play?

Practical considerations/realism:

- Makes no difference whether monetary instrument is r_t or m_t
- Central banks monitor economic conditions continuously, adjust policy as needed

Theoretical considerations:

- Why treat central bank, private sector so asymmetrically?
- LQ literature (Svensson-Woodford 2003, 2004, Woodford 2003, Pearlman 1994, etc.) assumes simultaneous play
- Investigate sensitivity of multiple equilibria to within-period timing

Game Γ_1: Action Spaces

In defining the game Γ_1, we assume repeated simultaneous play:

- firms i, households j, and central bank all play simultaneously in each period t
- action spaces of firms, households are same as in Γ_0
- for central bank, action space each period is set of measurable functions $r_t(X_t)$ (simultaneous play)
- Walrasian auctioneer clears markets, aggregate resource constraints

Again, do not confuse action spaces with strategies:

- strategies are unrestricted, may depend on arbitrary history of aggregate variables (until we impose Markovian restriction)

Policymaker Bellman Equation

$$V_t = \max_{\{r_t\}} \left\{ \frac{Y_t^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\}$$

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subject to:

$$\frac{L_t}{Y_t} = 2^{\theta} \frac{1 + x_t^{(1+\theta)/\theta}}{\left(1 + x_t^{1/\theta}\right)^{1+\theta}},$$

$$Y_t^{-\varphi}(1 + x_t^{1/\theta}) = \beta(1 + r_t)h_{1t},$$

$$2^{-\theta}\left(1+x_t^{1/\theta}\right)^{\theta}\left[Y_t^{1-\varphi} + \beta(1 + x_t^{1/\theta})h_{2t}\right] = (1+\theta)\chi_0\left[Y_t L_t^{\chi} + \beta\left(1+x_t^{1/\theta}\right)^{1+\theta} h_{3t}\right].$$

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where expectations of next period variables are given functions of this period's economic state: h_{1t}, h_{2t}, h_{3t} (discretion)

Markov Perfect Equilibria of the Game Γ_1

Along the equilibrium path of any Markov Perfect Equilibrium of Γ_1, state variables are degenerate (only operative off equilibrium path)

As a result, along the equilibrium path:

$$\begin{aligned} h_{1t} &= E_t Y_{t+1}^{-\varphi}(1 + x_{t+1}^{-1/\theta}) = h_1 \\ h_{2t} &= E_t \frac{Y_{t+1}^{1-\varphi}}{1 + x_{t+1}^{-1/\theta}} = h_2 \\ h_{3t} &= E_t \frac{Y_{t+1} L_{t+1}^{\chi}}{\left(1 + x_{t+1}^{-1/\theta}\right)^{1+\theta}} = h_3 \end{aligned}$$

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\end{aligned}
$$

Note: we will not write out how play evolves off of the equilibrium path, but simply assert that it agents will continue to play optimally (Phelan-Stachetti, 2001)

Solving for Markov Perfect Equilibria

Solve: $$V_t = \max_{\{r_t\}} \left\{ \frac{Y_t^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\}$$

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where h_1, h_2, h_3 are exogenous constants.

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Finally, impose equilibrium conditions:

$$h_1 = E_t Y_{t+1}^{-\varphi}(1 + x_{t+1}^{-1/\theta}),\ \ h_2 = E_t \frac{Y_{t+1}^{1-\varphi}}{1+x_{t+1}^{-1/\theta}},\ \ h_3 = E_t \frac{Y_{t+1} L_{t+1}^{\chi}}{\left(1+x_{t+1}^{-1/\theta}\right)^{1+\theta}}.$$

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Finally, impose equilibrium conditions:

$$h_1 = E_t Y_{t+1}^{-\varphi}(1 + x_{t+1}^{-1/\theta}), \;\; h_2 = E_t \frac{Y_{t+1}^{1-\varphi}}{1+x_{t+1}^{-1/\theta}}, \;\; h_3 = E_t \frac{Y_{t+1} L_{t+1}^\chi}{\left(1+x_{t+1}^{-1/\theta}\right)^{1+\theta}}.$$

Note: there can still be multiplicity here, e.g. if h_1, h_2, h_3 are "bad"

Solving for Markov Perfect Equilibria

Solve policymaker's problem via Lagrangean, yielding:

$$
\begin{aligned}
\lambda_t^{\text{Euler}} &= 0 \\
\chi_0 L_t^{1+\chi} &= \lambda_t^Y \frac{L_t}{Y_t} - \lambda_t^x (1+\theta)\chi_0 Y_t \chi L_t^\chi \\
\lambda_t^Y \frac{L_t}{Y_t} &= Y_t^{1-\varphi} + \lambda_t^x \left[(1-\varphi) 2^{-\theta} (1 + x_t^{1/\theta})^\theta Y_t^{1-\varphi} - (1+\theta)\chi_0 Y_t L_t^\chi \right] \\
\lambda_t^Y 2^\theta \frac{1+\theta}{\theta} \frac{x_t - 1}{(1 + x_t^{1/\theta})^{2(1+\theta)}} &= \lambda_t^x \left\{ 2^{-\theta} \left[\frac{Y_t^{1-\varphi}}{1 + x_t^{1/\theta}} + \frac{1+\theta}{\theta} \beta h_2 \right] - \chi_0 \beta \frac{(1+\theta)^2}{\theta} h_1 \right\}
\end{aligned}
$$

Combine these first-order conditions with private sector optimality constraints

Results

Proposition 7: The inflation rate π in any Markov Perfect Equilibrium of the game Γ_1 must satisfy the condition:

$$\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\,\frac{1+\pi^{1/\theta}}{1+\pi^{(1+\theta)/\theta}}\times$$

$$\left\{1-\frac{(\pi-1)\left[1+\chi-(1-\varphi)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right]}{(\pi-1)\left[1-(1-\varphi)\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right]+(1+\pi^{(1+\theta)/\theta})\left[1-\frac{1}{1+\theta}\frac{1+\beta\pi^{(1+\theta)/\theta}}{1+\beta\pi^{1/\theta}}\right]}\right\}=\frac{1}{1+\theta} \qquad (*)$$

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Proposition 8: Let $\varphi = 1$, $\chi = 0$, and $\beta > \max\{1/2, 1/(1+2\theta)\}$. *Then there is precisely one value of π that satisfies equation $(*)$.*

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Proposition 8: Let $\varphi=1$, $\chi=0$, and $\beta>\max\{1/2,1/(1+2\theta)\}$. *Then there is precisely one value of π that satisfies equation $(*)$.*

Note:

- $\varphi=1$, $\chi=0$ are not special, but simplify algebra in proofs
- there is a unique equilibrium for wide range of parameters
- confirmed by extensive numerical simulation in Matlab

Repeated Stackelberg Play, with Money

Given money supply m_t, expectations h_1, h_2, h_3, and private sector optimality conditions:

$$\frac{L_t}{Y_t} = 2^{\theta} \frac{1 + x_t^{(1+\theta)/\theta}}{\left(1 + x_t^{1/\theta}\right)^{1+\theta}},$$

$$Y_t^{-\varphi}(1 + x_t^{1/\theta}) = \beta(1 + r_t)h_1,$$

$$2^{-\theta}\left(1+x_t^{1/\theta}\right)^{\theta}\left[Y_t^{1-\varphi} + \beta(1 + x_t^{1/\theta})h_2\right] = (1+\theta)\chi_0\left[Y_t L_t^{\chi} + \beta\left(1+x_t^{1/\theta}\right)^{1+\theta} h_3\right],$$

$$m_t = Y_t \frac{2^{\theta} x_t}{(1 + x_t^{1/\theta})^{\theta}}$$

Solve for:

$$Y_t = Y(m_t), \quad x_t = x(m_t), \quad L_t = L(m_t), \quad r_t = r(m_t).$$

Repeated Stackelberg Play, with Money

Then solve: $V_t = \max_{\{m_t\}} \left\{ \frac{Y_t^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\}$ subject to:

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$$Y_t = Y(m_t), \quad x_t = x(m_t), \quad L_t = L(m_t), \quad r_t = r(m_t).$$

King and Wolman (2004): There are "good" and "bad" expectations h_1, h_2, h_3, which result in "good" and "bad" private sector equilibria $Y_t = Y(m_t)$, $x_t = x(m_t)$, $L_t = L(m_t)$, $r_t = r(m_t)$.

Repeated Simultaneous Play, with Money

Solve: $$V_t = \max_{\{m_t\}} \left\{ \frac{Y_t^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\}$$ subject to:

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$$m_t = Y_t \frac{2^\theta x_t}{(1 + x_t^{1/\theta})^\theta}$$

Repeated Simultaneous Play, with Money

Solve: $$V_t = \max_{\{m_t\}} \left\{ \frac{Y_t^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\}$$ subject to:

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$$m_t = Y_t \frac{2^\theta x_t}{(1 + x_t^{1/\theta})^\theta}$$

But first-order condition with respect to m_t:

$$\lambda_t^m = 0$$

Conclusions

- There are two definitions of "discretion" in the literature
- These definitions differ in terms of within-period timing of play
- Within-period timing has *major* equilibrium implications
- In the New Keynesian model with repeated Stackelberg play, there are multiple equilibria (King-Wolman, 2004)
- In the New Keyneisan model with repeated simultaneous play, there is a unique equilibrium (this paper)
- Open questions: other NK models, models with a (nondegenerate) state variable